# FURTHER RESULTS ON RADIAL GRAPHS 

Kumarappan Kathiresan<br>Center for Research and Post Graduate Studies in Mathematics<br>Ayya Nadar Janaki Ammal College<br>Sivakasi - 626 124, Tamil Nadu, India<br>e-mail: kathir2esan@yahoo.com

AND<br>G. Marimuthu<br>Department of Mathematics<br>The Madura College<br>Madurai - 625 011, Tamil Nadu, India<br>e-mail: yellowmuthu@yahoo.com


#### Abstract

In a graph $G$, the distance $d(u, v)$ between a pair of vertices $u$ and $v$ is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex $u$ is the distance to a vertex farthest from $u$. The minimum eccentricity is called the radius of the graph and the maximum eccentricity is called the diameter of the graph. The radial graph $R(G)$ based on $G$ has the vertex set as in $G$, two vertices $u$ and $v$ are adjacent in $R(G)$ if the distance between them in $G$ is equal to the radius of $G$. If $G$ is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components. The main objective of this paper is to characterize graphs $G$ with specified radius for its radial graph.


Keywords: radius, diameter, radial graph.
2010 Mathematics Subject Classification: 05C12.

## 1. Introduction

The graphs considered here are nontrivial and simple. For other graph theoretic notation and terminology, we follow [4, 9]. In a graph $G$, the distance $d(u, v)$ between a pair of vertices $u$ and $v$ is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex $u$ is the distance to a vertex farthest from $u$. The radius $r(G)$ of $G$ is defined by $r(G)=\min \{e(u)$ : $u \in V(G)\}$ and the diameter $d(G)$ of $G$ is defined by $d(G)=\max \{e(u)$ : $u \in V(G)\}$. A graph for which $r(G)=d(G)$ is called a self-centered graph of radius $r(G)$. A vertex $v$ is called an eccentric vertex of a vertex $u$ if $d(u, v)=e(u)$. A vertex $v$ of $G$ is called an eccentric vertex of $G$ if it is an eccentric vertex of some vertex of $G$. Let $S_{i}$ denote the subset of vertices of $G$ whose eccentricity is equal to $i$. The concept of antipodal graph was initially introduced by [8] and was further expanded by $[2,3]$. The antipodal graph of a graph $G$, denoted by $A(G)$, is the graph on the same vertices as of $G$, two vertices being adjacent if the distance between them is equal to the diameter of $G$. A graph is said to be antipodal if it is the antipodal graph $A(H)$ of some graph $H$. The concept of eccentric graph was introduced by [1]. The eccentric graph based on $G$ is denoted by $G_{e}$, whose vertex set is $V(G)$ and two vertices $u$ and $v$ are adjacent in $G_{e}$ if and only if $d(u, v)=\min \{e(u), e(v)\}$. Also Chartrand et al., [5] studied the concept of eccentric graphs. The subgraph of $G$ induced by its eccentric vertices is called the eccentric subgraph of $G$. In [5] a characterization of all graphs that are eccentric subgraph of some connected graph was presented. Kathiresan and Marimuthu [6] introduced a new type of graph called radial graph. Two vertices of a graph $G$ are said to be radial to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph $G$, denoted by $R(G)$, has the vertex set as in $G$ and two vertices are adjacent in $R(G)$ if and only if they are radial in $G$. If $G$ is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of $G$. A graph $G$ is called a radial graph if $R(H)=G$ for some graph $H$. We denote $G_{1}=G_{2}$ if the two graphs $G_{1}$ and $G_{2}$ are the same graphs and $G_{1} \subset G_{2}$ if $G_{1}$ is a proper subgraph of $G_{2}$. Next we provide some results which will be used to prove some theorems in this paper.

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}$ and $F_{3}$ denote the set of all connected graphs $G$ for which $r(G)=d(G)=1 ; r(G)=1$ and $d(G)=2 ; r(G)=d(G)=2$; $r(G)=2$ and $d(G)=3 ; r(G)=2$ and $d(G)=4$ and $r(G) \geq 3$ respectively. Let $F_{4}$ denote the set of all disconnected graphs.

Theorem A [4]. If $G$ is a simple graph with diameter at least 3 , then $\bar{G}$ has diameter at most 3 .

