Discussiones Mathematicae Graph Theory 30 (2010) 75–83

FURTHER RESULTS ON RADIAL GRAPHS

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Abstract

In a graph G, the distance d(u, v) between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity e(u) of a vertex u is the distance to a vertex farthest from u. The minimum eccentricity is called the radius of the graph and the maximum eccentricity is called the *diameter* of the graph. The radial graph R(G) based on G has the vertex set as in G, two vertices u and v are adjacent in R(G) if the distance between them in G is equal to the radius of G. If G is disconnected, then two vertices are adjacent in R(G) if they belong to different components. The main objective of this paper is to characterize graphs G with specified radius for its radial graph.

Keywords: radius, diameter, radial graph.

2010 Mathematics Subject Classification: 05C12.

1. INTRODUCTION

The graphs considered here are nontrivial and simple. For other graph theoretic notation and terminology, we follow [4, 9]. In a graph G, the distance d(u, v) between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity e(u) of a vertex u is the distance to a vertex farthest from u. The radius r(G) of G is defined by $r(G) = \min\{e(u) :$ $u \in V(G)$ and the diameter d(G) of G is defined by $d(G) = \max\{e(u) :$ $u \in V(G)$. A graph for which r(G) = d(G) is called a *self-centered graph* of radius r(G). A vertex v is called an eccentric vertex of a vertex u if d(u, v) = e(u). A vertex v of G is called an eccentric vertex of G if it is an eccentric vertex of some vertex of G. Let S_i denote the subset of vertices of G whose eccentricity is equal to i. The concept of antipodal graph was initially introduced by [8] and was further expanded by [2, 3]. The antipodal graph of a graph G, denoted by A(G), is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G. A graph is said to be antipodal if it is the antipodal graph A(H) of some graph H. The concept of eccentric graph was introduced by [1]. The eccentric graph based on G is denoted by G_e , whose vertex set is V(G) and two vertices u and v are adjacent in G_e if and only if $d(u, v) = \min\{e(u), e(v)\}$. Also Chartrand *et al.*, [5] studied the concept of eccentric graphs. The subgraph of G induced by its eccentric vertices is called the eccentric subgraph of G. In [5] a characterization of all graphs that are eccentric subgraph of some connected graph was presented. Kathiresan and Marimuthu [6] introduced a new type of graph called *radial* graph. Two vertices of a graph G are said to be radial to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph G, denoted by R(G), has the vertex set as in G and two vertices are adjacent in R(G) if and only if they are radial in G. If G is disconnected, then two vertices are adjacent in R(G) if they belong to different components of G. A graph G is called a radial graph if R(H) = G for some graph H. We denote $G_1 = G_2$ if the two graphs G_1 and G_2 are the same graphs and $G_1 \subset G_2$ if G_1 is a proper subgraph of G_2 . Next we provide some results which will be used to prove some theorems in this paper.

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}$ and F_3 denote the set of all connected graphs G for which r(G) = d(G) = 1; r(G) = 1 and d(G) = 2; r(G) = d(G) = 2; r(G) = 2 and d(G) = 3; r(G) = 2 and d(G) = 4 and $r(G) \ge 3$ respectively. Let F_4 denote the set of all disconnected graphs.

Theorem A [4]. If G is a simple graph with diameter at least 3, then \overline{G} has diameter at most 3.

Theorem B [4]. If G is a simple graph with diameter at least 4, then \overline{G} has diameter at most 2.

Theorem C [4]. If G is a simple graph with diameter at least 3, then \overline{G} has radius at most 2.

Theorem D [9]. If G is a self-centered graph with $r(G) \ge 3$, then \overline{G} is a self-centered graph of radius 2.

Theorem E [6]. A graph G is a radial graph if and only if G is the radial graph of itself or the radial graph of its complement.

Theorem F [3]. A graph G is an antipodal graph if and only if G is the antipodal graph of its complement.

The ladder graph L_n [7] with *n* steps is defined by $L_n = P_n \times K_2$ where P_n is a path on *n* vertices and × denotes the Cartesian product of graphs.

