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RECURSIVE GENERATION OF SIMPLE PLANAR QUADRANGULATIONS WITH VERTICES OF DEGREE 3 AND 4

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Abstract

We describe how the simple planar quadrangulations with vertices of degree 3 and 4, whose duals are known as octahedrites, can all be obtained from an elementary family of starting graphs by repeatedly applying two expansion operations. This allows for construction of a linear time generator of all graphs in the class with at most a given order, up to isomorphism.

Keywords: planar graph, octahedrite, quadrangulation, generation.

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1. Introduction

By a *simple planar quadrangulation with vertices of degree* 3 *and* 4 (SPQ(3,4)) we mean a connected simple graph imbedded on the sphere whose faces have size 4 and whose vertices have degree 3 or 4. We do not distinguish an outer face. The duals of SPQ(3,4)s are sometimes called *octahedrites*. They are the connected simple 4-regular planar graphs whose faces have size 3 and 4. Figure 10 shows all the SPQ(3,4)s with size at most 16.

Two planar graphs are regarded as the same if there is an embedding-preserving isomorphism (possibly reflectional) between them. This is the same as graph isomorphism in the case of 3-connected graphs, which is the case for the graphs in this paper. Table 1 gives the numbers of SPQ(3,4)s up to 133 vertices.

Let $\mathcal{C}$ be a class of planar graphs, $\mathcal{S}$ a subset of $\mathcal{C}$, and $\mathcal{F}$ a set of mappings from $\mathcal{C}$ to the power set $2^{\mathcal{C}}$. We say that $(\mathcal{S}, \mathcal{F})$ *recursively generates* $\mathcal{C}$ if for every $G \in \mathcal{C}$ there is a sequence $G_1, G_2, \ldots, G_k = G$ in $\mathcal{C}$ where $G_1 \in \mathcal{S}$ and, for each i, $G_{i+1} \in F(G_i)$ for some $F \in \mathcal{F}$. In many practical examples, including that in this paper, there is some integral graph parameter (such as the number of vertices) which is always increased by mappings in $\mathcal{F}$; in this case we refer to these mappings as *expansions*, their inverses as *reductions*, and the graphs in $\mathcal{S}$ as *irreducible*. In this terminology, $(\mathcal{S}, \mathcal{F})$ recursively generates $\mathcal{C}$ if every graph in $\mathcal{C} - \mathcal{S}$ is reducible.

Recursive generation algorithms for very many classes of planar graphs have appeared in the literature. Expansions usually take the form of removing some small subgraph and replacing it by a larger subgraph. We will note the examples of 3-connected [16], 3-regular [8], minimum degree 4 [1], 4-regular [7, 13], 5-regular [12], minimum degree 5 [6], and fullerenes (whose duals have minimum degree 5 and maximum degree 6) [2, 11]. Such constructions can be used to build practical generators [5] as well as to prove properties of graph classes by induction.

The structure and generation of different classes of planar quadrangulations are studied in [3, 4, 15]. The quadrangulations considered in this paper have been studied by Deza, Shtogrin and Dutour [9, 10]. Generation of them is included in a more general algorithm of Brinkmann, Harmuth and Heidemeier [4]. Their algorithm generates the 4-regular graphs dual to these quadrangulations by stitching together "patches" formed by straight-ahead paths. Our approach is the more traditional one of expansions as described generally above.

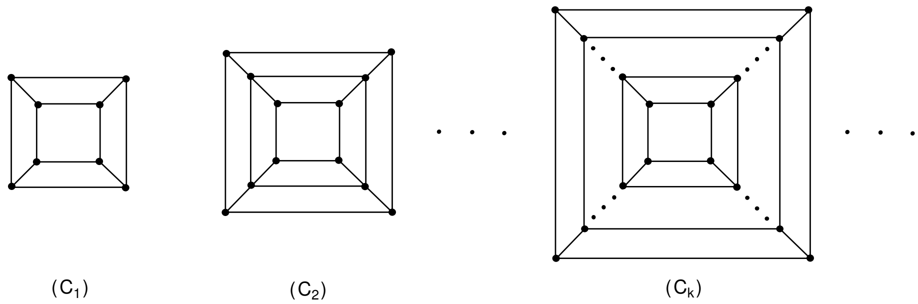

Figure 1. The set S of starting graphs.