Theorem B [4]. If $G$ is a simple graph with diameter at least 4, then $\bar{G}$ has diameter at most 2 .

Theorem C [4]. If $G$ is a simple graph with diameter at least 3, then $\bar{G}$ has radius at most 2 .

Theorem D [9]. If $G$ is a self-centered graph with $r(G) \geq 3$, then $\bar{G}$ is a self-centered graph of radius 2 .

Theorem E [6]. A graph $G$ is a radial graph if and only if $G$ is the radial graph of itself or the radial graph of its complement.

Theorem $\mathbf{F}$ [3]. A graph $G$ is an antipodal graph if and only if $G$ is the antipodal graph of its complement.

The ladder graph $L_{n}[7]$ with $n$ steps is defined by $L_{n}=P_{n} \times K_{2}$ where $P_{n}$ is a path on $n$ vertices and $\times$ denotes the Cartesian product of graphs.

## 2. Graph Equations Involving Radial Graphs

Result 2.1. Let $L_{n}$ be a ladder with $n$ steps. Then

$$
r\left(L_{n}\right)= \begin{cases}\frac{n+2}{2} & \text { if } n \equiv 0,2(\bmod 4), \\ \frac{n+1}{2} & \text { if } n \equiv 1,3(\bmod 4) .\end{cases}
$$

Result 2.2. Let $L_{n}$ be a ladder graph with $n$ steps. Then

$$
R\left(L_{n}\right)= \begin{cases}2 P_{n} & \text { if } n \equiv 0,2(\bmod 4) \\ C_{2 n} & \text { if } n \equiv 1(\bmod 4) \\ 2 C_{n} & \text { if } n \equiv 3(\bmod 4)\end{cases}
$$

Proof. Let $u_{1}, u_{2}, \ldots, u_{n}$ and let $v_{1}, v_{2}, \ldots, v_{n}$ be the vertices of the ladder $L_{n}$. The edge set of $L_{n}$ is $E\left(L_{n}\right)=\left\{u_{i} v_{i}, u_{i} u_{i+1}, v_{i} v_{i+1}: i=1,2, \ldots\right.$, $n-1\} \cup\left\{u_{n} v_{n}\right\}$.

Case i. $n \equiv 0(\bmod 4)$.
The radius $r$ of $L_{n}$ is $\frac{n+2}{2}$. The radial pairs are as follows:
$\left(u_{i}, u_{r+i}\right),\left(u_{i}, v_{r+i-1}\right)$ where $i=1,2, \ldots, \frac{n}{2}-1 ;\left(u_{n / 2+1}, v_{n}\right) ;\left(u_{n / 2}, v_{1}\right)$;
$\left(u_{j}, u_{j-r}\right),\left(u_{j}, v_{j-r+1}\right)$ where $j=r+1, r+2, \ldots, n$.
These radial pairs give the radial graph as the union of two path graphs $P_{n}^{1}$ and $P_{n}^{2}$, where
$P_{n}^{1}: u_{r-1}, v_{2 r-2}, v_{r-2}, u_{2 r-3}, u_{r-3}, v_{2 r-4}, v_{r-4}, u_{2 r-5}, u_{r-5}, v_{2 r-6}, v_{r-6}$, $\ldots, u_{2 r-(r-2)}, u_{r-(r-2)}, v_{2 r-(r-1)}, v_{r-(r-1)}, u_{r}$ and
$P_{n}^{2}: v_{r-1}, u_{2 r-2}, u_{r-2}, v_{2 r-3}, v_{r-3}, u_{2 r-4} u_{r-4}, v_{2 r-5}, v_{r-5}, u_{2 r-6}, u_{r-6}$, $\ldots, v_{2 r(r-2)}, v_{r-(r-2)}, u_{2 r-(r-1)}, u_{r-(r-1)}, v_{r}$.


Figure 1. The graph $L_{4}$ with $r=3$.


Figure 2. The graph $R\left(L_{4}\right)$.