2. GRAPH EQUATIONS INVOLVING RADIAL GRAPHS

Result 2.1. Let L_n be a ladder with n steps. Then

$$r(L_n) = \begin{cases} \frac{n+2}{2} & \text{if } n \equiv 0,2 \pmod{4}, \\ \frac{n+1}{2} & \text{if } n \equiv 1,3 \pmod{4}. \end{cases}$$

Result 2.2. Let L_n be a ladder graph with n steps. Then

$$R(L_n) = \begin{cases} 2P_n & \text{if } n \equiv 0, 2 \pmod{4}, \\ C_{2n} & \text{if } n \equiv 1 \pmod{4}, \\ 2C_n & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Proof. Let u_1, u_2, \ldots, u_n and let v_1, v_2, \ldots, v_n be the vertices of the ladder L_n . The edge set of L_n is $E(L_n) = \{u_i v_i, u_i u_{i+1}, v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{u_n v_n\}.$

Case i. $n \equiv 0 \pmod{4}$.

The radius r of L_n is $\frac{n+2}{2}$. The radial pairs are as follows:

 $(u_i, u_{r+i}), (u_i, v_{r+i-1})$ where $i = 1, 2, \dots, \frac{n}{2} - 1; (u_{n/2+1}, v_n); (u_{n/2}, v_1); (u_j, u_{j-r}), (u_j, v_{j-r+1})$ where $j = r+1, r+2, \dots, n$.

These radial pairs give the radial graph as the union of two path graphs P_n^1 and $P_n^2,\,{\rm where}$

$$P_n^{\mathbf{1}}: u_{r-1}, v_{2r-2}, v_{r-2}, u_{2r-3}, u_{r-3}, v_{2r-4}, v_{r-4}, u_{2r-5}, u_{r-5}, v_{2r-6}, v_{r-6}, \dots, u_{2r-(r-2)}, u_{r-(r-2)}, v_{2r-(r-1)}, v_{r-(r-1)}, u_r \text{ and }$$

 $P_n^2: v_{r-1}, u_{2r-2}, u_{r-2}, v_{2r-3}, v_{r-3}, u_{2r-4}u_{r-4}, v_{2r-5}, v_{r-5}, u_{2r-6}, u_{r-6}, \dots, v_{2r(r-2)}, v_{r-(r-2)}, u_{2r-(r-1)}, u_{r-(r-1)}, v_r.$



Figure 1. The graph L_4 with r = 3.



Figure 2. The graph $R(L_4)$.

Case ii. $n \equiv 2 \pmod{4}$. We can prove the result as in the Case i.

Case iii. $n \equiv 3 \pmod{4}$.

The radius r of L_n is $\frac{n+1}{2}$. The radial pairs are as follows: $(u_i, u_{r+i}), (u_i, v_{r+i-1})$ where $i = 1, 2, \ldots, r-1$; (u_r, v_1) ; (u_r, v_n) ; (u_j, u_{j-r}) , (u_j, v_{j-r+1}) where $j = r+1, r+2, \ldots, n$. The radial graph corresponding

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to the above radial pairs is the union of two disjoint cycle graphs ${\cal C}_n^1$ and ${\cal C}_n^2$ where

 $C_n^{1}: u_1, u_{r+1}, v_{r-(n-r-1)}, v_{2r-(n-r-1)}, u_{r-(n-r-2)}, u_{2r-(n-r-2)}, v_{r-(n-r-3)}, u_{2r-(n-r-3)}, u_{2r-(n-r-3)}, u_{2r-(n-r-3)}, u_{2r-(n-r-2)}, \dots, v_{r-2}, v_{2r-2}, u_{r-1}, u_{2r-1}, v_r, u_1.$ $C_n^{2}: v_1, v_{r+1}, u_{r-(n-r-1)}, u_{2r-(n-r-1)}, v_{r-(n-r-2)}, v_{2r-(n-r-2)}, u_{r-(n-r-3)}, u_{2r-(n-r-3)}, u_{2r-(n-r-3)}, v_{r-(n-r-4)}, v_{2r-(n-r-2)}, \dots, u_{r-2}, u_{2r-2}, v_{r-1}, v_{2r-1}, u_r, v_1.$

Case iv. $n \equiv 1 \pmod{3}$.

We can prove the result as in Case iii.

Proposition 2.3. Let G be a graph of order n. Then R(G) = G if and only if $G \in F_{11}$ or F_{12} .

Proof. Follows from the definition.

To improve the readability of the paper, we offer an outline of the proof of the following two results which are found in [6].

