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THE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF GRAPHS WITH GIVEN NUMBER OF CUT VERTICES*

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Abstract

In this paper, we determine the graph with maximal signless Laplacian spectral radius among all connected graphs with fixed order and given number of cut vertices.

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1. Introduction

In this paper, we consider only undirected simple connected graphs. Let $G = (V, E)$ be a graph of order n with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \ldots, e_m\}$. The *adjacency matrix* of G is $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent in G and $a_{ij} = 0$, otherwise. Let $D(G)$ be the degree diagonal matrix of G, i.e., $D(G) = \mathrm{diag}\{d(v_1), d(v_2), \ldots, d(v_n)\}$, where $d(v)$ denotes the degree of the vertex v in the graph G. The matrix $L(G) = D(G) - A(G)$ is known as the *Laplacian matrix* of G and is studied extensively in the literature; see, e.g. [1, 9, 14, 15].

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The matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* or *Q-matrix* of G in [12], and appears very rarely in published papers (see [3]). The paper [7] is one of the very few research papers concerning this matrix. Let $M = M(G) = [m_{ij}]$ be the *vertex-edge incidence matrix* of the graph G, i.e., $m_{ij} = 1$ if v_i is incident to e_j, and $m_{ij} = 0$, otherwise. Then $Q(G) = MM^T$, which implies $Q(G)$ is symmetric and positive semidefinite so that its eigenvalues can be arranged as

$$0 \leq \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G).$$

As $Q(G)$ is (entrywisely) nonnegative, by Perron-Frobenius Theorem, the spectral radius of $Q(G)$, denoted by $\mu(G)$, is exactly the largest eigenvalue $\mu_n(G)$. Similarly, the spectral radius of $A(G)$, denoted by $\rho(G)$, is the largest eigenvalue of $A(G)$. If, in addition, G is connected, then $\mu(G)$ (respectively, $\rho(G)$) is simple and has a (up to a factor) unique corresponding (entrywisely) positive eigenvector, known as *Perron vector* of $Q(G)$ (respectively, $A(G)$). We call $\rho(G), \mu(G)$ the *adjacency spectral radius* and the *signless Laplacian spectral radius* of G, respectively. In addition, $M^T M = 2I_m + A(L_G)$ and hence $\mu(G) = 2 + \rho(L_G)$, where L_G denotes the line graph of G.

Recently, the signless Laplacian matrices of graphs are received much attention. In [8, 16], the authors studied the signless Laplacian spectral radii of bicyclic graphs and all graphs with fixed order, respectively. In [7], Desai and Rao discussed the smallest eigenvalue of $Q(G)$ as a parameter reflecting the nonbipartiteness of the graph G. Some other results of the signless Laplacian matrices can be found in [4, 11]. For a survey paper of this direction, see [5]. One main goal of studying the eigenvalues of graphs is to investigate the structures of graphs. The papers [6, 12] provide spectral uncertainties with respect to the adjacency matrix and with respect to the signless Laplacian of sets of all graphs on n vertices when $n \leq 11$, which implies the spectra of signless Laplacian matrices are more closely related to the graph structures than those of adjacency matrices. An idea was expressed in [6] that, among matrices associated with a graph, the signless Laplacian matrix seems to be the most convenient for use in studying graph properties. Maybe this is a strong basis for our work on signless Laplacian matrices of graphs.

Recall a *cut vertex* in a connected graph is one whose deletion breaks the graph into two or more connected components. Denote by $\mathscr{G}_{n,k}$ the set of connected graphs on n vertices and with k cut vertices. In [2] Berman and Zhang have characterized the graph with maximal adjacency spectral radius among all graphs in $\mathscr{G}_{n,k}$. In this paper, we discuss this problem with respect to signless spectral radius, and show that the maximal signless spectral radius of graphs in $\mathscr{G}_{n,k}$ is attained uniquely at the graph $\mathbf{G}_{n,k}$, which is obtained by adding $(n-k)$ paths of almost equal lengths (that is, the absolute value of the difference of the lengths of any two paths is at most 1) to the vertices of the complete graph K_{n-k}, respectively.