Our starting set $\mathcal{S}$ consists of the infinite family $\{C_i \mid i \geq 1\}$ depicted in Figure 1. We employ two types of expansion, which we define via their corresponding reductions, as shown in Figure 2.

Let $p = v_1 \cdots v_k$ be a path in an SPQ(3,4) such that all the internal vertices of p have degree 4. We say that p has a *bend* at vertex v_i, $i \in \{2, \ldots, k-1\}$, if v_{i-1} and v_{i+1} appear consecutively around v_i. In that case, v_i is a *right* bend if v_{i+1} appears in anticlockwise order after v_{i-1} around v_i and otherwise a *left* bend.

Let $\mathcal{F}$ be the set of expansions inverse to the reductions $\{P_1, P_2\}$ shown in Figure 2. Our aim is to prove that the class of all SPQ(3,4)s is generated by $(\mathcal{S}, \mathcal{F})$.

Reduction $P_1(p)$ requires a path without bends between two distinct vertices of degree 3. Reduction $P_2(p)$ requires such a path with exactly one bend. The mirror image of $P_1(p)$ is also allowed and we will not consider it different from $P_1(p)$.

For all the reductions, the vertices on the path and their neighbours, plus the outside corner vertex drawn as an open circle in the figure, must be distinct.

The following rules should be considered in interpreting the pictures of this paper:

- half-edges indicate that at an edge must occur at this position in the cyclic order around the vertex;
- a triangle indicates that zero or more edges may occur at this position in the cyclic order around the vertex;

- if neither a half-edge nor a triangle is present in the angle between two edges in the picture, then these two edges must follow each other directly in the cyclic ordering of edges around that vertex.

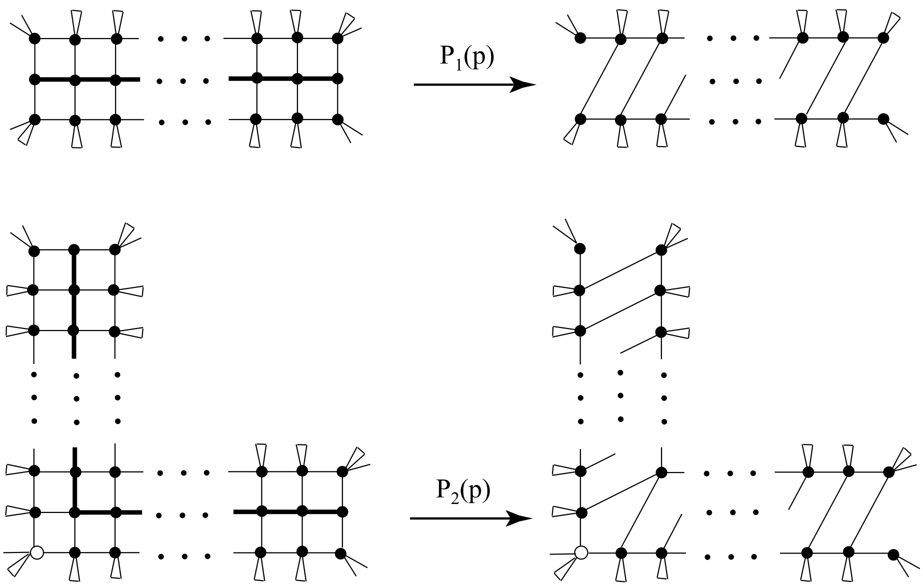

Figure 2. Reductions P_1 and P_2 (path p is drawn as a thick line).

2. Generation Algorithm

A cycle with length k of a plane graph is a *separating k-cycle* if it is not a face.

