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FURTHER RESULTS ON RADIAL GRAPHS

Kumarappan Kathiresan

Center for Research and Post Graduate Studies in Mathematics
Ayya Nadar Janaki Ammal College
Sivakasi – 626 124, Tamil Nadu, India

e-mail: kathir2esan@yahoo.com

and

G. Marimuthu

Department of Mathematics
The Madura College
Madurai – 625 011, Tamil Nadu, India

e-mail: yellowmuthu@yahoo.com

Abstract

In a graph G, the distance $d(u, v)$ between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The minimum eccentricity is called the radius of the graph and the maximum eccentricity is called the *diameter* of the graph. The radial graph $R(G)$ based on G has the vertex set as in G, two vertices u and v are adjacent in $R(G)$ if the distance between them in G is equal to the radius of G. If G is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components. The main objective of this paper is to characterize graphs G with specified radius for its radial graph.

Keywords: radius, diameter, radial graph.

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1. Introduction

The graphs considered here are nontrivial and simple. For other graph theoretic notation and terminology, we follow [4, 9]. In a graph G, the distance $d(u, v)$ between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(G)$ of G is defined by $r(G) = \min\{e(u) : u \in V(G)\}$ and the diameter $d(G)$ of G is defined by $d(G) = \max\{e(u) : u \in V(G)\}$. A graph for which $r(G) = d(G)$ is called a *self-centered graph* of radius $r(G)$. A vertex v is called an eccentric vertex of a vertex u if $d(u, v) = e(u)$. A vertex v of G is called an eccentric vertex of G if it is an eccentric vertex of some vertex of G. Let S_i denote the subset of vertices of G whose eccentricity is equal to i. The concept of antipodal graph was initially introduced by [8] and was further expanded by [2, 3]. The antipodal graph of a graph G, denoted by $A(G)$, is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G. A graph is said to be antipodal if it is the antipodal graph $A(H)$ of some graph H. The concept of eccentric graph was introduced by [1]. The eccentric graph based on G is denoted by G_e, whose vertex set is $V(G)$ and two vertices u and v are adjacent in G_e if and only if $d(u, v) = \min\{e(u), e(v)\}$. Also Chartrand *et al.,* [5] studied the concept of eccentric graphs. The subgraph of G induced by its eccentric vertices is called the eccentric subgraph of G. In [5] a characterization of all graphs that are eccentric subgraph of some connected graph was presented. Kathiresan and Marimuthu [6] introduced a new type of graph called *radial graph.* Two vertices of a graph G are said to be radial to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph G, denoted by $R(G)$, has the vertex set as in G and two vertices are adjacent in $R(G)$ if and only if they are radial in G. If G is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of G. A graph G is called a radial graph if $R(H) = G$ for some graph H. We denote $G_1 = G_2$ if the two graphs G_1 and G_2 are the same graphs and $G_1 \subset G_2$ if G_1 is a proper subgraph of G_2. Next we provide some results which will be used to prove some theorems in this paper.

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}$ and F_3 denote the set of all connected graphs G for which $r(G) = d(G) = 1$; $r(G) = 1$ and $d(G) = 2$; $r(G) = d(G) = 2$; $r(G) = 2$ and $d(G) = 3$; $r(G) = 2$ and $d(G) = 4$ and $r(G) \geq 3$ respectively. Let F_4 denote the set of all disconnected graphs.

Theorem A [4]. *If G is a simple graph with diameter at least 3, then $\overline{G}$ has diameter at most 3.*

Theorem B [4]. *If G is a simple graph with diameter at least 4, then $\overline{G}$ has diameter at most 2.*

Theorem C [4]. *If G is a simple graph with diameter at least 3, then $\overline{G}$ has radius at most 2.*

Theorem D [9]. *If G is a self-centered graph with $r(G) \geq 3$, then $\overline{G}$ is a self-centered graph of radius 2.*

Theorem E [6]. *A graph G is a radial graph if and only if G is the radial graph of itself or the radial graph of its complement.*

Theorem F [3]. *A graph G is an antipodal graph if and only if G is the antipodal graph of its complement.*

The ladder graph L_n [7] with n steps is defined by $L_n = P_n \times K_2$ where P_n is a path on n vertices and $\times$ denotes the Cartesian product of graphs.