Case ii. $n \equiv 2(\bmod 4)$.
We can prove the result as in the Case i.
Case iii. $n \equiv 3(\bmod 4)$.
The radius $r$ of $L_{n}$ is $\frac{n+1}{2}$. The radial pairs are as follows: $\left(u_{i}, u_{r+i}\right),\left(u_{i}, v_{r+i-1}\right)$ where $i=1,2, \ldots, r-1 ;\left(u_{r}, v_{1}\right) ;\left(u_{r}, v_{n}\right) ;\left(u_{j}, u_{j-r}\right)$, $\left(u_{j}, v_{j-r+1}\right)$ where $j=r+1, r+2, \ldots, n$. The radial graph corresponding
to the above radial pairs is the union of two disjoint cycle graphs $C_{n}^{1}$ and $C_{n}^{2}$ where
$C_{n}^{1}: u_{1}, u_{r+1}, v_{r-(n-r-1)}, v_{2 r-(n-r-1)}, u_{r-(n-r-2)}, u_{2 r-(n-r-2)}, v_{r-(n-r-3)}$,
$v_{2 r-(n-r-3)}, u_{r-(n-r-4)}, u_{2 r-(n-r-2)}, \ldots, v_{r-2}, v_{2 r-2}, u_{r-1}, u_{2 r-1}, v_{r}, u_{1}$.
$C_{n}^{2}: v_{1}, v_{r+1}, u_{r-(n-r-1)}, u_{2 r-(n-r-1)}, v_{r-(n-r-2)}, v_{2 r-(n-r-2)}, u_{r-(n-r-3)}$,
$u_{2 r-(n-r-3)}, v_{r-(n-r-4)}, v_{2 r-(n-r-2)}, \ldots, u_{r-2}, u_{2 r-2}, v_{r-1}, v_{2 r-1}, u_{r}, v_{1}$.
Case iv. $n \equiv 1(\bmod 3)$.
We can prove the result as in Case iii.
Proposition 2.3. Let $G$ be a graph of order $n$. Then $R(G)=G$ if and only if $G \in F_{11}$ or $F_{12}$.

Proof. Follows from the definition.
To improve the readability of the paper, we offer an outline of the proof of the following two results which are found in [6].

Proposition 2.4. If $r(G)>1$, then $R(G) \subseteq \bar{G}$.
Proof. If two vertices $u$ and $v$ are adjacent in $R(G)$, then they are nonadjacent in $G$, since $r(G)>1$.

Lemma 2.5. Let $G$ be a graph of order n. Then $R(G)=\bar{G}$ if and only if either $S_{2}(G)=V(G)$ or $G$ is disconnected in which each component is complete.

Proof. If $S_{2}(G)=V(G)$, then $R(G)=\bar{G}$. If $G$ is the union of complete graphs, that is $G=K_{n_{1}} \cup K_{n_{2}} \cup \cdots \cup K_{n_{t}}$, then its radial graph $R(G)$ is the complete multipartite graph $K_{n_{1}, n_{2}, \ldots n_{t}}$, and consequently, $R(G)=\bar{G}$.

If $R(G)=\bar{G}$, then $d(u, v)=1$ or $r$ in $G$ for any two distinct vertices $u$ and $v$ where $r$ is the radius of $G$. By Proposition 2.3, $r(G) \neq 1$. If $2<r<$ $\infty$, then there exist at least two vertices $x$ and $y$ in $G$ such that they are nonadjacent in both $R(G)$ and $G$, which contradicts $R(G)=\bar{G}$.

If there is a pair of vertices $x$ and $y$ such that $d(x, y)>2$, then they are nonadjacent in $G$ and $R(G)$. If $G$ has a noncomplete component then $R(G) \neq \bar{G}$.

Now, we provide some graph equations involving radial graphs.

Proposition 2.6. Let $G$ be a graph. Then $R(G)=R(\bar{G})$ if $G$ is any one of the following graphs.

1. $G$ or $\bar{G}$ is complete.
2. $G$ or $\bar{G}$ is disconnected with each component complete out of which one is an isolated vertex.

Proof. If $G$ is complete, then by Proposition 2.3, $R(G)=G$. Also $R(\bar{G})=G$.

If $\bar{G}$ is complete, then by Proposition $2.3, R(\bar{G})=\bar{G}$. Also $R(G)=\bar{G}$.
If $G$ is disconnected with each component complete out of which one is an isolated vertex, then $R(G)=\bar{G}$ by Lemma 2.5. But $R(\bar{G})=\bar{G}$ since $\bar{G}$ has a vertex of degree $n-1$.