Theorem 1 in [9] proves that all SPQ(3,4)s are 3-connected. Every SPQ(3,4) is a bipartite graph, so it does not have any separating 3-cycles. The following lemma proves the absence of separating 4-cycles in every SPQ(3,4) which is not a member of $\mathcal{S}$.

Lemma 1. *If G is an* SPQ(3,4) *which is not a member of $\mathcal{S}$, then G does not have any separating 4-cycles.*

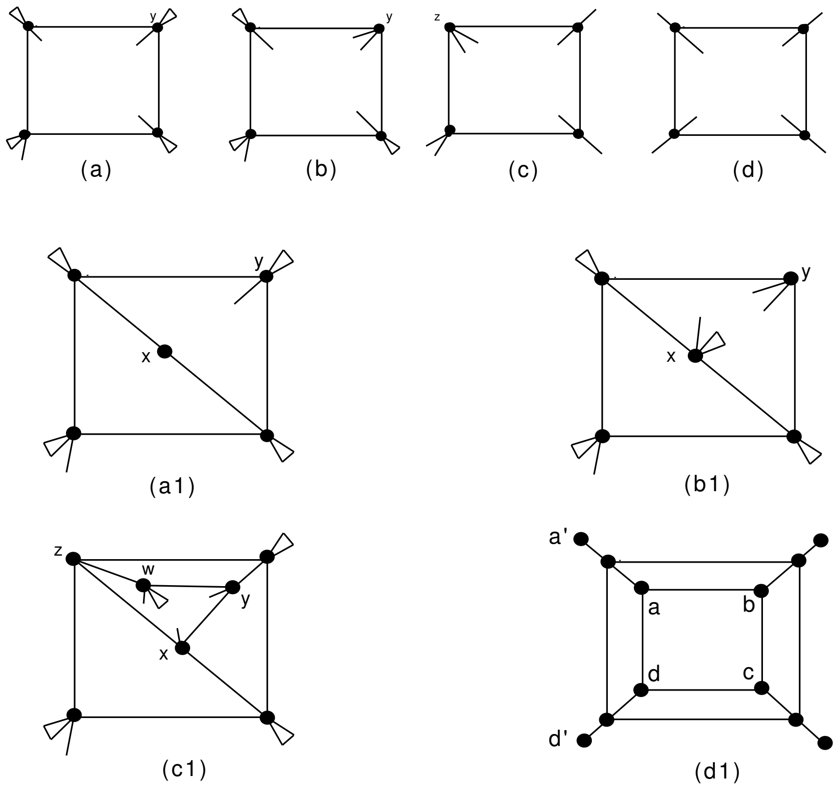

Figure 3. Cases for a separating 4-cycle.

Proof. Let G be a smallest SPQ(3,4) which has a separating 4-cycle but is not in S. By the symmetry between the inside and outside of the separating 4-cycle, four cases can occur as shown in Figure 3. Cases (a) and (b) do not happen because of the 3-connectivity (in Figures 3(a1) and (b1), $\{x, y\}$ is a cut). Let C be a separating 4-cycle of G such that there is no other separating 4-cycle inside C. Then C is not a separating 4-cycle of type (c), since cycle $C' = xywz$ (Figure 3(c1)) is a separating 4-cycle inside C. In case (d), because there is no separating 4-cycle inside C, vertices a, b, c and d are distinct and have degree 3 (Figure 3(d1)). Construct graph G' from G by deleting vertices a, b, c and d. Then G' is an SPQ(3,4) which is smaller than G and so does not have any separating 4-cycles by the definition of G. Therefore, the degrees of vertices a', b', c' and d' are 3 and G' is a cube. This proves that G is C_2 (the second member of $\mathcal{S}$), which is a contradiction. ■

The next lemma, due to Nakamoto, explains how vertices of degree 3 appear in a simple planar quadrangulation with minimum degree 3.