2. Graph Equations Involving Radial Graphs

Result 2.1. Let L_n be a ladder with n steps. Then

$$r(L_n) = \begin{cases} \frac{n+2}{2} & \text{if } n \equiv 0, 2 \pmod 4, \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod 4. \end{cases}$$

Result 2.2. Let L_n be a ladder graph with n steps. Then

$$R(L_n) = \begin{cases} 2P_n & \text{if } n \equiv 0, 2 \pmod 4, \\ C_{2n} & \text{if } n \equiv 1 \pmod 4, \\ 2C_n & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Let $u_1, u_2, \ldots, u_n$ and let $v_1, v_2, \ldots, v_n$ be the vertices of the ladder L_n. The edge set of L_n is $E(L_n) = \{u_i v_i, u_i u_{i+1}, v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{u_n v_n\}$.

Case i. $n \equiv 0 \pmod 4$.
The radius r of L_n is $\frac{n+2}{2}$. The radial pairs are as follows:
$(u_i, u_{r+i}), (u_i, v_{r+i-1})$ where $i = 1, 2, \ldots, \frac{n}{2} - 1$; $(u_{n/2+1}, v_n)$; $(u_{n/2}, v_1)$; (u_j, u_{j-r}), (u_j, v_{j-r+1}) where $j = r+1, r+2, \ldots, n$.

These radial pairs give the radial graph as the union of two path graphs P_n^1 and P_n^2, where

P_n^1 : $u_{r-1}, v_{2r-2}, v_{r-2}, u_{2r-3}, u_{r-3}, v_{2r-4}, v_{r-4}, u_{2r-5}, u_{r-5}, v_{2r-6}, v_{r-6}, \ldots, u_{2r-(r-2)}, u_{r-(r-2})), v_{2r-(r-1)}, v_{r-(r-1)}, u_r$ and
P_n^2 : $v_{r-1}, u_{2r-2}, u_{r-2}, v_{2r-3}, v_{r-3}, u_{2r-4}u_{r-4}, v_{2r-5}, v_{r-5}, u_{2r-6}, u_{r-6}, \ldots, v_{2r(r-2)}, v_{r-(r-2)}, u_{2r-(r-1)}, u_{r-(r-1)}, v_r$.

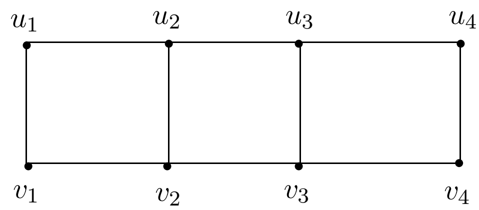

Figure 1. The graph L_4 with $r = 3$.

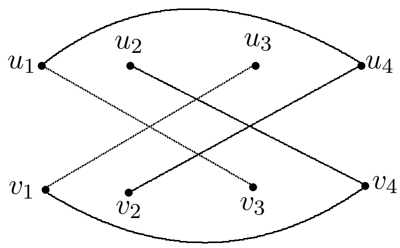

Figure 2. The graph $R(L_4)$.

Case ii. $n \equiv 2 \pmod 4$.
We can prove the result as in the Case i.

Case iii. $n \equiv 3 \pmod 4$.
The radius r of L_n is $\frac{n+1}{2}$. The radial pairs are as follows:
$(u_i, u_{r+i}), (u_i, v_{r+i-1})$ where $i = 1, 2, \ldots, r-1$; (u_r, v_1); (u_r, v_n); (u_j, u_{j-r}), (u_j, v_{j-r+1}) where $j = r+1, r+2, \ldots, n$. The radial graph corresponding

to the above radial pairs is the union of two disjoint cycle graphs C_n^1 and C_n^2 where
$C_n^1 : u_1, u_{r+1}, v_{r-(n-r-1)}, v_{2r-(n-r-1)}, u_{r-(n-r-2)}, u_{2r-(n-r-2)}, v_{r-(n-r-3)}, v_{2r-(n-r-3)}, u_{r-(n-r-4)}, u_{2r-(n-r-2)}, \ldots, v_{r-2}, v_{2r-2}, u_{r-1}, u_{2r-1}, v_r, u_1.$
$C_n^2 : v_1, v_{r+1}, u_{r-(n-r-1)}, u_{2r-(n-r-1)}, v_{r-(n-r-2)}, v_{2r-(n-r-2)}, u_{r-(n-r-3)}, u_{2r-(n-r-3)}, v_{r-(n-r-4)}, v_{2r-(n-r-2)}, \ldots, u_{r-2}, u_{2r-2}, v_{r-1}, v_{2r-1}, u_r, v_1.$

Case iv. $n \equiv 1 \pmod 3$.
We can prove the result as in Case iii. ■

Proposition 2.3. *Let G be a graph of order n. Then $R(G) = G$ if and only if $G \in F_{11}$ or F_{12}.*

Proof. Follows from the definition. ■

To improve the readability of the paper, we offer an outline of the proof of the following two results which are found in [6].

