Discussiones Mathematicae
Graph Theory 30 (2010) 55–73

THE EDGE GEODETIC NUMBER AND CARTESIAN PRODUCT OF GRAPHS *

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Abstract

For a nontrivial connected graph $G = (V(G), E(G))$, a set $S \subseteq V(G)$ is called an edge geodetic set of G if every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_1(G)$ of G is the minimum order of its edge geodetic sets. Bounds for the edge geodetic number of Cartesian product graphs are proved and improved upper bounds are determined for a special class of graphs. Exact values of the edge geodetic number of Cartesian product are obtained for several classes of graphs. Also we obtain a necessary condition of G for which $g_1(G \Box K_2) = g_1(G)$.

Keywords: geodetic number, edge geodetic number, linear edge geodetic set, perfect edge geodetic set, (edge, vertex)-geodetic set, superior edge geodetic set.

2010 Mathematics Subject Classification: 05C12.

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*Research supported by DST Project No. SR/S4/MS: 319/06.

1. Introduction

The geodetic number of a graph was introduced in [2, 5] and further studied in [1, 3]. The edge geodetic number of a graph was introduced and studied in [7]. Although the edge geodetic number is greater than or equal to the geodetic number for an arbitrary graph, the properties of the edge geodetic sets and results regarding edge geodetic number are quite different from that of geodetic concepts. These concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. In the case of designing the route for a shuttle, although all the vertices are covered by the shuttle when considering geodetic sets, some of the edges may be left out. This drawback is rectified in the case of edge geodetic sets and hence considering edge geodetic sets is more advantageous to the real life application of routing problem. In particular, the edge geodetic sets are more useful than geodetic sets in the case of regulating and routing the goods vehicles to tranport the commodities to important places.

The results in [1, 7] motivate us to investigate the behaviour of edge geodetic sets in Cartesian product of two graphs. In section 2, we first obtain a lower bound for the edge geodetic number of Cartesian product of two graphs. Then we obtain a necessary and sufficient condition for an edge to lie on a geodesic of $G\Box H$ and use this to obtain an upper bound for the edge geodetic number of $G\Box H$. We also improve the upper bound of $g_1(G\Box H)$ when both G and H posses linear minimum edge geodetic sets. In section 3, we obtain the exact value of $g_1(G\Box H)$ for several classes of graphs. We prove, in particular, that $g_1(K_m\Box K_n) = mn - \min\{m, n\}$ and $g_1(P_m\Box K_n) = 2n-2$ for $m, n \geq 2$ and also that $g_1(T_1\Box T_2) = \max\{g_1(T_1), g_1(T_2)\}$ for any two trees T_1 and T_2. We also prove that $g_1(G\Box H) = \max\{g_1(G), g_1(H)\}$ when both G and H posses the so called perfect minimum edge geodetic sets. Further, we prove that $g_1(G\Box G) = g_1(G)$ if G posseses an (edge, vertex)-geodetic set of cardinality $g_1(G)$. The question of when $g_1(G\Box K_2) = g_1(G)$ is also partially answered. From the results given in [1], we observe that the edge geodetic number and the geodetic number have significant difference in products of graphs.

By a graph $G = (V(G), E(G))$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. The *distance* $d(u, v)$ between two vertices u and

v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. It is known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G, and the maximum eccentricity is its *diameter*, *diam* G of G. A *geodetic* set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets. An *edge geodetic set* of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_1(G)$ of G is the minimum order of its edge geodetic sets. For the graph G given in Figure 1.1, $S = \{v_1, v_2, v_4\}$ is a minimum edge geodetic set of G so that $g_1(G) = 3$. Also $S' = \{v_3, v_5\}$ is a minimum geodetic set of G so that $g(G) = 2$. Thus the geodetic number and the edge geodetic number of a graph are different.

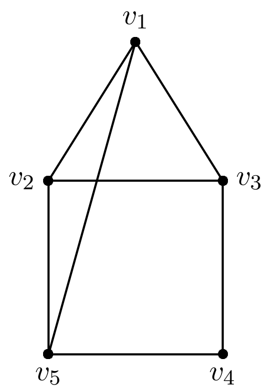

Figure 1.1. G

The *Cartesian product* of graphs G and H, denoted by $G \square H$, has vertex set $V(G) \times V(H)$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ and $y_1y_2 \in E(H)$, or $y_1 = y_2$ and $x_1x_2 \in E(G)$. The mappings $\pi_G : (x, y) \mapsto x$ and $\pi_H : (x, y) \mapsto y$ from $V(G \square H)$ onto G and H respectively are called *projections*. For a set $S \subseteq V(G \square H)$, we define the *G-projection* on G as $\pi_G(S) = \{x \in V(G) : (x, y) \in S$ for some $y \in V(H)\}$, and the *H-projection* $\pi_H(S) = \{y \in V(H) : (x, y) \in S$ for some $x \in V(G)\}$. For any $y \in V(H)$, the subgraph of $G \square H$ induced by $\{(x, y) : x \in V(G)\}$ is isomorphic to G. We denote it by G_y and call it the *copy* of G corresponding to y. Similarly, for any x in $V(G)$ the subgraph of $G\Box H$ induced by $\{(x,y) : y \in V(H)\}$ is isomorphic to H, and we denote it by H_x and call it the *copy* of H corresponding to x. Given a path P in a graph and two vertices x, y on P, we use $P[x,y]$ to denote the portion of P between x and y, inclusive of x and y. The geodetic number of Cartesian product of graphs was studied in [1]. For basic graph theoretic terminology, we refer to [4]. We also refer to [2] for results on distance in graphs and to [6] for metric structures in Cartesian product of graphs. Throughout the following G denotes a connected graph with at least two vertices. The following theorems will be used in the sequel.

Theorem 1.1 [6]. *Let G and H be connected graphs with (u,v) and (x,y) arbitrary vertices of the Cartesian product $G\Box H$ of G and H. Then $d_{G\Box H}((u,v),\,(x,y)) = d_G(u,x) + d_H(v,y)$. Moreover, if P is a $(u,v)-(x,y)$ geodesic in $G\Box H$, then the G-projection $\pi_G(P)$ is a $u-x$ geodesic in G and the H-projection $\pi_H(P)$ is $v-y$ geodesic in H.*

Theorem 1.2 [7]. *For the complete graph K_n, $g_1(K_n) = n$.*

Theorem 1.3 [7]. *For any tree T, the edge geodetic number $g_1(T)$ equals the number of end vertices in T. In fact, the set of all end vertices of T is the unique minimum edge geodetic set of T.*

Theorem 1.4 [7]. *Every edge geodetic set of a connected graph G is a geodetic set of G.*

2. Bounds for the Edge Geodetic Number

In this section we determine possible bounds for the edge geodetic number of the Cartesian product of two connected graphs.

