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VERTEX-DISTINGUISHING EDGE-COLORINGS OF LINEAR FORESTS

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Abstract

In the PhD thesis by Burris (Memphis (1993)), a conjecture was made concerning the number of colors $c(G)$ required to edge-color a simple graph G so that no two distinct vertices are incident to the same multiset of colors. We find the exact value of $c(G)$ — the irregular coloring number, and hence verify the conjecture when G is a vertex-disjoint union of paths. We also investigate the point-distinguishing chromatic index, $\chi_0(G)$, where sets, instead of multisets, are required to be distinct, and determine its value for the same family of graphs.

Keywords: irregular edge-coloring, vertex-distinguishing edge-coloring, point-distinguishing chromatic index.

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1. Introduction

Consider a simple graph G. In [1] Aigner and Triesch introduced the following problem. Let C be a color set, and let $w\colon E(G) \to C$ be an *edge-coloring* (not necessarily proper) of G. Let then $S(v)$ (or $S_w(v)$ if the coloring w is not obvious) denote the multiset of colors assigned to the edges incident to $v \in V(G)$. Such an edge-coloring is called *irregular* or *vertex-distinguishing* if $S(u) \neq S(v)$ for any two distinct vertices $u, v \in V(G)$. It exists if and only

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if G contains no more than one isolated vertex and no isolated edges. Then such a graph is said to be *vertex-distinguishing edge-colorable* (*vdec*-graph), while the minimal number of colors necessary to obtain its irregular edge-coloring is called the *irregular coloring number* and is denoted by $c(G)$. As one may check in [1], this parameter is just a simple modification of the well known *irregularity strength* of a graph, see e.g. [7].

Let $\Delta(G)$ denote the maximum degree, and let n_d (or $n_d(G)$) denote the number of vertices of degree d in G. Note that if there is an irregular edge-coloring of G with k colors, then, by the standard combinatorial formula for the number of multisets of a given size, we must have that

$$\binom{k+d-1}{d} \geqslant n_d \tag{1}$$

for each $d \geqslant 1$. The following conjecture was posed by Burris in [4].

Conjecture 1. *Let G be a vdec-graph, and let k be the minimum integer such that $\binom{k+d-1}{d} \geqslant n_d(G)$ for $1 \leqslant d \leqslant \Delta(G)$. Then $c(G) = k$ or $k+1$.*

In the general case, this conjecture appears to be difficult. Several results, mostly for connected graphs, are included in [4]. In [8, 9] we investigated some aspects of the irregular edge-coloring of 2-regular graphs, which are simply (vertex-disjoint) unions of cycles. In this paper, we prove that Conjecture 1 holds for the unions of paths. Some special cases of this problem were already investigated by Aigner and Triesch [1], but only when all the paths were of the same lengths, but they did not study all such cases. In our research we apply a similar method as Balister, Bollobás and Schelp in [3], where the graph parameter $\chi'_s(G)$, in the case when $\Delta(G) = 2$, is investigated. It is called the *strong coloring number* (or the *observability*) of a graph and may be viewed as the modification of $c(G)$ by the restriction that an edge-coloring has to be proper (see also [4, 5, 6]).

It will appear in the next section that in the case when G is a disjoint union of paths — a *linear forest*, the problem of irregular edge-coloring is equivalent to a certain problem of packing of the line graph of G into a special pseudograph. We solve it in Section 3. In the last section, we study the *point-distinguishing chromatic index*, $\chi_0(G)$ (see [10]), which is another modification of $c(G)$ where sets, instead of multisets, are required to be distinct (while an edge-coloring does not have to be proper). We determine its value also for all linear forests.

2. Paths Packing Problem

Let M_n denote the complete graph K_n with a single loop added at each vertex. Though M_n is actually a pseudograph, we shall call it a graph. Moreover, write P_{n+1} for a path of length n (on $n+1$ vertices) and $P(v_1, v_2, \ldots, v_{n+1})$ for the *trail* of length n on the vertices v_i and with edges $v_i v_{i+1}$, $1 \leqslant i \leqslant n$. We do not require v_i to be distinct. Such a trail is called *open* if $v_1 \neq v_{n+1}$, otherwise it is called *closed*. For any two graphs G_1 and G_2, write $G_1 \cup G_2$ for the vertex-disjoint union of G_1 and G_2.

If G_1 and G_2 are graphs, a *packing* of G_1 into G_2 is a map $f \colon V(G_1) \to V(G_2)$ such that $xy \in E(G_1)$ implies $f(x)f(y) \in E(G_2)$ and the induced map on edges $xy \mapsto f(x)f(y)$ is an injection from $E(G_1)$ to $E(G_2)$. We do not require f to be injective on vertices, so if G_1 contains a path, its image in G_2 will be a trail.