If $\bar{G}$ is disconnected with each component complete out of which one is an isolated vertex, then $R(\bar{G})=G$ and $R(G)=G$.

Lemma 2.7. If $G$ and $\bar{G}$ are members of $F_{22}$, then $\overline{R(G)}=R(\bar{G})$.
Proof. Since $G$ and $\bar{G}$ are members of $F_{22}$, by Lemma 2.5, $R(G)=\bar{G}$ and $R(\bar{G})=G$ and hence $\overline{R(G)}=R(\bar{G})$.

Lemma 2.8. Let $G$ be a disconnected graph with each component complete and has no isolates. Then $\overline{R(G)}=R(\bar{G})$.

Proof. Since each component of $G$ is complete, by Lemma 2.5, $R(G)=\bar{G}$. Also $\bar{G} \in F_{22}$. Again by Lemma 2.5, $R(\bar{G})=G$. Thus $\overline{R(G)}=R(\bar{G})$.

Lemma 2.9. Let $G$ be a connected graph such that $\overline{R(G)}=R(\bar{G})$. Then either $G$ or $\bar{G}$ is a member of $F_{22}$.

Proof. We prove this lemma by assuming that $G \notin F_{22}$. It suffices to show that $\bar{G} \in F_{22}$. If not, then $\bar{G} \in \mathcal{A}=F_{11} \cup F_{12} \cup F_{23} \cup F_{24} \cup F_{3} \cup F_{4}$. It is impossible that $G$ is connected if $\bar{G} \in F_{11} \cup F_{12}$.

From Theorems A and C, if $\bar{G} \in F_{23}$, then $G \in F_{22} \cup F_{23}$. But, since $G \notin F_{22}$, only the case $G \in F_{23}$ has to be considered. By Proposition 2.4, $\underline{R(\bar{G})} \subset G$. If $G \in F_{23}$, then $R(G) \subset \bar{G}$ and hence $G \subset \overline{R(G)}$. Thus $R(\bar{G}) \subset$ $\overline{R(G)}$, a contradiction. Since $G$ is connected $G \notin F_{4}$.

If $\bar{G} \in F_{24} \cup F_{3}$, then by Theorem B, $G \in F_{22}$, a contradiction.
Now let $\bar{G} \in F_{4}$. If $\bar{G}$ has at least one isolated vertex, then $R(G)=G$. But $R(\bar{G}) \subseteq G$. Thus $R(\bar{G}) \subseteq R(G) \subseteq \bar{G}$. This contradicts the fact that
$R(\bar{G}) \subseteq G$. If $\bar{G}$ has no isolated vertices, then $G \in F_{22}$, a contradiction. The above argument force us to conclude that $\bar{G}$ is a member of $F_{22}$.

Lemma 2.10. Let $G$ be a disconnected graph such that $\overline{R(G)}=R(\bar{G})$. Then each component of $G$ is a complete graph and has no isolates.

Proof. Suppose at least one of the components of $G$ is not complete. Then $R(G) \subset \bar{G}$. This implies that $G \subset \overline{R(G)}$.

The following examples show that the notions of radial graph and antipodal graph are independent.


Figure 3. A radial graph but not an antipodal graph.


Figure 4. An antipodal graph but not a radial graph.
Next we characterize those antipodal which are radial graphs.
Theorem 2.11. A graph $G$ is both radial and antipodal if and only if either $\bar{G} \in F_{22}$ or each component of $\bar{G}$ is complete.

Proof. If either $\bar{G} \in F_{22}$ or each component of $\bar{G}$ is complete, then $A(\bar{G})=$ $R(\bar{G})=G$, by Lemma 2.5.

Conversely assume that $G$ is both radial and antipodal. Assume that $\bar{G} \notin F_{22}$. We must claim that each component of $\bar{G}$ is complete. Suppose $\bar{G}$ has at least one noncomplete component. Then $A(\bar{G}) \neq G$. This is a contradiction.

## 3. The Radius of Radial Graphs

Theorem G [6]. If both $G$ and $\bar{G}$ are of self-centered graphs of radius 2, then so is $R(G)$.