Lemma 2.1. *Let S be an edge geodetic set of $G\Box H$. Then $\pi_G(S)$ and $\pi_H(S)$ are edge geodetic sets of G and H respectively.*

Proof. Let $e = ux$ be an edge in G. Then $e_y = (u,y)(x,y)$ is an edge in $G\Box H$ for each vertex y in H. Since S is an edge geodetic set of $G\Box H$, e_y lies on some $(g_1,h_1)-(g_2,h_2)$ geodesic P of $G\Box H$ with $(g_1,h_1) \in S$ and $(g_2,h_2) \in S$. Let $\pi_G(P)$ be the projection of P on G. Then, by Theorem 1.1,

$\pi_G(P)$ is a $g_1 - g_2$ geodesic in G with $g_1, g_2 \in \pi_G(S)$ and it is clear that the edge $e = ux$ lies on $\pi_G(P)$. Hence $\pi_G(S)$ is an edge geodetic set of G. Similarly, we can prove that $\pi_H(S)$ is an edge geodetic set of H. ∎

Remark 2.2. The converse of Lemma 2.1 is not true. By Theorem 1.2, the vertex sets $V(G) = \{x_1, x_2, x_3\}$ and $V(H) = \{y_1, y_2\}$ are the edge geodetic sets of the complete graphs $G = K_3$ and $H = K_2$ respectively. It is clear that the edge $(x_1, y_2)(x_2, y_2)$ does not lie on a geodesic joining any pair of vertices in S, where $S = \{(x_1, y_1), (x_2, y_1), (x_3, y_2)\}$, and so S not an edge geodetic set of $G\square H$.

Theorem 2.3. *Let G and H be connected graphs. Then* $\max\{g_1(G), g_1(H)\} \leq g_1(G\square H)$.

Proof. Let S be a minimum edge geodetic set of $G\square H$. Then $g_1(G\square H) = |S|$. Let $\pi_G(S)$ and $\pi_H(S)$ be the projections of S on G and H respectively. By Lemma 2.1, $\pi_G(S)$ and $\pi_H(S)$ are edge geodetic sets of G and H respectively and so $g_1(G) \leq |\pi_G(S)|$ and $g_1(H) \leq |\pi_H(S)|$. Since $|\pi_G(S)| \leq |S|$ and $|\pi_H(S)| \leq |S|$, it follows that $g_1(G) \leq |S|$ and $g_1(H) \leq |S|$. Therefore, $\max\{g_1(G), g_1(H)\} \leq g_1(G\square H)$. ∎

Lemma 2.4 *Let G and H be connected graphs with $e = (x_1, y)(x_2, y)$ an edge of $G\square H$. Then e lies on a $(g,h) - (g', h')$ geodesic of $G\square H$ if and only if the edge x_1x_2 lies on a $g - g'$ geodesic of G and the vertex y lies on a $h - h'$ geodesic of H.*

Proof. Suppose that the edge $e = (x_1, y)(x_2, y)$ in $G\square H$ lies on some $(g,h) - (g', h')$ geodesic P of $G\square H$. Let $\pi_G(P)$ and $\pi_H(P)$ be the projections of P on G and H respectively. Then it follows from Theorem 1.1 that $\pi_G(P)$ is a $g - g'$ geodesic in G containing the edge x_1x_2 and $\pi_H(P)$ is a $h - h'$ geodesic in H containing the vertex y.

Conversely, suppose that the edge $e_1 = x_1x_2$ of G lies on some $g - g'$ geodesic P of G and the vertex y of H lies on some $h - h'$ geodesic Q of H. Let L_1 be the copy of P in the copy G_h of G corresponding to h, L_2 be the copy of Q in the copy H_{x_1} of H corresponding to x_1, L_3 be the copy of P in the copy G_y of G corresponding to y and L_4 be the copy of Q in the copy $H_{g'}$ of H corresponding to g'. Let $e = (x_1, y)(x_2, y)$. Then e is an edge of $G\square H$ and it is clear that $R : L_1[(g,h),(x_1,h)] \cup L_2[(x_1,h),(x_1,y)] \cup L_3[(x_1,y),(g',y)] \cup L_4[(g',y),(g',h')]$ is a $(g,h) - (g',h')$

path in $G\Box H$ that contains the edge e. Also each of $L_1[(g,h),(x_1,h)]$, $L_2[(x_1,h),(x_1,y)]$, $L_3[(x_1,y),(g',y)]$ and $L_4[(g',y),(g',h')]$ is a geodesic between the respective vertices. Now, it follows from Theorem 1.1 and the fact that x_1 lies on a $g-g'$ geodesic and y lies on a $h-h'$ geodesic that the length of R,

$$\begin{aligned} l(R) &= l(L_1[(g,h),(x_1,h)]) + l(L_2[(x_1,h),(x_1,y)]) + l(L_3[(x_1,y),(g',y)]) \\ &\quad + \ l(L_4[(g',y),(g',h')]) \\ &= d_G(g,x_1) + d_H(h,y) + d_G(x_1,g') + d_H(y,h') \\ &= d_G(g,x_1) + d_G(x_1,g') + d_H(h,y) + d_H(y,h') \\ &= d_G(g,g') + d_H(h,h') \\ &= d_{G\Box H}[(g,h),(g',h')]. \end{aligned}$$

Thus, R is a $(g,h)-(g',h')$ geodesic of $G\Box H$ such that it contains the edge $e=(x_1,y)(x_2,y)$. ∎

Theorem 2.5. *Let G and H be connected graphs such that $S \subseteq V(G)$ and $T \subseteq V(H)$. Then S and T are edge geodetic sets of G and H respectively if and only if $S\times T$ is an edge geodetic set of $G\Box H$.*