Proposition 2.4. *If $r(G) > 1$, then $R(G) \subseteq \overline{G}$.*

Proof. If two vertices u and v are adjacent in $R(G)$, then they are non-adjacent in G, since $r(G) > 1$. ■

Lemma 2.5. *Let G be a graph of order n. Then $R(G) = \overline{G}$ if and only if either $S_2(G) = V(G)$ or G is disconnected in which each component is complete.*

Proof. If $S_2(G) = V(G)$, then $R(G) = \overline{G}$. If G is the union of complete graphs, that is $G = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t}$, then its radial graph $R(G)$ is the complete multipartite graph $K_{n_1,n_2,\ldots n_t}$, and consequently, $R(G) = \overline{G}$.

If $R(G) = \overline{G}$, then $d(u, v) = 1$ or r in G for any two distinct vertices u and v where r is the radius of G. By Proposition 2.3, $r(G) \neq 1$. If $2 < r < \infty$, then there exist at least two vertices x and y in G such that they are nonadjacent in both $R(G)$ and G, which contradicts $R(G) = \overline{G}$.

If there is a pair of vertices x and y such that $d(x, y) > 2$, then they are nonadjacent in G and $R(G)$. If G has a noncomplete component then $R(G) \neq \overline{G}$. ■

Now, we provide some graph equations involving radial graphs.

Proposition 2.6. *Let G be a graph. Then $R(G) = R(\overline{G})$ if G is any one of the following graphs.*

1. *G or $\overline{G}$ is complete.*
2. *G or $\overline{G}$ is disconnected with each component complete out of which one is an isolated vertex.*

Proof. If G is complete, then by Proposition 2.3, $R(G) = G$. Also $R(\overline{G}) = G$.

If $\overline{G}$ is complete, then by Proposition 2.3, $R(\overline{G}) = \overline{G}$. Also $R(G) = \overline{G}$.

If G is disconnected with each component complete out of which one is an isolated vertex, then $R(G) = \overline{G}$ by Lemma 2.5. But $R(\overline{G}) = \overline{G}$ since $\overline{G}$ has a vertex of degree $n-1$.

If $\overline{G}$ is disconnected with each component complete out of which one is an isolated vertex, then $R(\overline{G}) = G$ and $R(G) = G$. ■

Lemma 2.7. *If G and $\overline{G}$ are members of F_{22}, then $\overline{R(G)} = R(\overline{G})$.*

Proof. Since G and $\overline{G}$ are members of F_{22}, by Lemma 2.5, $R(G) = \overline{G}$ and $R(\overline{G}) = G$ and hence $\overline{R(G)} = R(\overline{G})$. ■

Lemma 2.8. *Let G be a disconnected graph with each component complete and has no isolates. Then $\overline{R(G)} = R(\overline{G})$.*

Proof. Since each component of G is complete, by Lemma 2.5, $R(G) = \overline{G}$. Also $\overline{G} \in F_{22}$. Again by Lemma 2.5, $R(\overline{G}) = G$. Thus $\overline{R(G)} = R(\overline{G})$. ■

Lemma 2.9. *Let G be a connected graph such that $\overline{R(G)} = R(\overline{G})$. Then either G or $\overline{G}$ is a member of F_{22}.*

Proof. We prove this lemma by assuming that $G \notin F_{22}$. It suffices to show that $\overline{G} \in F_{22}$. If not, then $\overline{G} \in \mathcal{A} = F_{11} \cup F_{12} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. It is impossible that G is connected if $\overline{G} \in F_{11} \cup F_{12}$.

From Theorems A and C, if $\overline{G} \in F_{23}$, then $G \in F_{22} \cup F_{23}$. But, since $G \notin F_{22}$, only the case $G \in F_{23}$ has to be considered. By Proposition 2.4, $R(\overline{G}) \subset G$. If $G \in F_{23}$, then $R(G) \subset \overline{G}$ and hence $G \subset \overline{R(G)}$. Thus $R(\overline{G}) \subset \overline{R(G)}$, a contradiction. Since G is connected $G \notin F_4$.

If $\overline{G} \in F_{24} \cup F_3$, then by Theorem B, $G \in F_{22}$, a contradiction.