2. Results

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be a Perron vector of $Q(G)$ of a graph G of order n. Then x can be considered as a function defined on the vertex set of G, that is, for any vertex v_i, we map it to $x_i = x(v_i)$. We often say x_i is a *value* of the vertex v_i given by x. One can find that

$$x^T Q(G)x = \sum_{uv\in E(G)} [x(u) + x(v)]^2, \tag{2.1}$$

and

$$[\mu(G) - d(v)]x(v) = \sum_{u\in N(v)} x(u), \quad \text{for each} \quad v \in V(G), \tag{2.2}$$

where $N(v) = \{w : vw \in E(G)\}$, the *neighborhood* of v in the graph G.

Denote by P_n and K_n a path and a complete graph of order n, respectively. Denote by $\Delta(G)$ the maximal degree of all vertices of a graph G. A graph is said *trival* if it consists of only one vertex.

Lemma 2.1 ([5, 18]). *Let G be a graph with signless Laplacian spectral radius $\mu(G)$. Then*

$$\Delta(G) + 1 \le \mu(G) \le \max\{d(u) + d(v) : uv \in E(G)\};$$

and for a connected graph G, the left equality holds if and only if G is a star, and the right equality holds if and only if G is regular or semi-regular bipartite.

In following, we often say "*adding a path to some vertex of a graph*", which means identifying one pendant vertex of the path with a specified vertex of the graph (disjoint to the path).

Lemma 2.2 ([13]). (1) *Let G be a simple graph containing a non-isolated vertex u, and let $G_{k,l}$ denote the graph obtained from G by adding two paths P_{k+1} and P_{l+1} at u, respectively. Then for $k \geq l \geq 1$, $\rho(G_{k,l}) > \rho(G_{k+1,l-1})$.*

(2) *Let G be a simple graph containing two adjacent vertices u, v both of degree greater than one, and let $G^1_{k,l}$ denote the graph obtained from G by adding a path P_{k+1} at u and a path P_{l+1} at v. Then for $k \geq l \geq 1$, $\rho(G^1_{k,l}) > \rho(G^1_{k+1,l-1})$.*

Tan and Wang [17] proved that Lemma 2.2 also holds for the spectral radius of the signless Laplacian matrix of a graph.

Lemma 2.3 [17]. *Let $G(k,l)$ and $G^1(k,l)$ be defined as in Lemma 2.2, respectively. Then for $k \geq l \geq 1$, $\mu(G_{k,l}) > \mu(G_{k+1,l-1})$ and $\mu(G^1_{k,l}) > \mu(G^1_{k+1,l-1})$.*

The following result is a simple fact and its proof is ommited.

Lemma 2.4. *Let G be a connected graph on n vertices with k $(k \geq 1)$ cut vertices. Then $k \leq n-2$, with equality if and only if $G = P_n$. If, in addition, each cut vertex is contained in exactly two blocks, then G contains exactly $k+1$ block, and $k = n-3$ if and only if G is obtained from a triangle by adding one path (possibly being trivial) at each vertex of the triangle.*

The notion $G_1 v G_2$ will mean a graph consisting of two connected subgraphs G_1 and G_2 sharing with exactly one common vertex v.

Proposition 2.5. (1) *The signless Laplacian spectral radius of $K_m v K_3$ is $\frac{1+2m+\sqrt{(2m-5)^2+16}}{2}$.*

(2) *Let G be a graph obtained from $K_m v K_3$ $(m \geq 3)$ by adding one path (possibly being trival) at each vertex except v. Then there exists a graph H with the same order as G, which is obtained from K_{m+1} by adding one path of some length (possibly being trival) at each vertex, such that*

$$\mu(G) < \mu(H).$$

Proof. (1) Let x be a Perron vector of the signless Laplacian matrix $Q(K_m v K_3)$ corresponding the spectral radius μ. By symmetry of the graph, except v, all vertices of K_m (and respectively K_3) have the same value given by x, denoted by α (and respectively, β). Denote the value of v by γ. Then by Equation (2.2),

$$[\mu-(m-1)]\alpha = (m-2)\alpha+\gamma,\ [\mu-(m+1)]\gamma = (m-1)\alpha+2\beta,\ (\mu-2)\beta = \beta+\gamma.$$