Proof. Suppose that $S\times T$ is an edge geodetic set of $G\Box H$. Then S and T are the projections of $S\times T$ on G and H respectively. Hence by Lemma 2.1, S and T are edge geodetic sets of G and H respectively. Conversely, suppose that S and T are edge geodetic sets of G and H respectively. Let $e = (x_1,y_1)(x_2,y_2)$ be any edge in $G\Box H$. Assume that $e_1 = x_1x_2$ is an edge in G. Then $y_1 = y_2 = y$ (say). Since S is an edge geodetic set of G, there exist $g_1, g_2 \in S$ such that e_1 lies on some $g_1 - g_2$ geodesic of G. Since T is an edge geodetic set of H, by Theorem 1.4, T is also a geodetic set of H and so there exist $h_1, h_2 \in T$ such that the vertex y lies on some $h_1 - h_2$ geodesic of H. Hence by Lemma 2.4, the edge $e = (x_1,y_1)(x_2,y_2)$ lies on some $(g_1,h_1)-(g_2,h_2)$ geodesic of $G\Box H$ with $(g_1,h_1),(g_2,h_2) \in S\times T$. Thus, $S\times T$ is an edge geodetic set of $G\Box H$. ∎

Theorem 2.6. *Let G and H be connected graphs with $g_1(G) = p$ and $g_1(H) = q$ such that $p \geq q \geq 2$. Then $g_1(G\Box H) \leq pq - q$.*

Proof. Let $S = \{g_1, g_2, \ldots, g_p\}$ and $T = \{h_1, h_2, \ldots, h_q\}$ be edge geodetic sets of G and H respectively. Let $U = S\times T - \bigcup_{i=1}^{q}\{(g_i,h_i)\}$. We claim that U is an edge geodetic set of $G\Box H$. Let $e = (x, y)(x', y') \in E(G\Box H)$. Without loss of generality we assume that $e_1 = xx' \in E(G)$. Then $y = y'$ in H. Since S is an edge geodetic set of G, there exist indices i and j with $1 \leq i$, $j \leq p$ and $i \neq j$ such that the edge $e_1 = xx'$ lies on some $g_i - g_j$ geodesic P of G. By Theorem 1.4, T is a geodetic set of H. If $y \in T$, then $y = h_k$ for some $1 \leq k \leq q$. Since $q \geq 2$, y lies on a $h_k - h_l$ geodesic for any l with $l \neq k$ and $1 \leq l \leq q$. If $y \notin T$, then by Theorem 1.4, y lies on a $h_k - h_l$ geodesic of H with $k \neq l$ and $1 \leq k$, $l \leq q$. Let $B = \{(g_i, h_k), (g_i, h_l), (g_j, h_k), (g_j, h_l)\}$. We consider the following cases.

Case 1. Suppose that $B \subseteq U$. Then $(g_i, h_k) \in U$ and $(g_j, h_l) \in U$. Since P is a $g_i - g_j$ geodesic of G containing the edge $e_1 = xx'$ and Q is a $h_k - h_l$ geodesic of H containing the vertex y, by Lemma 2.4, there exists a $(g_i, h_k) - (g_j, h_l)$ geodesic of $G\Box H$ containing the edge $e = (x, y)(x', y')$. Hence U is an edge geodetic set of $G\Box H$.

Case 2. Suppose that $B \not\subseteq U$.

Subcase 2.1. First suppose that $(g_i, h_k) \notin U$. Then $i = k$ and so $i \neq l$ and $j \neq k$. Thus $(g_i, h_l) \in U$ and $(g_j, h_k) \in U$. Since the edge $e_1 = xx'$ lies on the $g_i - g_j$ geodesic P of G and the vertex y lies on the $h_l - h_k$ geodesic Q^{-1} of H, by Lemma 2.4, the edge $e = (x, y)(x', y')$ lies on some $(g_i, h_l) - (g_j, h_k)$ geodesic of $G\Box H$. The other subcases are similar. Thus U is an edge geodetic set of $G\Box H$. ■

Corollary 2.7. *For any connected graphs G and H,* $\max\{g_1(G), g_1(H)\} \leq g_1(G\Box H) \leq g_1(G)g_1(H) - \min\{g_1(G), g_1(H)\}$.

Proof. This follows from Theorems 2.3 and 2.6. ■

Corollary 2.8. *If G and H are connected graphs with $g_1(G) = g_1(H) = 2$, then $g_1(G\Box H) = 2$. Thus the bounds in Corollary 2.7 are sharp.*

Proof. This follows from Corollary 2.7. ■

In the following we introduce a class of graphs G and H for which the upper bound of the edge geodetic number $g_1(G\Box H)$ of $G\Box H$ is further improved. A *linear geodetic set* is defined in [1]. We now define *linear edge geodetic set* and proceed.

Definition 2.9. An edge geodetic set $S = \{x_1, x_2, \ldots, x_k\}$ of a graph G is called a *linear edge geodetic set* if for any edge e of G, there exists an index $i, 1 \leq i < k$ such that the edge e lies on some $x_i - x_{i+1}$ geodesic of G.

If G is any graph with $g_1(G) = 2$, then every minimum edge geodetic set is linear. For the graph G given in Figure 2.1, $S = \{u, v, w, x\}$ is the unique linear minimum edge geodetic set. The complete graph $K_n (n \geq 3)$ does not admit a linear edge geodetic set. For the double star, the set of all end vertices is the unique linear minimum edge geodetic set.

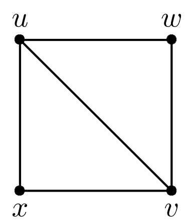

Figure 2.1. G

Theorem 2.10. *For the complete bipartite graph $K_{r,s}(2 \leq r \leq s)$ with bipartition $(X, Y), |X| = r$ and $|Y| = s$, X is a linear minimum edge geodetic set.*

Proof. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$. Let x_iy_j be any edge of $K_{r,s}$. For $1 \leq i < r$, x_iy_j lies on the geodesic x_i, y_j, x_{i+1}. For $i = r$, x_iy_j lies on the geodesic x_{r-1}, y_j, x_r. It follows that X is a linear edge geodetic set. Now, let T be any set of vertices of $K_{r,s}$ such that $|T| < |X|$. Then there exist vertices $x_i \in X$ and $y_j \in Y$ such that $x_i, y_j \notin T$. Since $diam(K_{r,s}) = 2$, it follows that the edge x_iy_j cannot lie on any geodesic joining a pair of vertices in T. Thus T is not an edge geodetic set. Hence X is a linear minimum edge geodetic set of $K_{r,s}$. ■