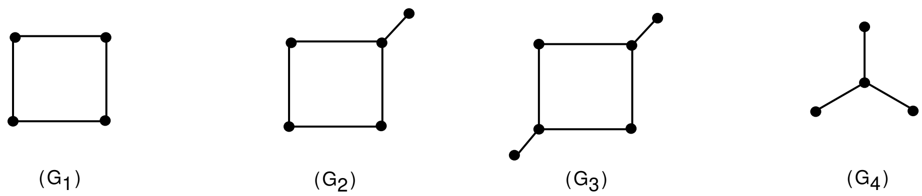

Figure 4. Subgraphs for Lemma 2.

Lemma 2 [15]**.** *Let G be a simple quadrangulation with minimum degree* 3 *and let H be a component of the subgraph induced by the vertices of degree* 3*. Then H is one of the following graphs:*

- *a cycle of even length at least* 8;
- *a path* (*possibly of a single vertex*);
- *a cube, in which case $G = H$*;
- *one of the four graphs of Figure* 4*.*

By Euler's formula, each SPQ(3,4) with n vertices has exactly 8 vertices with degree 3, $n - 2$ faces and $2n - 4$ edges. This implies the following restriction of Lemma 2 to the case of SPQ(3,4)s.

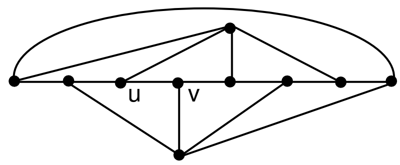

Figure 5. Pseudo-double wheel.

Lemma 3. *Let G be an* SPQ(3,4) *and let D be the subgraph induced by the vertices of degree* 3*. Then D is one of the following graphs:*

- *a cycle of length* 8*, in which case G is a pseudo-double wheel* (*Figure* 5);
- *a cube, in which case $G = D$*;
- *a union of two disjoint copies of G_4* (*Figure* 4);

- *a union of two disjoint cycles of length* 4, *in which case* G *is a member of* $\mathcal{S}$;
- *a union of disjoint paths.*

Proof. If D is a cycle of length 8, then G is a pseudo-double wheel (as noted in [15]). If G is not a cube or a pseudo-double wheel and D is not a union of disjoint paths, then by Lemma 2, D has a component H which is one of the graphs in Figure 4. If H is one of G_1, G_2 or G_3, then G has a separating 4-cycle. Therefore, by Lemma 1, G is a member of $\mathcal{S}$. This implies that H is G_1 and D consists of two disjoint cycles with length 4.

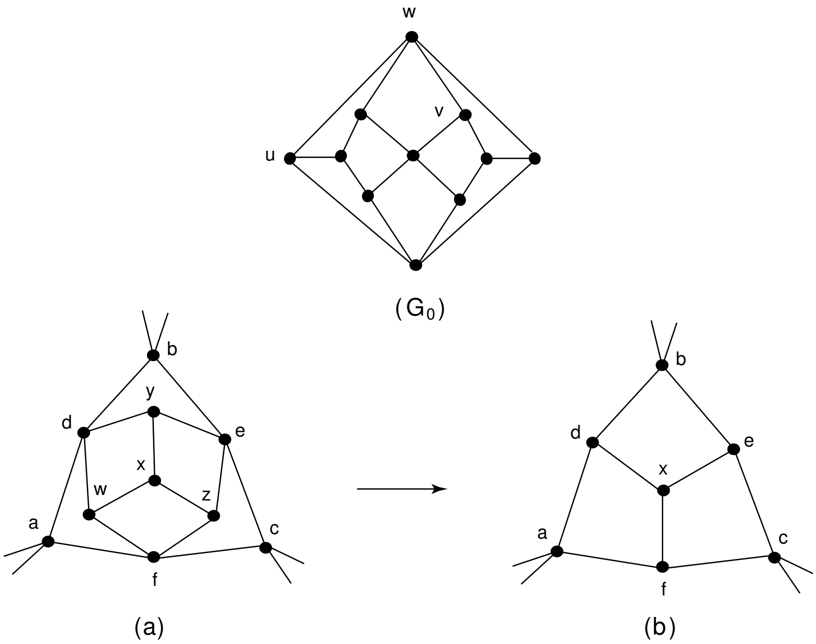

Figure 6. Graph G_0 and the method of constructing G' from G in the proofs of Lemma 3 and Theorem 1.