Solving the equations, we get the result. ■

(2) Consider the case of $m = 3$ first. Except v, let the vertices of one K_3 by w_1, w_2 and let the vertices of another K_3 by u_1, u_2. Let x be a Perron vector of the signless Laplacian matrix $Q(G)$. We may assume that $x(w_1) = \max\{x(w_i), x(u_i), i = 1, 2\}$. Now deleting the edge u_1u_2 and adding edges u_1w_1, u_1w_2, we get a new graph denoted by H. By Equation (2.1),

$$\begin{aligned} &x^TQ(H)x - x^TQ(G)x \\ &= [x(u_1) + x(w_1)]^2 + [x(u_1) + x(w_2)]^2 - [x(u_1) + x(u_2)]^2 > 0, \end{aligned}$$

which implies the desired result.

Now consider the case of $m \geq 4$. Except the vertex v, let the vertices of K_3 be u_1, u_2 and those of K_m be $w_1, w_2, \ldots, w_{m-1}$. By the first result just proved, $\mu := \mu(G) \geq \mu(K_m v K_3) \geq 7$. Let x be a Perron vector of $Q(G)$. Assume that $x(u_1) \geq x(u_2)$. We first prove $x(v) > (\mu - 5)x(u_1)$. If the path added to u_1 is nontrival and is denoted by $u_1^0 u_1^1 \cdots u_1^p$, where $u_1^0 = u_1, p \geq 1$ and u_1^{i-1} is adjacent to u_1^i for each $i = 1, 2, \ldots, p$, by Equation (2.2),

$$x\big(u_1^{p-1}\big) = (\mu - 1)x\big(u_1^p\big) > x\big(u_1^p\big),$$

and if $p \geq 2$,

$$\begin{aligned} x\big(u_1^{p-2}\big) &= (\mu - 2)x\big(u_1^{p-1}\big) - x\big(u_1^p\big) \\ &= (\mu - 3)x\big(u_1^{p-1}\big) + \big[x\big(u_1^{p-1}\big) - x\big(u_1^p\big)\big] > x\big(u_1^{p-1}\big). \end{aligned}$$

Repeating the discussion if necessary, we at last get $x(u_1) = x(u_1^0) > x(u_1^1)$, and then $x(v) = (\mu-3)x(u_1)-x(u_1^1)-x(u_2) > (\mu-5)x(u_1)$. If the path added to v_1 is trival, we also have $x(v) = (\mu - 2)x(u_1) - x(u_2) \geq (\mu - 3)x(u_1) > (\mu - 5)x(u_1)$. We next show

$$\gamma := \sum_{i=1}^{m-1} x(w_i) > x(u_1).$$

By Equation (2.2) and the fact $x(v) > (\mu - 5)x(u_1)$ and $x(u_1) \geq x(u_2)$,

$$\gamma = [\mu - (m+1)]x(v) - [x(u_1) + x(u_2)] > [(\mu - m - 1)(\mu - 5) - 2]x(u_1).$$

If $m = 4$, then $\mu \geq 7$ and then $\gamma > 2x(u_1) > x(u_1)$. If $m \geq 5$, as $\mu \geq \Delta(G) + 1 \geq m + 2$ by Lemma 2.1 and $\mu > 8$ by the first result, we get $\gamma > x(u_1)$.