For any real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Theorem 2.11. *Let G and H be connected graphs with $g_1(G) = p$ and $g_1(H) = q$. Suppose that both G and H contain linear minimum edge geodetic sets. Then $g_1(G \Box H) \leq \left\lfloor \frac{pq}{2} \right\rfloor$.*

Proof. Let $S = \{g_1, g_2, \ldots, g_p\}$ and $T = \{h_1, h_2, \ldots, h_q\}$ be linear minimum edge geodetic sets of G and H respectively. Let $U = S \times T - \bigcup_{i+j\ even}\{(g_i, h_j)\}$. Then $|U| = \left\lfloor \frac{pq}{2} \right\rfloor$. We claim that U is an edge geodetic set of $G \Box H$. Let $e = (x, y)(x', y')$ be an arbitrary edge of $G \Box H$. We assume that $e_1 = xx' \in E(G)$. Then $y = y'$. Since S is a linear edge geodetic set of G, there exists an index $i, 1 \leq i < p$ such that the edge $e_1 = xx'$ lies on some $g_i - g_{i+1}$ geodesic P of G. Since T is a linear edge geodetic set of H, it follows that there exists an index $j, 1 \leq j < q$ such that the vertex y lies on some $h_j - h_{j+1}$ geodesic Q of H. Now we consider two cases.

Case 1. Suppose that $i + j$ is odd. Then$(i + 1) + (j + 1)$ is odd and so $(g_i, h_j) \in U$ and $(g_{i+1}, h_{j+1}) \in U$. By Lemma 2.4, the edge $e = (x, y)(x', y')$ lies on some $(g_i, h_j) - (g_{i+1}, h_{j+1})$ geodesic of $G \Box H$.

Case 2. Suppose that $i + j$ is even. Then $i + (j + 1)$ and $(i + 1) + j$ are odd and so $(g_i, h_{j+1}) \in U$ and $(g_{i+1}, h_j) \in U$. Now, since the vertex y lies on the $h_j - h_{j+1}$ geodesic Q of H, y also lies on the $h_{j+1} - h_j$ geodesic Q^{-1} of H. Hence, by Lemma 2.4, the edge $e = (x, y)(x', y')$ lies on some $(g_i, h_{j+1}) - (g_{i+1}, h_j)$ geodesic of $G \Box H$. Thus in both cases, U is an edge geodetic set of $G \Box H$ and so $g_1(G \Box H) \leq |U| = \left\lfloor \frac{pq}{2} \right\rfloor$. ∎

Corollary 2.12. *Let G and H be connected graphs such that G contains a linear minimum edge geodetic set and $g_1(H) = 2$, then $g_1(G \Box H) = g_1(G)$.*

Proof. Let $g_1(G) = p$. Since $g_1(H) = 2$, if follows that every minimum edge geodetic set of H is linear and so by Theorem 2.11, $g_1(G \Box H) \leq \left\lfloor \frac{2p}{2} \right\rfloor = p = g_1(G)$. Also, by Theorem 2.3, $g_1(G) \leq g_1(G \Box H)$. Hence $g_1(G \Box H) = g_1(G)$. ∎

Corollary 2.13. *For the complete bipartite graph $K_{r,s}$ $(2 \leq r \leq s)$, $g_1(K_{r,s} \Box K_{r,s}) \leq \left\lfloor \frac{r^2}{2} \right\rfloor$.*

Proof. This follows from Theorems 2.10 and 2.11. ∎

3. EXACT EDGE GEODETIC NUMBERS

In this section we determine the exact values of the edge geodetic numbers of the Cartesian product for several classes of graphs. We also give several classes of graphs G and H with $g_1(G \Box H) = g_1(G)$. It is to be noted that the graphs given in Corollary 2.12 belong to this class. Further, we determine a necessary condition on G for which $g_1(G\Box K_2) = g_1(G)$.

Observation 3.1. *Let G be a connected graph of diameter 2. Then any edge in G has at least one end in every edge geodetic set of G.*

Theorem 3.2. *For integers $m \geq n \geq 2$, $g_1(K_m\Box K_n) = mn - n$.*

Proof. It follows from Theorems 1.2 and 2.6 that $g_1(K_m\Box K_n) \leq mn - n$. Now, we prove that $g_1(K_m\Box K_n) \geq mn - n$. Let $V(K_m) = \{x_1, x_2, \ldots, x_m\}$ and $V(K_n) = \{y_1, y_2, \ldots, y_n\}$. Let G_i be the copy of K_m corresponding to $y_i (1 \leq i \leq n)$ on $K_m\Box K_n$. Let S be an edge geodetic set of $K_m\Box K_n$ of minimum cardinality. Then $g_1(K_m\Box K_n) = |S|$. We claim that $|S\cap V(G_i)| \geq m - 1$ for all $i = 1, 2, \ldots, n$. Suppose that $|S \cap V(G_i)| < m - 1$ for some i $(1 \leq i \leq n)$. Then we can choose vertices $u = (x_j, y_i)$ and $v = (x_k, y_i)$ in $V(G_i)$ with $1 \leq j \neq k \leq m$ such that $u, v \notin S$. Since $G_i \cong K_m$, it follows that uv is an edge of $K_m\Box K_n$ and since diameter of $K_m\Box K_n$ is 2, by Observation 3.1, uv has at least one end in S, which is a contradiction to our choice. Thus $|S \cap V(G_i)| \geq m - 1$ for all $i = 1, 2, \ldots, n$. It follows that $|S| \geq n(m - 1) = mn - n$. Thus, $g_1(K_m\Box K_n) = mn - n$. ■

Observation 3.3. *Let T be a nontrivial tree with k end vertices and $n \geq 2$ be any integer. Let G_v be the copy of K_n on $T\Box K_n$ corresponding to an end vertex v of T. Then, every edge e of G_v is either an initial edge or terminal edge of any geodesic containing e.*

Theorem 3.4. *Let T be a nontrivial tree with k end vertices and $n \geq 2$ be any integer. Then*