Now, suppose H is a copy of G_4. We complete the proof by induction on the number of vertices. Graph G has a subgraph as shown in Figure 6(a). Because of the absence of separating 4-cycles, if the degree of one of a, b and c is 3 then the degrees of all of them are 3 and G is G_0 (Figure 6) which is the smallest graph whose subgraph induced by the vertices of degree 3 has G_4 as a component. Suppose that G is not graph G_0. Obtain graph G' from

G by removing vertices y, w and z and adding edges xd, xe and xf (Figure 6(b)). By induction the subgraph of G' induced by the vertices of degree 3 consists of two copies of G_4, which proves D also consists of two copies of G_4. ∎

Theorem 1. *The class of all* SPQ(3,4)s *is generated by* $(\mathcal{S}, \mathcal{F})$.

Proof. Denote by $\mathcal{B}$ the set of all SPQ(3,4)s whose subgraphs induced by their vertices of degree 3 consist of two copies of G_4.

Let G be a graph in $\mathcal{B}$, and let H_1 and H_2 be the two components of the subgraph induced by the vertices of degree 3. Let u be a vertex of H_1 and v a vertex of H_2 whose distance in G is least among all such pairs. We claim that there is a shortest path p from u to v such that $P_1(p)$ applies.

We prove the claim by induction on the number of vertices. The smallest member of $\mathcal{B}$ is G_0 (Figure 6) and $P_1(uwv)$ applies to it. Suppose that G is the smallest member of $\mathcal{B}$ which does not satisfy the claim. By the definition of $\mathcal{B}$, G has a subgraph as shown in Figure 6(a). The degrees of a, b and c are 4 since G is not G_0. Obtain graph G' from G as explained in the proof of Lemma 3. Graph G' is a member of $\mathcal{B}$ which is smaller than G. So, it satisfies the claim and has two vertices u and v with a shortest path p between them such that $P_1(p)$ applies. Without loss of generality, suppose $v = d$. It is easy to see that $P_1(pw)$ applies in G, which proves the claim.

Let G be an SPQ(3,4) which is not in $\mathcal{B} \cup \mathcal{S}$. If G is a pseudo-double wheel then $P_1(uv)$ applies (Figure 5), and otherwise if G has some adjacent vertices of degree 3, then by Lemma 3 the subgraph D induced by the vertices of degree 3 is a union of paths. Let u be a vertex which has degree 1 in D, and let v be its neighbour in D. Then $P_1(uv)$ applies to G.

Finally suppose that G does not have any adjacent vertices of degree 3. Let p be a path in G such that:

(i) p is a shortest path among all the paths between two vertices of degree 3, say u and v;

(ii) subject to condition (i), the segment of p from u to the first bend is as long as possible.

We claim that p has at most one bend. Suppose $p = v_0 v_2 \cdots v_n$ where $v_0 = u$ and $v_n = v$. For $i \in \{0, \ldots, n\}$, if v_i is not a bend of p, let u_i be the neighbour of v_i on the right and w_i be the neighbour of v_i on the left when we move along p. Otherwise let x_i and y_i be the neighbours of v_i other than v_{i-1} and v_{i+1}, in anticlockwise order.

Suppose that p has more than one bend.

Let v_i and v_j, $0 < i < j < n$, be the first two bends of p. If p has the same type of bend, say a right bend, at v_i and v_j, then the walk $v_0 \cdots v_{i-1} u_{i+1} \cdots u_{j-1} v_{j+1} \cdots v_n$ is shorter than p which is a contradiction (see Figure 7(a)). If instead p has a left bend at v_i and a right bend at v_j then consider the walk $q = v_0 \cdots v_i y_i u_{i+1} \cdots u_{j-1} v_{j+1} \cdots v_n$. Walk q has the same length as p and so is a shortest path but the first straight segment of q is longer than that of p (see Figure 7(b)). This is also a contradiction. Therefore, p has at most one bend.