In this paper we investigate the case when a graph G is a (vertex-disjoint) union of p paths, hence $G = P_{l_1+2} \cup \cdots \cup P_{l_p+2}$ with $l_i \geqslant 1$ (G is a vdec-graph and we ignore the presence or absence of an isolated vertex, since this does not influence a coloring). The line graph $L(G) = P_{l_1+1} \cup \cdots \cup P_{l_p+1}$ of such a graph is also a union of p paths. If G is given a vertex-distinguishing edge-coloring by n colors, say $C = \{1, \ldots, n\}$, then we get a packing of $L(G)$ as p edge-disjoint trails in M_n. Each edge of G corresponds to a vertex of $L(G)$ which is mapped to a (color) vertex of M_n, where $V(M_n) = \{1, \ldots, n\}$. Since the coloring is vertex-distinguishing, the trails being images of the paths are indeed edge disjoint in M_n. Moreover, they are open, since the $2p$ endpoints of the paths have to be mapped to distinct vertices. Conversely, if we have a packing of $L(G)$ into M_n such that each endpoint of the paths in $L(G)$ is mapped to a different vertex, then we can color each edge of G with the image of the corresponding vertex of $L(G)$ in M_n. The obtained edge-coloring is vertex-distinguishing, hence the value of $c(G)$ is equal to the smallest n such that a packing of $L(G)$ into M_n with all endpoints mapped to distinct vertices exists, see e.g. [1].

3. Irregular Edge-Coloring

We make use of the following result by Balister from [2] to solve the packing problem to which our problem was reduced and thus determine the exact value of $c(G)$ in the case described above.

Theorem 2. *Let $L = \sum_{i=1}^{p} t_i$, $t_i \geqslant 3$, with $L = \binom{n}{2}$ when n is odd and $\binom{n}{2} - \frac{n}{2} - 2 \leqslant L \leqslant \binom{n}{2} - \frac{n}{2}$ when n is even. Then we can write some subgraph of K_n as an edge-disjoint union of closed trails of lengths $t_1, \ldots, t_p$.* ■

Theorem 3. *The following conditions are both necessary and sufficient for packing $\bigcup_{i=1}^{p} P_{l_i+1}$, $l_i \geqslant 1$, into M_n with endpoints mapped to distinct vertices:*

$$(1) \quad L \leqslant \binom{n+1}{2} - \frac{r}{2} \qquad \text{if } r \text{ (or } n\text{) is even,}$$

$$(2) \quad L \leqslant \binom{n+1}{2} - p \qquad \text{if } r \text{ (or } n\text{) is odd,}$$

where $n = 2p + r$, $r \geqslant 0$, and $L = \sum_{i=1}^{p} l_i$. In particular, $L \leqslant \binom{n}{2}$ is always sufficient.

Proof. First we prove that the conditions are necessary.

Clearly, we cannot pack paths of total length L greater than the size of M_n, $\binom{n+1}{2}$. Moreover, if G' is the image of the described packing in M_n, it consists of p open trails, whose ends form a set of $2p$ vertices of odd degrees in G'. The remaining r vertices have even degrees in G'. Therefore, if r is odd (hence n is odd and the degrees of all vertices of M_n are even), we must delete at least p edges from M_n to obtain G'. Analogously, if r is even (hence n is also even and the degrees of all vertices of M_n are odd), we need to remove at least $\frac{r}{2}$ edges from M_n to obtain G'.

Now we prove the sufficiency of the conditions by induction on n.

We verified the cases for $n \leqslant 9$ by a computer program we created[1] (we might have done it without using a computer, but then the proof gets longer and more unclear), thus now let $n \geqslant 10$. Let $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_p$. If all the paths of $L(G)$ are of length one, we are done, since in M_n there is a set of $\lfloor \frac{n}{2} \rfloor \geqslant p$ independent edges. Therefore, we may assume $l_1 \geqslant 2$.