(i) *$g_1(T\Box K_n) = kn - k$ for $k \leq n$ and*

(ii) *$kn - k \leq g_1(T\Box K_n) \leq kn - n$ for $k > n$.*

Proof. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$. First we prove that $g_1(T\Box K_n) \geq kn - k$. Let S be a minimum edge geodetic set of $T\Box K_n$. Let G_v be the copy of K_n corresponding to an end vertex v of T. Now, we claim that $|S \cap V(G_v)| \geq n - 1$. If $|S \cap V(G_v)| < n - 1$, then there exist at least two vertices say $(v, x_1), (v, x_2)$ not in S. Since $G_v \cong K_n$, it follows that $(v, x_1)(v, x_2)$ is an edge of G_v. Since S is an edge geodetic set of $T\Box K_n$, it follows from Observation 3.3 that $(v, x_1) \in S$ or $(v, x_2) \in S$, which is a contradiction. Thus $|S \cap V(G_v)| \geq n - 1$. Since T has k end vertices, it follows that $|S| \geq k(n-1) = kn - k$. Now, by Theorem 1.3, $g_1(T) = k$ and by Theorem 1.2, $g_1(K_n) = n$ and it follows from Theorem 2.6 that $g_1(T \square K_n) \leq kn - \min\{k, n\}$. Now the result follows. ■

Corollary 3.5. *For integers $m, n \geq 2$, $g_1(P_m \square K_n) = 2n - 2$.*

Proof. This follows from Theorems 1.3 and 3.4. ■

Let S and T be disjoint nonempty subsets of $V(G)$. Often, we use the terminology that a vertex v (or an edge e) of G lies on an S geodesic of G if v (edge e) lies on a $x-y$ geodesic of G with $x, y \in S$ and that v (edge e) lies on an $S-T$ *geodesic* of G if v (edge e) lies on a $x-y$ geodesic of G with $x \in S$ and $y \in T$.

Theorem 3.6. *Let G be a connected graph. If G has a minimum edge geodetic set S, which can be partitioned into pairwise disjoint non-empty subsets $S_1, S_2, \ldots, S_n (n \geq 2)$ such that every edge of G lies on an $S_i - S_j$ geodesic for every i, j with $i \neq j$, then $g_1(G \square H) = g_1(G)$ for every connected graph H with $g_1(H) = n$.*

Proof. Let $T = \{h_1, h_2, \ldots, h_n\}$ be a minimum edge geodetic set of H. Let $W_i = \{(s_i, h_i) : s_i \in S_i\}$ for $1 \leq i \leq n$. Then $|W_i| = |S_i|$ for $i = 1, 2, \ldots, n$. Let $W = \bigcup_{i=1}^{n} W_i$. Then $|W| = \sum_{i=1}^{n} |W_i| = \sum_{i=1}^{n} |S_i| = |S| = g_1(G)$. We claim that W is an edge geodetic set of $G \square H$. Let $e = (x_1, y_1)(x_2, y_2)$ be any edge of $G \square H$.

Case 1. Suppose that $x_1 x_2 \in E(G)$. Then $y_1 = y_2$. Since T is an edge geodetic set of H, it follows that y_1 lies on some $h_k - h_l$ geodesic of H with $1 \leq k \neq l \leq n$. By hypothesis, $x_1 x_2$ lies on some $s_k - s_l$ geodesic of G with $s_k \in S_k$ and $s_l \in S_l$. Hence by Lemma 2.4, the edge $e = (x_1, y_1)(x_2, y_2)$ lies on some $(s_k, h_k) - (s_l, h_l)$ geodesic of $G \square H$, where $(s_k, h_k), (s_l, h_l) \in W$. Thus, W is an edge geodetic set of $G \square H$.

Case 2. Suppose that $y_1 y_2 \in E(H)$. Then $x_1 = x_2$. Since T is an edge geodetic set of H, the edge $y_1 y_2$ lies on some $h_k - h_l$ geodesic of H with $1 \leq k \neq l \leq n$. Now, it follows from the hypothesis that the vertex x_1 lies on some $s_k - s_l$ geodesic of G with $s_k \in S_k$ and $s_l \in S_l$. Hence by Lemma 2.4, the edge $e = (x_1, y_1)(x_2, y_2)$ lies on a $(s_k, h_k) - (s_l, h_l)$ geodesic of $G \square H$, where (s_k, h_k),$(s_l, h_l) \in W$. Thus W is an edge geodetic set of $G \square H$. Therefore, $g_1(G \square H) \leq |W| = g_1(G)$. Now the result follows from Theorem 2.3. ■

If a connected graph G has a minimum edge geodetic set S with a vertex x in S such that every edge of G lies on a $x - w$ geodesic of G for some $w \in S$, then it follows from Theorem 3.6 that $g_1(G \Box H) = g_1(G)$ for any connected graph H with $g_1(H) = 2$. Now, for the complete bipartite graph $K_{r,s}(2 \leq r \leq s)$, it follows from Theorem 2.10 that $X = \{v_1, v_2, \ldots, v_r\}$ is a minimum edge geodetic set. Letting $S_1 = \{v_1\}$ and $S_2 = \{v_2, v_3, \ldots, v_r\}$, we see that every edge of $K_{r,s}$ lies on a $S_1 - S_2$ geodesic and hence it follows from Theorems 1.3 and 3.6 that $g_1(K_{r,s} \Box P) = r$ for any path P.

Definition 3.7. An edge geodetic set S of a graph G is called a perfect edge geodetic set if for every edge e of G, there exists a vertex $x \in S$ such that the edge e lies on a $x - w$ geodesic of G for every $w \in S$, where $w \neq x$.

If G is graph with $g_1(G) = 2$, then every minimum edge geodetic set is perfect. For the graph G given in Figure 3.1, $S = \{a, d, e\}$ is an edge geodetic set of G, which is perfect. For the complete graph $K_n(n \geq 3)$, the unique edge geodetic set $V(K_n)$ is not perfect. For the graph G given in the Figure 2.1, $S = \{u, v, w, x\}$ is the unique edge geodetic set, which is not perfect.

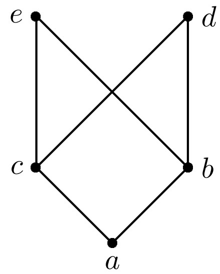

Figure 3.1. G

Theorem 3.8. *For connected graphs G and H, each having a perfect minimum edge geodetic set,* $g_1(G \Box H) = \max\{g_1(G), g_1(H)\}$.

Proof. Let $S = \{g_1, g_2, \ldots, g_p\}$ and $T = \{h_1, h_2, \ldots, h_q\}$ be perfect minimum edge geodetic sets of G and H respectively. Then $g_1(G) = p$ and $g_1(H) = q$.