Now deleting the edge u_1u_2 and adding the edges u_1w_i for $i = 1, 2, \ldots, m-1$, we get a new graph, denoted by H, which holds that

$$\begin{aligned} & x^TQ(H)x - x^TQ(G)x \\ & = \sum_{i=1}^{m-1} [x(u_1) + x(w_i)]^2 - [x(u_1) + x(u_2)]^2 \\ & = (m-2)x(u_1)^2 - x(u_2)^2 + 2x(u_1)[\gamma - x(u_2)] + \sum_{i=1}^{m-1} x(w_i)^2. \end{aligned}$$

As $\gamma > x(u_1) \geq x(u_2)$ and $(m-2)x(u_1)^2 - x(u_2)^2 \geq (m-3)x(u_1)^2 > 0$, $x^TQ(H)x > x^TQ(G)x$, which implies that $\mu(H) > \mu(G)$. ■

We now get the main result of this paper.

Theorem 2.6. *Among all the connected graphs with n vertices and k cut vertices, the maximal signless Laplacian spectral radius of graph G is attained uniquely at the graph $\mathbf{G}_{n,k}$, namely, a graph obtained from the complete graph K_{n-k} by adding $(n-k)$ paths of almost equal lengths to its vertices respectively.*

Proof. We have to prove that if $G \in \mathscr{G}_{n,k}$, then $\mu(G) \leq \mu(\mathbf{G}_{n,k})$, with equality only when $G = \mathbf{G}_{n,k}$. Noting that the signless Laplacian matrix of a connected graph is nonnegative and irreducible, so if we add an edge e to a connected graph G, $\mu(G+e) > \mu(G)$. Thus we can assume that each cut vertex of G connects exactly two blocks and that all of these blocks are cliques.

If G has no cut vertices, i.e., $k = 0$, clearly $\mu(G) \leq \mu(K_n)$ with equality if and only if $G = K_n = \mathbf{G}_{n,0}$. Now assume that G has cut vertices. Then G contains exactly $k+1$ blocks $B_1, B_2, \ldots, B_{k+1}$ with cardinalities arranged as

$$a_1 \geq a_2 \geq \cdots \geq a_{k+1} \geq 2,$$

where a_i denotes the cardinality of the block B_i for $i = 1, 2, \ldots, k+1$. Noting that each cut vertex is counted twice in the sum $\sum_{i=1}^{k+1} a_i$, we have

$$\sum_{i=1}^{k+1} a_i - k = n. \tag{2.3}$$

If $k = n-2$, by Lemma 2.4, G is the path $P_n = \mathbf{G}_{n,n-2}$. If $k = n-3$, also by Lemma 2.4, G is obtained from a triangle by adding one path (possibly being trivial) at each vertex of the triangle. Hence the the result follows by repeated using Lemma 2.3. Thus we may assume that $1 \le k \le n-4$. We observe that (or by Equation (2.3))

$$a_1 = n + k - (a_2 + \cdots + a_{k+1}) \le n + k - 2k = n - k.$$

By Lemma 2.1,

$$\begin{aligned}
\mu(G) &\le (a_1 + a_2 - 2) + (a_1 + a_3 - 2) \\
&= (a_1 + a_2 + a_3) + a_1 - 4 \\
&= n + k - (a_4 + \cdots + a_{k+1}) + a_1 - 4 \\
&\le n + k - 2(k-2) + a_1 - 4 \\
&= n - k + a_1.
\end{aligned}$$

So, if $a_1 \le n-k-2$, then $\mu(G) \le 2(n-k-1)$. Note that $Q(\mathbf{G}_{n,k})$ contains a proper subgraph K_{n-k} of $G_{n,k}$. Hence $\mu(\mathbf{G}_{n,k}) > \rho(K_{n-k}) = 2(n-k-1)$. Therefore, if $a_1 \le n-k-2$, then $\mu(G) < \mu(\mathbf{G}_{n,k})$. So it suffices to consider only the case $a_1 = n-k$ or $a_1 = n-k-1$.

If $a_1 = n-k$, then by Equation (2.3), $a_2 = a_3 \cdots = a_{k+1} = 2$. So G is obtained from a complete graph K_{n-k} by adding one path of some length (possibly being trivial) at each vertex of the complete graph. Now the result follows by repeatedly using Lemma 2.3.