Let us first consider the case $p = 1$. If n is odd (even) and $l_1 = \binom{n+1}{2} - 1$ ($l_1 = \binom{n+1}{2} - \frac{n}{2} + 1$), remove one edge from the Eulerian graph M_n (remove $\frac{n}{2} - 1$ independent edges from M_n) to form a desired open trail. If $l_1 = \binom{n+1}{2} - 1 - q$ ($l_1 = \binom{n+1}{2} - \frac{n}{2} + 1 - q$), where $1 \leqslant q \leqslant n$, it is enough to remove q loops from the trail described above. Finally, if $l_1 \leqslant \binom{n}{2} - 2$ ($l_1 \leqslant \binom{n}{2} - \frac{n}{2}$) and $l_1 \geqslant 4$, observe that for $t_1 = l_1 - 1$ and $t_2 = \binom{n}{2} - (l_1 - 1)$ ($t_2 = \binom{n}{2} - \frac{n}{2} - (l_1 - 1)$), $L = t_1 + t_2$ ($L = t_1$ or $L = t_1 + t_2$) satisfies the

[1]The source code along with other necessary files are available at *http://home.agh.edu.pl/~cichacz/ang/preprints.php*.

assumptions of Theorem 2. Therefore, by its thesis, there is a closed trail T_1 of length $l_1 - 1$ in K_n. Clearly, there must also be some edge $uv \in E(K_n)$ that does not belong to this trail with u being one of its vertices. Adding this edge to T_1 yields an open trail of length l_1. If $l_1 \leqslant 3$, the result is obvious.

Let then $p \geqslant 2$ and denote $l_0 := l_1 + l_2 - 2$. Since $l_1 \geqslant 2$ (and $l_2 \geqslant 1$), we have $l_0 \geqslant 1$. We consider two cases; first, when the sum of the lengths of the paths is relatively small and then when this sum is large enough.

Case 1. Assume that the lengths $l_0, l_3, l_4, \ldots, l_p$ satisfy the assumptions of the theorem for M_{n-2}.

In such a case we can pack $P_{l_0+1} \cup \bigcup_{i=3}^{p} P_{l_i+1}$ into M_{n-2} (M_n with two vertices, say a and b, removed) by induction, obtaining as a resulting image of this packing edge-disjoint trails of lengths $l_0, l_3, l_4, \ldots, l_p$. Let u and v be the ends of one of these trails of length l_0 in M_{n-2} and let u' be a vertex of this trail such that the distance along this trail between u and u' equals $l_1 - 1$ (hence the distance along this trail between v and u' equals $l_2 - 1$). Then, by adding the edge $u'a$ to the part of this trail between u and u', we obtain a trail of length l_1 with endpoints u, a, and by adding the edge bu' to the rest of the trail of length l_0, we obtain a trail of length l_2 with endpoints v, b. Thus we obtain the desired packing.

Case 2. Assume then that the lengths $l_0, l_3, l_4, \ldots, l_p$ do not satisfy the assumptions of the theorem for M_{n-2}, i.e., that their sum exceeds the bound from (1) or (2) for M_{n-2} ($n \geqslant 10$).

Then, however, one can easily show that the length of at least one path, $l_1, l_2, \ldots, l_{p-1}$ or l_p, must exceed 7. On the contrary, suppose that $l_0 + l_3 + l_4 + \cdots + l_p \leqslant (7 + 7 - 2) + (p - 2)7 = 7p - 2$. For even n we would then by (1°) have $7p - 2 > \binom{(n-2)+1}{2} - \frac{r}{2}$ (since we have now $p - 1$ paths, r remains the same), hence, since $\frac{n}{2} = p + \frac{r}{2}$, $6p - 2 > \binom{n-1}{2} - \frac{n}{2}$, and thus (since $\frac{n}{2} \geqslant p$) $6\frac{n}{2} - 2 > \binom{n-1}{2} - \frac{n}{2}$. The obtained inequality, however,

$$n^2 - 10n + 6 < 0$$

does not hold for $n \geqslant 10$. Analogously, for odd n, by (2°) we would have $7p - 2 > \binom{n-1}{2} - (p - 1)$, hence $7\frac{n}{2} - 2 > \binom{n-1}{2} - (\frac{n}{2} - 1)$, while this inequality