Assume without loss of generality that $p \geq q$. Let $U = \{(g_1, h_1), (g_2, h_2), \ldots, (g_q, h_q), (g_{q+1}, h_q), \ldots, (g_p, h_q)\}$. Then $|U| = p$. We claim that U is an edge geodetic set of $G \Box H$. Let $e = (x_1, y_1)(x_2, y_2)$ be an edge of $G \Box H$.

We assume that $e_1 = x_1x_2$ is an edge in G. Then $y_1 = y_2 = y$ (say). Since S is a perfect edge geodetic set of G, there exists $i_0 \in \{1, 2, \ldots, p\}$ such that the edge e_1 lies on some $g_{i_0} - g_s$ geodesic of G for all $s \neq i_0$, $s \in \{1, 2, \ldots, p\}$. Also since T is a perfect edge geodetic set of H, it follows that there exists $j_0 \in \{1, 2, \ldots, q\}$ such that the vertex y lies on a $h_{j_0} - h_t$ geodesic of H for all $t \neq j_0$, $t \in \{1, 2, \ldots, q\}$. Hence by Lemma 2.4, the edge e lies on some $(g_{i_0}, h_{j_0}) - (g_s, h_t)$ geodesic of $G\Box H$ for all $s \neq i_0$ and $t \neq j_0$.

Case 1. Suppose that $i_0 = j_0$. Now, choose $k \in \{1, 2, 3, \ldots, q\}$ different from i_0. Then $k \neq j_0$ and both (g_{i_0}, h_{j_0}) and (g_k, h_k) belong to U. Thus e lies on a $(g_{i_0}, h_{j_0}) - (g_k, h_k)$ geodesic joining a pair of vertices of U and so U is an edge geodetic set of $G\Box H$.

Case 2. Suppose that $i_0 \neq j_0$. We consider two subcases.

Subcase 2.1. Suppose that $1 \leq i_0 \leq q$. Then $1 \leq i_0$, $j_0 \leq q$ and $i_0 \neq j_0$. Since $i_0 \neq j_0$, the edge e_1 lies on a $g_{i_0} - g_{j_0}$ geodesic P of G. Also since $1 \leq i_0 \leq q$, $i_0 \neq j_0$, the vertex y lies on a $h_{j_0} - h_{i_0}$ geodesic Q of H. Thus, y lies on the $h_{i_0} - h_{j_0}$ geodesic Q^{-1} of H. By Lemma 2.4, the edge $e = (x_1, y_1)(x_2, y_2)$ lies on some $(g_{i_0}, h_{i_0}) - (g_{j_0}, h_{j_0})$ geodesic of $G\Box H$. Since $(g_{i_0}, h_{i_0}), (g_{j_0}, h_{j_0}) \in U$, it follows that U is an edge geodetic set of $G\Box H$.

Subcase 2.2. Suppose that $q + 1 \leq i_0 \leq p$. Then $1 \leq j_0 \leq q < q + 1 \leq i_0 \leq p$. Suppose that $j_0 \neq q$. Then the vertex y lies on a $h_{j_0} - h_q$ geodesic Q of H. Thus y lies on the $h_q - h_{j_0}$ geodesic Q^{-1} of H. Also since $i_0 \neq j_0$, the edge e_1 lies on a $g_{i_0} - g_{j_0}$ geodesic P of G. Thus, by Lemma 2.4, the edge $e = (x_1, y_1)(x_2, y_2)$ lies on some $(g_{i_0}, h_q) - (g_{j_0}, h_{j_0})$ geodesic of $G\Box H$. Since $(g_{i_0}, h_q), (g_{j_o}, h_{j_o}) \in U$, it follows that U is an edge geodetic set of $G\Box H$. Suppose that $j_0 = q$. Since $q \geq 2$, the vertex y lies on a $h_q - h_1$ geodesic Q of H. Also since $i_0 \geq q + 1 \geq 3$, the edge e_1 lies on a $g_{i_0} - g_1$ geodesic P of G. By Lemma 2.4, the edge $e = (x_1, y_1)(x_2, y_2)$ lies on some $(g_{i_0}, h_q) - (g_1, h_1)$ geodesic of $G\Box H$. Since $(g_{i_0}, h_q), (g_1, h_1) \in U$, it follows that U is an edge geodetic set of $G\Box H$. Hence $g_1(G\Box H) \leq |U| = p = g_1(G)$ and it follows from Theorem 2.3 that $g_1(G\Box H) = g_1(G) = \max\{g_1(G), g_1(H)\}$. ■

Corollary 3.9. *For integers* $r, s \geq 1$, $g_1(K_{1,r}\Box K_{1,s}) = \max\{r, s\}$.

Proof. The respective end vertices of $K_{1,r}$ and $K_{1,s}$ are the unique minimum perfect edge geodetic sets of $K_{1,r}$ and $K_{1,s}$. Now the result follows from Theorem 3.8. ■

In view of Theorem 2.3, a natural question that arises is of when $g_1(G\Box G) = g_1(G)$. In the following we introduce a special class of graphs G and prove that $g_1(G\Box G) = g_1(G)$.

Definition 3.10. For a connected graph G, a set $S \subseteq V(G)$ is called a (edge, vertex)- geodetic set if for every pair of an edge e and a vertex v of G, there exist x and y in S such that e and v lie on geodesics between x and y.

Note 3.11. The edge e and the vertex v in Definition 3.10 need not lie on a single $x - y$ geodesic. If S is an (edge, vertex)-geodetic set, it follows that S is an edge geodetic set and hence a geodetic set too.

If G is a graph with $g_1(G) = 2$, then any minimum edge geodetic set is a (edge, vertex)-geodetic set of G. The set of all end vertices of a tree T is a (edge, vertex)-geodetic set of T. For $n \geq 3$, the complete graph K_n has no (edge, vertex)-geodetic set. Given an integer $k \geq 2$, there exists a graph G with an (edge, vertex)-geodetic set of cardinality k. (The star $G = K_{1,k}$ works).