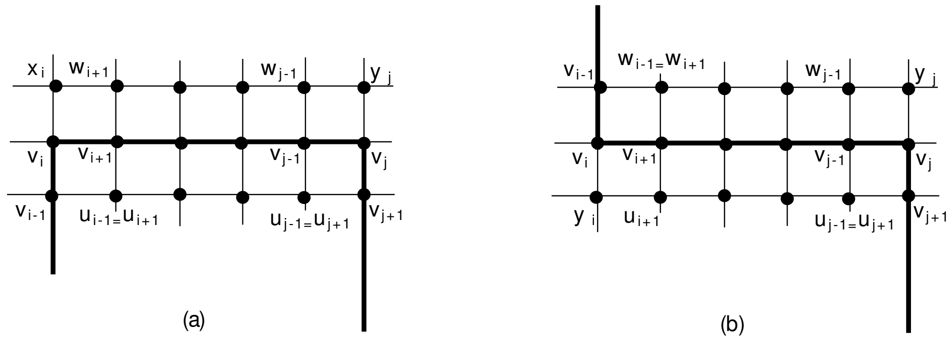

Figure 7. Bent paths in the proof of Theorem 1 (path p is drawn as a thick line).

It remains to show that the vertices of p and its neighbouring vertices are all distinct, apart from the two neighbours which are necessarily the same at the inside of a bend. Since p is a shortest path, it does not have any chords. If the u_i's and w_j's are all distinct, then p does not have any bend and $P_1(p)$ applies.

Suppose that $u_j = w_i$ for some $0 \leq i \neq j \leq n$. Since p is a shortest path from u to v and G does not have any cycle with odd length, $|j - i| = 2$. But then cycle $v_i v_{(i+j)/2} v_j u_j$ is a separating 4-cycle, which contradicts Lemma 1. By the same argument, $u_i \neq u_j$ and $w_i \neq w_j$ for $i \neq j$ except for the case of a bend at v_i where $u_{i-1} = u_{i+1}$ or $w_{i-1} = w_{i+1}$. So suppose without loss of generality that p has a left bend at v_i. Vertices x_i and y_j are different from u_j and w_k for all j, k by the same argument as before. Let z be the common neighbour of x_i and y_i other than v_i. Since there are no two vertices on p other than v_{i-1} and v_{i+1} which have a common neighbour not lying on p, z is not a vertex on p. Vertices x_i and y_i are different from v_j, w_k, u_ℓ for all j, k, ℓ, so z is different from u_j and w_j for $0 < j \neq i < n$ (otherwise z

must have degree at least 5). Suppose $z = u_0$. Since p is a shortest path and G does not have any cycle with odd length, $i = 3$. According to the definition of p, vertices u_1 and u_2 must have degree 4. So, cycle $u_0u_1u_2u_3$ is a separating 4-cycle of G, which proves $z \neq u_0$. Suppose instead that $z = w_0$. For the same reason, $i = 3$. Since G is a quadrangulation, u_3 is a neighbour of u_0. This shows that $u_0u_1u_2u_3$ is a separating 4-cycle of G. Similarly, $z \notin \{u_n, w_n\}$. Therefore all conditions required to apply $P_2(p)$ hold. ■

Figure 8 shows a graph which has only reductions of type P_1. It is easy to see how to make infinitely many SPQ(3,4)s with the same property.

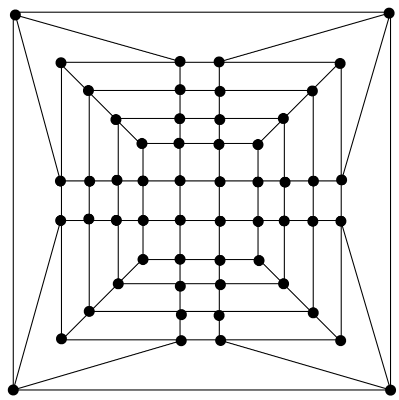

Figure 8. An SPQ(3,4) without any P_2 reductions.