Proposition 2.4. If r(G) > 1, then $R(G) \subseteq \overline{G}$.

Proof. If two vertices u and v are adjacent in R(G), then they are non-adjacent in G, since r(G) > 1.

Lemma 2.5. Let G be a graph of order n. Then $R(G) = \overline{G}$ if and only if either $S_2(G) = V(G)$ or G is disconnected in which each component is complete.

Proof. If $S_2(G) = V(G)$, then $R(G) = \overline{G}$. If G is the union of complete graphs, that is $G = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t}$, then its radial graph R(G) is the complete multipartite graph K_{n_1,n_2,\dots,n_t} , and consequently, $R(G) = \overline{G}$.

If $R(G) = \overline{G}$, then d(u, v) = 1 or r in G for any two distinct vertices uand v where r is the radius of G. By Proposition 2.3, $r(G) \neq 1$. If $2 < r < \infty$, then there exist at least two vertices x and y in G such that they are nonadjacent in both R(G) and G, which contradicts $R(G) = \overline{G}$.

If there is a pair of vertices x and y such that d(x, y) > 2, then they are nonadjacent in G and R(G). If G has a noncomplete component then $R(G) \neq \overline{G}$.

Now, we provide some graph equations involving radial graphs.

Proposition 2.6. Let G be a graph. Then $R(G) = R(\overline{G})$ if G is any one of the following graphs.

- 1. G or \overline{G} is complete.
- 2. G or \overline{G} is disconnected with each component complete out of which one is an isolated vertex.

Proof. If G is complete, then by Proposition 2.3, R(G) = G. Also $R(\overline{G}) = G$.

If \overline{G} is complete, then by Proposition 2.3, $R(\overline{G}) = \overline{G}$. Also $R(G) = \overline{G}$.

If G is disconnected with each component complete out of which one is an isolated vertex, then $R(G) = \overline{G}$ by Lemma 2.5. But $R(\overline{G}) = \overline{G}$ since \overline{G} has a vertex of degree n - 1.

If \overline{G} is disconnected with each component complete out of which one is an isolated vertex, then $R(\overline{G}) = G$ and R(G) = G.

Lemma 2.7. If G and \overline{G} are members of F_{22} , then $\overline{R(G)} = R(\overline{G})$.

Proof. Since G and \overline{G} are members of F_{22} , by Lemma 2.5, $R(G) = \overline{G}$ and $R(\overline{G}) = G$ and hence $\overline{R(G)} = R(\overline{G})$.

Lemma 2.8. Let G be a disconnected graph with each component complete and has no isolates. Then $\overline{R(G)} = R(\overline{G})$.

Proof. Since each component of G is complete, by Lemma 2.5, $R(G) = \overline{G}$. Also $\overline{G} \in F_{22}$. Again by Lemma 2.5, $R(\overline{G}) = G$. Thus $\overline{R(G)} = R(\overline{G})$.

Lemma 2.9. Let G be a connected graph such that $\overline{R(G)} = R(\overline{G})$. Then either G or \overline{G} is a member of F_{22} .

Proof. We prove this lemma by assuming that $G \notin F_{22}$. It suffices to show that $\overline{G} \in F_{22}$. If not, then $\overline{G} \in \mathcal{A} = F_{11} \cup F_{12} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. It is impossible that G is connected if $\overline{G} \in F_{11} \cup F_{12}$.

From Theorems A and C, if $\overline{G} \in F_{23}$, then $G \in F_{22} \cup F_{23}$. But, since $G \notin F_{22}$, only the case $G \in F_{23}$ has to be considered. By Proposition 2.4, $\underline{R}(\overline{G}) \subset G$. If $G \in F_{23}$, then $R(G) \subset \overline{G}$ and hence $G \subset \overline{R(G)}$. Thus $R(\overline{G}) \subset \overline{R(G)}$, a contradiction. Since G is connected $G \notin F_4$.

If $\overline{G} \in F_{24} \cup F_3$, then by Theorem B, $G \in F_{22}$, a contradiction.

Now let $\overline{G} \in F_4$. If \overline{G} has at least one isolated vertex, then R(G) = G. But $R(\overline{G}) \subseteq G$. Thus $R(\overline{G}) \subseteq R(G) \subseteq \overline{G}$. This contradicts the fact that $R(\overline{G}) \subseteq G$. If \overline{G} has no isolated vertices, then $G \in F_{22}$, a contradiction. The above argument force us to conclude that \overline{G} is a member of F_{22} .