Theorem 3.12. *If a connected graph G has an (edge, vertex)-geodetic set S of cardinality $g_1(G)$, then $g_1(G\Box G) = g_1(G)$.*

Proof. Let $S = \{g_1, g_2, \ldots, g_p\}$ be an (edge,vertex)-geodetic set of G of cardinality $g_1(G) = p$. Let $T = \{(g_1, g_1), (g_2, g_2), \ldots, (g_p, g_p)\}$. We claim that T is an edge geodetic set of $G\Box G$. Let $e = (x_1, y_1)(x_2, y_2)$ be an edge in $G\Box G$. Assume that $e_1 = x_1x_2 \in E(G)$. Then $y_1 = y_2 = y$ (say). Since S is an (edge, vertex)-geodetic set, there exist g_i and g_j in S such that the edge e_1 lies on some $g_i - g_j$ geodesic P of G and the vertex y lies on some $g_i - g_j$ geodesic Q of G. Therefore, by Lemma 2.4, the edge $e = (x_1, y_1)(x_2, y_2)$ lies on some $(g_i, g_i) - (g_j, g_j)$ geodesic of $G\Box G$. Thus, T is an edge geodetic set of $G\Box G$ and so $g_1(G\Box G) \leq |T| = g_1(G)$. Also by Theorem 2.3, $g_1(G) \leq g_1(G\Box G)$ and so $g_1(G\Box G) = g_1(G)$. ∎

Corollary 3.13. *For any tree T, $g_1(T\Box T) = g_1(T)$.*

Proof. Since the set of all end vertices of T is an (edge, vertex)-geodetic, the result follows from Theorems 1.3 and 3.12. ∎

Trees are yet another class of graphs that achieve the lower bound of Theorem 2.3. For the proof, we use Theorem 1.3 and the simple properties given in the following Lemma.

Lemma 3.14. *Let T be a tree and L be the set of all end vertices of T. Then the following properties hold:*

(P1) *If $x \in L$ and $v \in V(T)$, then there exists $y \in L$ with $y \neq x$ such that v lies on the $x - y$ geodesic of T.*

(P2) *If $x, y \in L$ and if the edge e lies on the $x - y$ geodesic of T, then for any $z \in L$ either e lies on the $x - z$ geodesic of T or e lies on the $y - z$ geodesic of T.*

(P3) *If $x, y \in L$ and if the vertex v lies on the $x - y$ geodesic of T, then for any $z \in L$ either v lies on the $x - z$ geodesic of T or v lies on the $y - z$ geodesic of T.*

Proof. (P1) is obvious. Since $T - e$ is disconnected and since e lies on the $x - y$ geodesic of T, x and y lie on different components of $T - e$. Let C be the component of $T - e$ that contains z. Then it is clear that not both x and y are in C. If $x \notin C$, then e lies on the $x - z$ geodesic of T, and otherwise, e lies on the $y - z$ geodesic of T. Thus (P2) is proved and (P3) follows from (P2). ■

Theorem 3.15. *For any trees T_1 and T_2, $g_1(T_1 \Box T_2) = \max\{g_1(T_1), g_1(T_2)\}$.*

Proof. Let L_1 and L_2 be the set of all end vertices of T_1 and T_2 respectively. Then by Theorem 1.3, $g_1(T_1) = |L_1|$ and $g_1(T_2) = |L_2|$. Let $p = g_1(T_1) \geq g_1(T_2)$ and let $f : L_1 \to L_2$ be an arbitrary onto mapping. Let $L_1 = \{x_1, x_2, \ldots, x_p\}$. We claim that $S = \{(x_i, f(x_i)) : i = 1, 2, \ldots, p\}$ is an edge geodetic set of $T_1 \Box T_2$. Let $e = (g_1, h_1)(g_2, h_2)$ be any edge in $T_1 \Box T_2$. Assume that $e_1 = g_1 g_2 \in E(T_1)$. Then $h_1 = h_2 = h$ (say). Since L_1 is an edge geodetic set of T_1, e_1 lies on the $x_i - x_j$ geodesic of T_1 for some $x_i, x_j \in L_1$. By Theorem 1.4, L_2 is a geodetic set T_2. Now, if h lies on the $f(x_i) - f(x_j)$ geodesic of T_2, then by Lemma 2.4, the edge e lies on a $(x_i, f(x_i)) - (x_j, f(x_j))$ geodesic of $T_1 \Box T_2$. Hence S is an edge geodetic set of $T_1 \Box T_2$. If h does not lie on the $f(x_i) - f(x_j)$ geodesic of T_2, then, by (P1) of Lemma 3.14, there exists $y \in L_2$ (different from $f(x_i)$) such that h lies on the $f(x_i) - y$ geodesic of T_2. Also, since h does not lie on the $f(x_i) - f(x_j)$ geodesic of T_2, by (P3) of Lemma 3.14, h lies on the $f(x_j) - y$ geodesic of T_2. Take any $x_k \in f^{-1}(y)(1 \leq k \leq p)$. Since the edge e_1 lies on the $x_i - x_j$ geodesic of T_1, by (P2) of Lemma 3.14, either e_1 lies on the $x_i - x_k$ geodesic of T_1 or e_1 lies on the $x_j - x_k$ geodesic of T_1. Since $y = f(x_k)$, it follows now from Lemma 2.4 that the edge e lies on either $(x_i, f(x_i)) - (x_k, f(x_k))$ geodesic of $T_1 \Box T_2$ or $(x_j, f(x_j)) - (x_k, f(x_k))$ geodesic of $T_1 \Box T_2$. Since $(x_i, f(x_i)), (x_j, f(x_j)), (x_k, f(x_k)) \in S$, it follows that S is an edge geodetic set of $T_1 \Box T_2$ so that $g_1(T_1 \Box T_2) \leq p = g_1(T_1)$. Now, it follows from Theorem 2.3 that $g_1(T_1 \Box T_2) = \max\{g_1(T_1), g_1(T_2)\}$. ■

Now, we proceed to investigate graphs G for which $g_1(G \Box K_2) = g_1(G)$. For this we introduce a class of graphs called superior edge geodetic graphs.

Definition 3.16. Let G be a connected graph. An edge geodetic set $S \subseteq V(G)$ is said to be a superior edge geodetic set of G if S can be partitioned into two disjoint non-empty subsets S_1 and S_2 such that every edge of G either lies on a $S_1 - S_2$ geodesic or it lies on both an S_1 geodesic and an S_2 geodesic of G. A graph G is called a superior edge geodetic graph if it has a superior minimum edge geodetic set.