The smallest SPQ(3,4) which has only reductions of type P_2 has 136 vertices. In Figure 9, we show a 158 vertex SPQ(3,4) which has only reductions of type P_2 even if we relax the definition of P_1 reductions to allow some of the vertices adjacent to the path to be equal. Further SPQ(3,4)s with the same property can be made by drawing a set of concentric closed curves in the place indicated by the dashed line and converting intersections to vertices. The graphs in Figures 8 and 9 were found using the program ENU described in [4].

Theorem 1 can be used in conjunction with the method of [14] to produce a generator of non-isomorphic SPQ(3,4)s. Briefly the method works as follows. For each SPQ(3,4) G, one expansion is attempted from each equivalence class of expansions under the automorphism group of G. If the new graph is H, then H is accepted if the reduction inverse to the expansion by which H was constructed is equivalent under the automorphism group of H to a "canonical" reduction of H. The essential algorithmic requirements are computation of automorphism groups and canonical labelling, which are both easy to do in linear time using a depth-first search starting at the vertices of degree 3. In addition, we note that by restricting reductions (and their inverse expansions) to the types proven to exist in the proof of Theorem 1, no SPQ(3,4) has more than 756 reductions (1 straight and 2 bent reductions for each pair of edges incident to different vertices of degree 3). By [14, Theorem 3], this means that the set of all isomorphism types of SPQ(3,4)s of order at most n can be found in amortised time $O(n)$ per graph.

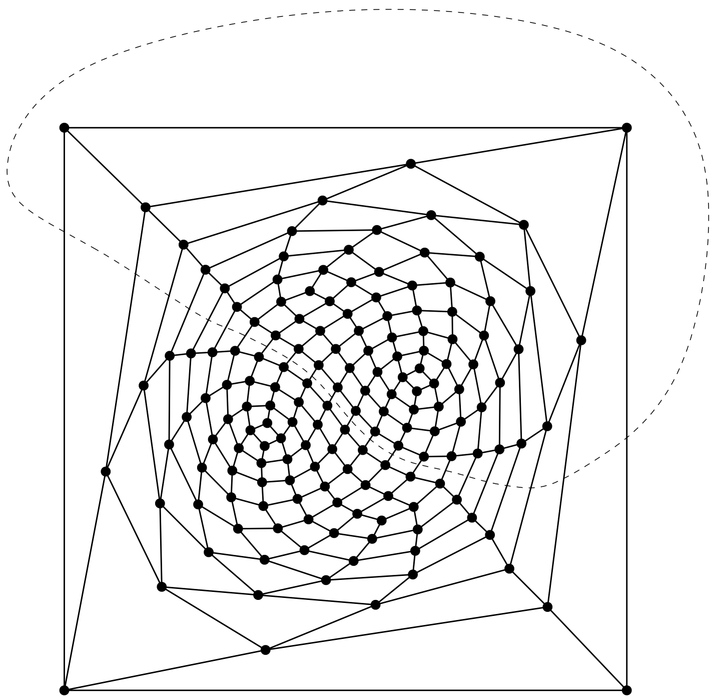

Figure 9. An SPQ(3,4) without any P_1 reductions.