$$n^2 - 11n + 8 < 0$$

is false for $n \geqslant 11$. Therefore, since $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_p$, we have $l_1 \geqslant 8$ and $l_0 - 6 \geqslant 1$. Let us reduce l_0 by 6 if n is odd or reduce it by 5 if n is even. Then keep reducing l_i, $i \geqslant 3$, and what is left of l_0 by multiples of four until we have reduced all the lengths to at most four or until we have removed total (taking into account the initial reduction by 6 or 5) length of $2n-4$ if n is odd, or $2n-3$ if n is even. If we then denote the reduced lengths by $l'_0, l'_3, \ldots, l'_p$, we have $l'_i \geqslant 1$ for all i and $l'_0 + \sum_{i=3}^{p} l'_i \leqslant \max\{4(p-1), L-(2n-2)\}$ if n is odd, or $l'_0 + \sum_{i=3}^{p} l'_i \leqslant \max\{4(p-1), L-(2n-1)\}$ if n is even. Therefore, the paths $P_{l'_0+1}, P_{l'_3+1}, \ldots, P_{l'_p+1}$ satisfy the assumptions of the theorem for M_{n-2} (since our initial paths satisfied them for M_n) and we may pack them into this graph by induction, obtaining as a resulting image of this packing edge-disjoint trails of lengths $l'_0, l'_3, l'_4, \ldots, l'_p$. Assume M_{n-2} was formed of M_n by removing vertices a and b from it and let the mentioned trail of length l'_0 have endvertices u and v in M_{n-2}. Fix a vertex u' on this trail which is at distance at most $l_1 - 6$ along the trail from u and distance at most $l_2 - 1$ from v and so that this distance from v is congruent to $(l_2 - 1) \bmod 2$. It is possible, since $l'_0 \leqslant (l_1 - 6) + (l_2 - 1)$ and $l'_0 \geqslant 1$. Now we will use edges (and loops) incident to a or b (avoiding ab if n is odd) to complete the trails, so that they make up the desired packing.

First, for each P_{l_i+1} with $i \geqslant 3$ that has been shortened fix an endvertex v_i of the corresponding trail of length l'_i in M_{n-2} not equal to u'. Now pick $\frac{l_i - l'_i - 2}{2}$ paths of length two of the form $P(a, x, b)$ where x is neither u' nor any of v_i. Joining these paths with the edges $v_i a$, $v_i b$ and the trail of length l'_i gives a trail of length l_i in M_n with the same endvertices (because $\frac{l_i - l'_i - 2}{2}$ is an odd integer) as the trail of length l'_i in M_{n-2}. Now we need only to form two trails corresponding to P_{l_1+1} and P_{l_2+1} in M_n. We construct a trail of length l_2 by joining the part of the trail of length l'_0 between v and u' with the edge $u'a$ and with some number of the paths of length two between a and b. This trail has endvertices v and a or b. Similarly, we construct a trail of length l_1 by joining subsequently the part of the trail of length l'_0 between u and u' with the edge $u'b$, then with the loop at b, one of the remaining paths of length two between a and b, the loop at a, and with some of the rest of the paths of length two between a and b. If n is even, we additionally add the edge ab to the trail at the end. By our construction, this trail has endvertices u and a or b distinct (since $l_2 + l_1 = l'_0 + 1 + 7 + 4t$ if n is odd or $l_2 + l_1 = l'_0 + 1 + 6 + 4t$ otherwise, $t \geqslant 0$) from the endvertices of the trail of length l_2. ■

Note that if $G = \bigcup_{i=1}^{p} P_{l_i+2}$ with $L = \sum_{i=1}^{p} l_i$, the inequality (1) for $d = 1$ is of the form $k \geqslant 2p$, hence it provides us with the restriction that we need to use at least as many colors as the number of endpoints of the paths. Analogously, since in G there are exactly L vertices of degree 2, we obtain $\binom{k+1}{2} \geqslant L$ for $d = 2$. Consequently, Conjecture 1 is valid for unions of paths.

Corollary 4. *Let G be a vertex-disjoint union of paths of lengths at least two. Let $n_1(G) \leqslant k$ and $n_2(G) \leqslant \binom{k+1}{2}$ with k chosen as small as possible. Then $c(G) = k$ or $k + 1$.* ■

4. Point-Distinguishing Coloring

The concept of the *point-distinguishing coloring*, an edge-coloring (not necessarily proper) distinguishing all vertices by sets of colors of their incident edges, was introduced in [10] by Harary and Plantholt. Note that since a vertex of degree d in a graph G may obtain a set of colors consisting of at most d elements, we must have that

$$\binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{d} \geqslant n_1(G) + n_2(G) + \cdots + n_d(G) \tag{2}$$

for $1 \leqslant d \leqslant \Delta(G)$ if there is a point-distinguishing coloring of G by k colors. As we mentioned, the smallest k for which there is such a coloring for G is called the point distinguishing chromatic index of a graph and is denoted by $\chi_0(G)$. Several results concerning evaluation of this parameter for some classes of connected graphs can be found in [10]. We focus on the case when a vdec-graph G does not have to be connected. Observe that