If $a_1 = n-k-1$, then also by Equation (2.3), $a_2 = 3$, and $a_3 = \cdots = a_{k+1} = 2$. We have two cases: (i) B_1 and B_2 are joined by a nontrival path, and (ii) B_1 and B_2 share a common cut vertex. Note that here $B_1 = K_{n-k-1}, B_2 = K_3$, both being complete. For the case (i), by Lemma 2.1, $\mu(G) \le 2(a_1 - 1 + 1) = 2(n-k-1) < \mu(\mathbf{G}_{n,k})$. For the case (ii), by Proposition 2.5, $\mu(G) < \mu(H)$, where H is one obtained from K_{n-k} by adding one path of some length (possibly being trivial) at each vertex and has the same order as G. Clearly H also has k cut vertices. The result follows by repeated using Lemma 2.3 on the graph H. ■

References

[1] W.N. Anderson and T.D. Morely, *Eigenvalues of the Laplacian of a graph*, Linear Multilinear Algebra **18** (1985) 141–145.

[2] A. Berman and X.-D. Zhang, *On the spectral radius of graphs with cut vertices*, J. Combin. Theory (B) **83** (2001) 233–240.

[3] D. Cvetković, M. Doob and H. Sachs, Spectra of graphs, (third ed., Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995).

[4] D. Cvetković, *Signless Laplacians and line graphs*, Bull. Acad. Serbe Sci. Ars. Cl. Sci. Math. Nat. Sci. Math. **131** (30) (2005) 85–92.

[5] D. Cvetković, P. Rowlinson and S.K. Simić, *Signless Laplacians of finite graphs*, Linear Algebra Appl. **423** (2007) 155–171.

[6] E.R. van Dam and W. Haemers, *Which graphs are determined by their spectrum*? Linear Algebra Appl. **373** (2003) 241–272.

[7] M. Desai and V. Rao, *A characterization of the smallest eigenvalue of a graph*, J. Graph Theory **18** (1994) 181–194.

[8] Y.-Z. Fan, B.-S. Tam and J. Zhou, *Maximizing spectral radius of unoriented Laplacian matrix over bicyclic graphs of a given order*, Linear Multilinear Algebra **56** (2008) 381–397.

[9] M. Fiedler, *Algebraic connectivity of graphs*, Czechoslovak Math. J. **23** (1973) 298–305.

[10] R. Grone and R. Merris, *The Laplacian spectrum of a graph*, SIAM J. Matrix Anal. Appl. **11** (1990) 218–238.

[11] J.W. Grossman, D.M. Kulkarni and I. Schochetman, *Algebraic graph theory without orientation*, Linear Algebra Appl. **212/213** (1994) 289–307.

[12] W. Haemer and E. Spence, *Enumeration of cospectral graphs*, Europ. J. Combin. **25** (2004) 199–211.

[13] Q. Li and K.-Q. Feng, *On the largest eigenvalues of graphs*, (in Chinese), Acta Math. Appl. Sin. **2** (1979) 167–175.

[14] R. Merris, *Laplacian matrices of graphs: A survey,* Linear Algebra Appl. **197/198** (1994) 143–176.

[15] B. Mohar, *Some applications of Laplacian eigenvalues of graphs*, in: Graph Symmetry (G. Hahn and G. Sabidussi Eds), (Kluwer Academic Publishers, Dordrecht, 1997), 225–275.

[16] B.-S. Tam, Y.-Z. Fan and J. Zhou, *Unoriented Laplacian maximizing graphs are degree maximal*, Linear Algebra Appl. **429** (2008) 735–758.

[17] Shangwang Tan and Xingke Wang, *On the largest eigenvalue of signless Laplacian matrix of a graph*, Journal of Mathematical Reserch and Exposion **29** (2009) 381–390.

[18] X.-D. Zhang and R. Luo, *The Laplacian eigenvalues of a mixed graph*, Linear Algebra Appl. **362** (2003) 109–119.

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