Now let $\overline{G} \in F_4$. If $\overline{G}$ has at least one isolated vertex, then $R(G) = G$. But $R(\overline{G}) \subseteq G$. Thus $R(\overline{G}) \subseteq R(G) \subseteq \overline{G}$. This contradicts the fact that

$R(\overline{G}) \subseteq G$. If $\overline{G}$ has no isolated vertices, then $G \in F_{22}$, a contradiction. The above argument force us to conclude that $\overline{G}$ is a member of F_{22}. ■

Lemma 2.10. *Let G be a disconnected graph such that $\overline{R(G)} = R(\overline{G})$. Then each component of G is a complete graph and has no isolates.*

Proof. Suppose at least one of the components of G is not complete. Then $R(G) \subset \overline{G}$. This implies that $G \subset \overline{R(G)}$. ■

The following examples show that the notions of radial graph and antipodal graph are independent.

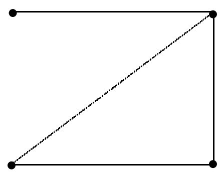

Figure 3. A radial graph but not an antipodal graph.

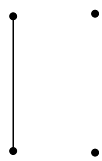

Figure 4. An antipodal graph but not a radial graph.

Next we characterize those antipodal which are radial graphs.

Theorem 2.11. *A graph G is both radial and antipodal if and only if either $\overline{G} \in F_{22}$ or each component of $\overline{G}$ is complete.*

Proof. If either $\overline{G} \in F_{22}$ or each component of $\overline{G}$ is complete, then $A(\overline{G}) = R(\overline{G}) = G$, by Lemma 2.5.

Conversely assume that G is both radial and antipodal. Assume that $\overline{G} \notin F_{22}$. We must claim that each component of $\overline{G}$ is complete. Suppose $\overline{G}$ has at least one noncomplete component. Then $A(\overline{G}) \neq G$. This is a contradiction. ■

3. The Radius of Radial Graphs

Theorem G [6]. *If both G and $\overline{G}$ are of self-centered graphs of radius 2, then so is $R(G)$.*

Proposition 3.1. *Let G be a graph of order n. Then $r(R(G)) = 1$ if and only if either $\Delta(G) = n - 1$ or G is disconnected with at least one isolated vertex.*

Corollary 3.2.
(a) *Let G be a connected graph. Then $R(G) \in F_{11}$ if and only if $G \in F_{11}$.*
(b) *Let G be a connected graph. Then $R(G) \in F_{12}$ if and only if $G \in F_{12}$.*
(c) *Let G be a disconnected graph. Then $R(G) \in F_{11}$ if and only if $G = \overline{K_n}$.*
(d) *Let G be a disconnected graph. Then $R(G) \in F_{12}$ if and only if G has at least one isolated vertex and has a nontrivial component.*

Proposition 3.3. *Let G be a disconnected graph. Then $R(G) \in F_{22}$ if and only if G has no isolated vertex.*

Proof. Follows from the definition. ■

Next we provide a characterization theorem for $R(G)$ and $R(\overline{G})$ to be members of F_{22}.

Theorem 3.4. *Let G be a connected graph. Then $R(G)$ and $R(\overline{G})$ are members of F_{22} if and only if G and $\overline{G}$ are members of F_{22}.*

Proof. If G and $\overline{G}$ are members of F_{22}, then by Lemma 2.5, $R(G) = \overline{G}$ and $R(\overline{G}) = G$.

Conversely assume that $R(G)$ and $R(\overline{G})$ are members of F_{22}. Assume that $G \notin F_{22}$. Then $G \in \mathcal{A} = F_{11} \cup F_{12} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$.

If $G \in F_{11}$, then by Theorem 2.3, $R(G) = G$, a contradiction to the fact that $R(G) \in F_{22}$. If $G \in F_{12}$, then $R(G) = G$, a contradiction. If $G \in F_{23}$, then by Theorem A, $\overline{G} \in F_{22}$ or $\overline{G} \in F_{23}$. Then $R(\overline{G}) = G$ or $R(\overline{G}) \subset G$, a contradiction, since $R(\overline{G}) \in F_{22}$. It is easy to obtain a contradiction if $G \in F_{24}$ or $G \in F_3$. $G \in F_4$ will not hold since G is connected. Therefore $G \in F_{22}$. Similarly we can prove that $\overline{G} \in F_{22}$. ■

Theorem 3.5. *Let G be a connected graph. Then $R(G) \in F_{23}$ if $G \in F_{22}$ and $\overline{G} \in F_{23}$.*

Proof. By Lemma 2.5, $R(G) = \overline{G}$. Also $\overline{G} \in F_{23}$ and hence $R(G) \in F_{23}$. ■

Theorem 3.6. *Let G be a connected graph. Then $R(G) \in F_3$ if $G \in F_{22}$ and $\overline{G} \in F_3$.*

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