Lemma 2.10. Let G be a disconnected graph such that $\overline{R(G)} = R(\overline{G})$. Then each component of G is a complete graph and has no isolates.

Proof. Suppose at least one of the components of G is not complete. Then $R(G) \subset \overline{G}$. This implies that $G \subset \overline{R(G)}$.

The following examples show that the notions of radial graph and antipodal graph are independent.



Figure 3. A radial graph but not an antipodal graph.



Figure 4. An antipodal graph but not a radial graph.

Next we characterize those antipodal which are radial graphs.

Theorem 2.11. A graph G is both radial and antipodal if and only if either $\overline{G} \in F_{22}$ or each component of \overline{G} is complete.

Proof. If either $\overline{G} \in F_{22}$ or each component of \overline{G} is complete, then $A(\overline{G}) = R(\overline{G}) = G$, by Lemma 2.5.

Conversely assume that G is both radial and antipodal. Assume that $\overline{G} \notin F_{22}$. We must claim that each component of \overline{G} is complete. Suppose \overline{G} has at least one noncomplete component. Then $A(\overline{G}) \neq G$. This is a contradiction.

3. The Radius of Radial Graphs

Theorem G [6]. If both G and \overline{G} are of self-centered graphs of radius 2, then so is R(G).

Proposition 3.1. Let G be a graph of order n. Then r(R(G)) = 1 if and only if either $\Delta(G) = n - 1$ or G is disconnected with at least one isolated vertex.

Corollary 3.2.

- (a) Let G be a connected graph. Then $R(G) \in F_{11}$ if and only if $G \in F_{11}$.
- (b) Let G be a connected graph. Then $R(G) \in F_{12}$ if and only if $G \in F_{12}$.
- (c) Let G be a disconnected graph. Then $R(G) \in F_{11}$ if and only if $G = \overline{K_n}$.
- (d) Let G be a disconnected graph. Then $R(G) \in F_{12}$ if and only if G has at least one isolated vertex and has a nontrivial component.

Proposition 3.3. Let G be a disconnected graph. Then $R(G) \in F_{22}$ if and only if G has no isolated vertex.

Proof. Follows from the definition.

Next we provide a characterization theorem for R(G) and $R(\overline{G})$ to be members of F_{22} .

Theorem 3.4. Let G be a connected graph. Then R(G) and $R(\overline{G})$ are members of F_{22} if and only if G and \overline{G} are members of F_{22} .

Proof. If G and \overline{G} are members of F_{22} , then by Lemma 2.5, $R(G) = \overline{G}$ and $R(\overline{G}) = G$.

Conversely assume that R(G) and $R(\overline{G})$ are members of F_{22} . Assume that $G \notin F_{22}$. Then $G \in \mathcal{A} = F_{11} \cup F_{12} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$.

If $G \in F_{11}$, then by Theorem 2.3, R(G) = G, a contradiction to the fact that $R(G) \in F_{22}$. If $G \in F_{12}$, then R(G) = G, a contradiction. If $G \in F_{23}$, then by Theorem A, $\overline{G} \in F_{22}$ or $\overline{G} \in F_{23}$. Then $R(\overline{G}) = G$ or $R(\overline{G}) \subset G$, a contradiction, since $R(\overline{G}) \in F_{22}$. It is easy to obtain a contradiction if $G \in F_{24}$ or $G \in F_3$. $G \in F_4$ will not hold since G is connected. Therefore $G \in F_{22}$. Similarly we can prove that $\overline{G} \in F_{22}$.

Theorem 3.5. Let G be a connected graph. Then $R(G) \in F_{23}$ if $G \in F_{22}$ and $\overline{G} \in F_{23}$. **Proof.** By Lemma 2.5, $R(G) = \overline{G}$. Also $\overline{G} \in F_{23}$ and hence $R(G) \in F_{23}$.

Theorem 3.6. Let G be a connected graph. Then $R(G) \in F_3$ if $G \in F_{22}$ and $\overline{G} \in F_3$.

Acknowledgement

The authors are thankful to the anonymous referee for valuable suggestions and comments for the paper resulted in an improved manner.

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Received 17 April 2008 Revised 22 January 2009 Accepted 22 January 2009