$$c(G) \leqslant \chi_0(G) \leqslant \chi'_s(G), \tag{3}$$

where $c(G)$ and $\chi_0(G)$ coincide when 2-regular graphs are considered, see [9]. Assume then $G = P_{l_1+2} \cup \cdots \cup P_{l_p+2}$, with $L = \sum_{i=1}^{p} l_i$, is a union of p paths of lengths at least two. Taking $d = 1$ and $d = 2$ in the inequality (2), we obtain $2p \leqslant k$ and $|V(G)| = 2p + L \leqslant \binom{k+1}{2}$. Note that the problem of the point-distinguishing coloring of G is equivalent to almost the same packing problem as in the case of the irregular edge-coloring, with one new restriction. Namely, we additionally require for each endpoint of a path from $L(G)$ that if it is mapped to a vertex $v \in V(M_n)$, then the loop at v does not appear in the image of the packing (it is because a set, in contrast to a multiset, of colors for a vertex of degree 2 can be the same as a set of colors for a vertex of degree 1), see also [10].

Theorem 5. *The following conditions are both necessary and sufficient for packing* $\bigcup_{i=1}^{p} P_{l_i+1}$, $l_i \geqslant 1$, *into* M_n *with endpoints mapped to distinct vertices and with loops at these vertices not appearing in the image of the packing:*

(1°) $L = \binom{n}{2}$ *or* $L \leqslant \binom{n}{2} - 3$ *if* $r = 0$,

(2°) $L \leqslant \binom{n+1}{2} - \frac{r}{2} - 2p$ *if* r *(or* n*) is even* $(r > 0)$,

(3°) $L \leqslant \binom{n+1}{2} - p - 2p$ *if* r *(or* n*) is odd,*

where $n = 2p + r$, $r \geqslant 0$, *and* $L = \sum_{i=1}^{p} l_i$. *In particular,* $L \leqslant \binom{n}{2} - 2p$ *is always sufficient.*

Proof. It is enough to make some modifications in the proof of Theorem 3. Here we only restrict ourselves to point out these changes.

The necessity of the conditions follows by almost the same argument as in the mentioned proof. The subtraction of additional $2p$ in the above inequalities (2°) and (3°) corresponds to $2p$ loops that cannot appear in the image of the packing, while for $r = 0$ (hence $n = 2p$ and $\binom{n}{2} = \binom{n+1}{2} - 2p$), we need to remove at least three or neither of the edges from M_n to obtain an even number of vertices of odd degree in the resulting image of the packing.

Now we comment on the proof of the sufficiency of the conditions. The cases for $n \leqslant 9$ were verified by a computer program we created[2], thus now let $n \geqslant 10$. For $p = 1$ we only remove the two loops at the ends of the trail of length $l_1 = \binom{n+1}{2} - 1$ ($l_1 = \binom{n+1}{2} - \frac{n}{2} + 1$ if n is even) described at the beginning of the paragraph. The rest remains (almost) the same. Let then $p \geqslant 2$. If the paths of lengths $l_0, l_3, l_4, \ldots, l_p$ satisfy the assumptions of this theorem for M_{n-2}, the proof does not change. Assume then it is otherwise. The main difference in *Case* 2 consists in the following: we initially reduce l_0 by 4 (instead of 6) if n is odd, or by 3 (instead of 5) if n is even. This time it is enough that $l_1 \geqslant 6$, while otherwise we would have $l_0+l_3+l_4+\cdots+l_p \leqslant 5p-2$, hence $5p - 2 > \binom{n-2}{2} - 3$ for $r = 0$, $5p - 2 > \binom{n-1}{2} - \frac{r}{2} - 2(p-1)$ for even n and $r > 0$, or $5p - 2 > \binom{n-1}{2} - (p-1) - 2(p-1)$ for odd n, a contradiction with $n \geqslant 10$ (it is enough to substitute $\frac{n}{2} = p + \frac{r}{2}$, $\frac{n}{2} \geqslant p$ and solve the

[2]The source code along with other necessary files are available at *http://home.agh.edu.pl/~cichacz/ang/preprints.php*.

three obtained inequalities with respect to n). Analogously, we subsequently reduce the lengths of the paths by at most $2n-6$ (not $2n-4$) if n is odd, or $2n-5$ (not $2n-3$) if n is even. Then, while fixing u' on the trail of length l'_0, we require it to be at distance at most $l_1 - 4$ (instead of $l_1 - 6$) along the trail from u, and finally, at the end of the proof, we omit the loops at a and b while constructing a trail of length l_1. ■

Corollary 6. *Let G be a vertex-disjoint union of paths of lengths at least two. Let $n_1(G) \leqslant k$ and $n_1(G) + n_2(G) \leqslant k + \binom{k}{2}$ with k chosen as small as possible. Then $\chi_0(G) = k$ or $k+1$.* ■

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