Proposition 3.1. Let $G$ be a graph of order $n$. Then $r(R(G))=1$ if and only if either $\Delta(G)=n-1$ or $G$ is disconnected with at least one isolated vertex.

## Corollary 3.2.

(a) Let $G$ be a connected graph. Then $R(G) \in F_{11}$ if and only if $G \in F_{11}$.
(b) Let $G$ be a connected graph. Then $R(G) \in F_{12}$ if and only if $G \in F_{12}$.
(c) Let $G$ be a disconnected graph. Then $R(G) \in F_{11}$ if and only if $G=\overline{K_{n}}$.
(d) Let $G$ be a disconnected graph. Then $R(G) \in F_{12}$ if and only if $G$ has at least one isolated vertex and has a nontrivial component.

Proposition 3.3. Let $G$ be a disconnected graph. Then $R(G) \in F_{22}$ if and only if $G$ has no isolated vertex.

Proof. Follows from the definition.
Next we provide a characterization theorem for $R(G)$ and $R(\bar{G})$ to be members of $F_{22}$.

Theorem 3.4. Let $G$ be a connected graph. Then $R(G)$ and $R(\bar{G})$ are members of $F_{22}$ if and only if $G$ and $\bar{G}$ are members of $F_{22}$.

Proof. If $G$ and $\bar{G}$ are members of $F_{22}$, then by Lemma $2.5, R(G)=\bar{G}$ and $R(\bar{G})=G$.

Conversely assume that $R(G)$ and $R(\bar{G})$ are members of $F_{22}$. Assume that $G \notin F_{22}$. Then $G \in \mathcal{A}=F_{11} \cup F_{12} \cup F_{23} \cup F_{24} \cup F_{3} \cup F_{4}$.

If $G \in F_{11}$, then by Theorem $2.3, R(G)=G$, a contradiction to the fact that $R(G) \in F_{22}$. If $G \in F_{12}$, then $R(G)=G$, a contradiction. If $G \in F_{23}$, then by Theorem A, $\bar{G} \in F_{22}$ or $\bar{G} \in F_{23}$. Then $R(\bar{G})=G$ or $R(\bar{G}) \subset G$, a contradiction, since $R(\bar{G}) \in F_{22}$. It is easy to obtain a contradiction if $G \in F_{24}$ or $G \in F_{3} . G \in F_{4}$ will not hold since $G$ is connected. Therefore $G \in F_{22}$. Similarly we can prove that $\bar{G} \in F_{22}$.

Theorem 3.5. Let $G$ be a connected graph. Then $R(G) \in F_{23}$ if $G \in F_{22}$ and $\bar{G} \in F_{23}$.

Proof. By Lemma 2.5, $R(G)=\bar{G}$. Also $\bar{G} \in F_{23}$ and hence $R(G) \in F_{23}$.
Theorem 3.6. Let $G$ be a connected graph. Then $R(G) \in F_{3}$ if $G \in F_{22}$ and $\bar{G} \in F_{3}$.

## Acknowledgement

The authors are thankful to the anonymous referee for valuable suggestions and comments for the paper resulted in an improved manner.

## References

[1] J. Akiyama, K. Ando and D. Avis, Eccentric graphs, Discrete Math. 16 (1976) 187-195.
[2] R. Aravamuthan and B. Rajendran, Graph equations involving antipodal graphs, Presented at the seminar on Combinatorics and applications held at ISI (Culcutta during 14-17 December 1982) 40-43.
[3] R. Aravamuthan and B. Rajendran, On antipodal graphs, Discrete Math. 49 (1984) 193-195.
[4] F. Buckley and F. Harary, Distance in Graphs (Addison-Wesley Reading, 1990).
[5] G. Chartrand, W. Gu, M. Schultz and S.J. Winters, Eccentric graphs, Networks 34 (1999) 115-121.
[6] KM. Kathiresan and G. Marimuthu, A study on radial graphs, Ars Combin. (to appear).
[7] KM. Kathiresan, Subdivision of ladders are graceful, Indian J. Pure Appl. Math. 23 (1992) 21-23.
[8] R.R. Singleton, There is no irregular Moore graph, Amer. Math. Monthly 7 (1968) 42-43.
[9] D.B. West, Introduction to Graph Theory (Prentice-Hall of India, New Delhi, 2003).