APPENDIX

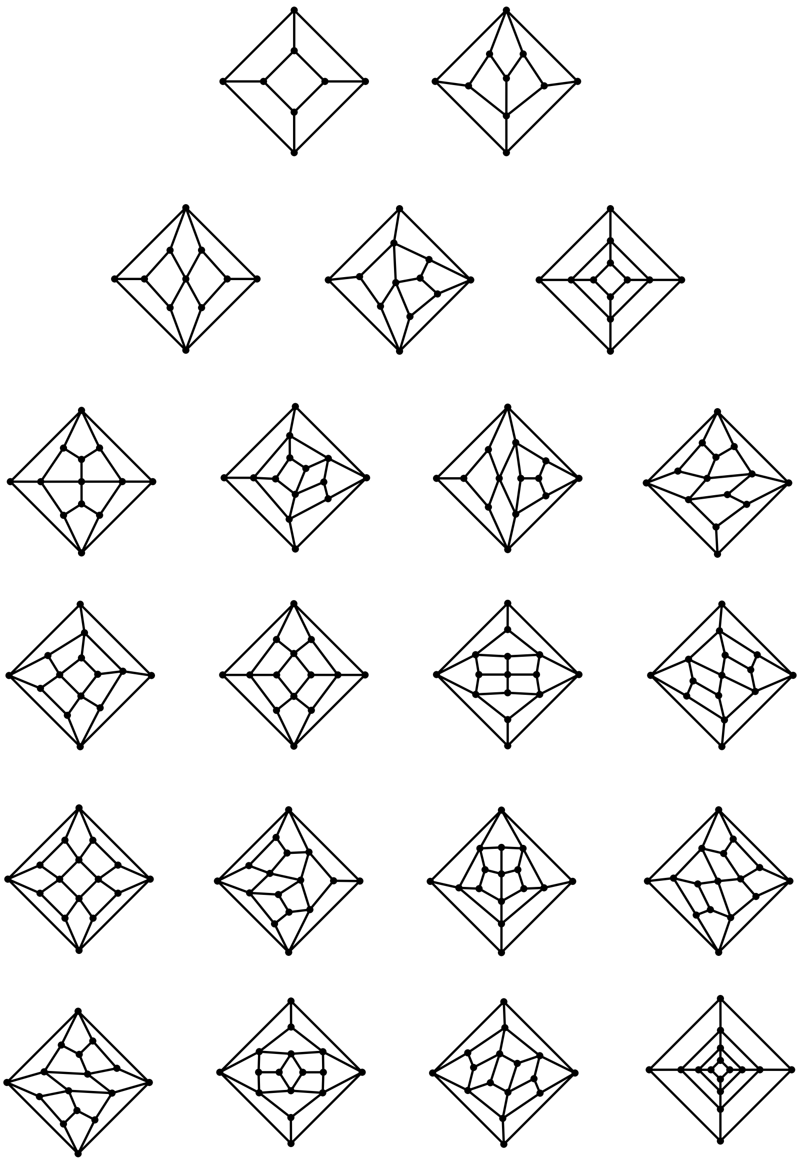

Figure 10. All SPQ(3,4)s with at most 16 vertices.

Table 1. The number N_n of SPQ(3,4)s with n vertices.

n	N_n	n	N_n	n	N_n	n	N_n	n	N_n	n	N_n
8	1	29	78	50	2045	71	9097	92	47928	113	103581
9	0	30	144	51	1554	72	13428	93	37362	114	143658
10	1	31	106	52	2505	73	10481	94	53183	115	112273
11	1	32	218	53	1946	74	15562	95	41861	116	157549
12	2	33	150	54	3008	75	12034	96	59160	117	123277
13	1	34	274	55	2322	76	17744	97	46518	118	171531
14	5	35	212	56	3713	77	14021	98	66396	119	134957
15	2	36	382	57	2829	78	20277	99	51531	120	186242
16	8	37	279	58	4354	79	15814	100	73024	121	146663
17	5	38	499	59	3418	80	23311	101	57843	122	204872
18	12	39	366	60	5233	81	18112	102	81084	123	159091
19	8	40	650	61	4063	82	26257	103	63334	124	220351
20	25	41	493	62	6234	83	20666	104	89938	125	174644
21	13	42	815	63	4784	84	29741	105	70269	126	240238
22	30	43	623	64	7301	85	23345	106	98783	127	187785
23	23	44	1083	65	5740	86	33881	107	77795	128	260958
24	51	45	800	66	8514	87	26228	108	108424	129	204168
25	33	46	1305	67	6631	88	37786	109	85359	130	281936
26	76	47	1020	68	10103	89	29911	110	120378	131	221774
27	51	48	1653	69	7794	90	42471	111	93426	132	303638
28	109	49	1261	70	11572	91	33187	112	130756	133	239100

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