Graphs G with $g_1(G) = 2$ and nontrivial trees are obvious instances of such graphs. Now, for the graph G in Figure 3.2, $S = \{v_1, v_5, v_6, v_{10}\}$ is a minimum edge geodetic set. For the partition $S_1 = \{v_1, v_{10}\}$ and $S_2 = \{v_5, v_6\}$ of S, all the edges except v_3v_8 lie on $S_1 - S_2$ geodesic and the edge v_3v_8 lies on both an S_1-geodesic and an S_2-geodesic so that S is a superior minimum edge geodetic set. Hence the graph in Figure 3.2 is a superior edge geodetic graph.

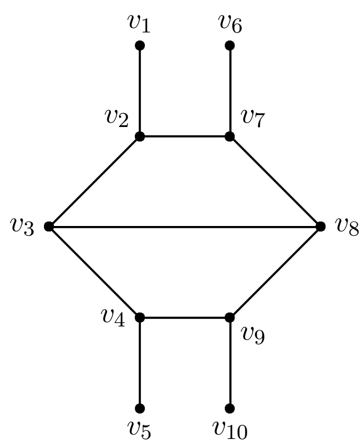

Figure 3.2. G

Proposition 3.17. *The complete graph $K_n(n \geq 3)$ is not a superior edge geodetic graph.*

Proof. By Theorem 1.2, the set S of all vertices of K_n is the unique minimum edge geodetic set of K_n. Let $S = S_1 \cup S_2$ be any partition of S. Since $n \geq 3$, it is clear that $|S_1| \geq 2$ or $|S_2| \geq 2$. Assume without loss of generality that $|S_1| \geq 2$. Now, the induced subgraph $\langle S_1 \rangle$ is complete and has at least one edge e. This edge e does not lie on any $S_1 - S_2$ geodesic of K_n. Also e does not lie on any S_2 geodetic of K_n. Hence S is not a superior edge geodetic set of K_n and so $K_n(n \geq 3)$ is not a superior edge geodetic graph. ■

We make use of the following simple observation to prove Theorem 3.19.

Observation 3.18. *Let G be any connected graph and let $V(K_2) = \{v_1, v_2\}$. Let G_1 and G_2 be the copies of G in $G\square K_2$ corresponding to the vertices v_1 and v_2 of K_2 respectively. Then every $u - v$ geodesic in $G\square K_2$, where both $u, v \in V(G_i)$ for either $i = 1$ or $i = 2$ lies completely in G_i.*

Theorem 3.19. *If G is a connected graph such that $g_1(G\square K_2) = g_1(G)$, then G is a superior edge geodetic graph.*

Proof. Let $V(K_2) = \{v_1, v_2\}$. Let G_1 and G_2 be the copies of G in $G\square K_2$ corresponding to the vertices v_1 and v_2 of K_2 respectively. Let T be a minimum edge geodetic set of $G\square K_2$. Then $|T| = g_1(G\square K_2)$. Let $T_i = T \cap V(G_i)$ for $i = 1, 2$. Since T is an edge geodetic set of $G\square K_2$, it follows from Observation 3.18 that $T_i \neq \emptyset$ for $i = 1, 2$. Also, $T = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$. Let $S = \pi_G(T)$ and $S_i = \pi_G(T_i)$ for $i = 1, 2$. Since $T_i \neq \emptyset$, it is clear that $S_i \neq \emptyset$ for $i = 1, 2$. We prove that S is a superior minimum edge geodetic set of $G\square K_2$. Now, $x \in S$ iff $(x, v_i) \in T$ for some $i \in \{1, 2\}$ iff $(x, v_i) \in T_1 \cup T_2$ for some $i \in \{1, 2\}$ iff $x \in S_1 \cup S_2$. Thus $S = S_1 \cup S_2$. We now show that $S_1 \cap S_2 = \emptyset$. Since $S = \pi_G(T)$, we have $|S| \leq |T|$...(1) and by Lemma 2.1, S is an edge geodetic set of G so that $g_1(G) \leq |S|$...(2). By hypothesis, $g_1(G\square K_2) = g_1(G)$...(3). Hence it follows from (1), (2) and (3) that $|S| \leq |T| = g_1(G\square K_2) = g_1(G) \leq |S|$. Thus $|S| = |T| = g_1(G)$ and this shows that S is a minimum edge geodetic set of G. Since $S_i = \pi_G(T_i)$ for $i = 1, 2$ and $T_1 \cap T_2 = \emptyset$, it follows that $|S| = |T| = |T_1| + |T_2| \geq |S_1| + |S_2|$. Also, since $S = S_1 \cup S_2$, we have $|S| \leq |S_1| + |S_2|$ and so $|S| = |S_1| + |S_2|$. Hence $S_1 \cap S_2 = \emptyset$. Therefore, S_1 and S_2 form a partition of S. Now, we prove that the minimum edge geodetic set S is superior. Let $e = uv \in E(G)$ be arbitrary. If e does not lie on any $S_1 - S_2$ geodesic of G, then, since S is an edge geodetic set of G, we may assume without loss of generality that e lies on a S_2 geodesic of G. Now, $e_1 = (u, v_1)(v, v_1)$ is an edge of G_1 and so the edge e_1 lies on some $g' - h'$ geodesic of $G \Box K_2$, where $g', h' \in T$. Hence it follows from Observation 3.18 that either $g' \in T_1$ or $h' \in T_1$. We claim that both g' and h' belong to T_1. Suppose that $g' \in T_1$ and $h' \in T_2$. Then $g' = (g, v_1)$ and $h' = (h, v_2)$ for some $g, h \in V(G)$. Hence it follows that $g \in S_1$ and $h \in S_2$. Since the edge e_1 lies on the $(g, v_1) - (h, v_2)$ geodesic of $G \Box K_2$, by Lemma 2.4, the edge $e = uv$ lies on some $g - h$ geodesic of G, where $g \in S_1$ and $h \in S_2$, which is a contradiction to the assumption that e does not lie on any $S_1 - S_2$ geodesic of G. Thus, both g' and h' belongs to T_1. Hence $g' = (g, v_1)$ and $h' = (h, v_1)$ for some $g, h \in S_1$. Thus $e_1 = (u, v_1)(v, v_1)$ lies on a $(g, v_1) - (h, v_1)$ geodesic of $G \Box K_2$, where $g, h \in S_1$. Hence by Lemma 2.4, the edge $e = uv$ lies on some $g - h$ geodesic of G, where $g, h \in S_1$. Thus e lies on a S_1 geodesic of G so that S is a superior minimum edge geodetic set of G. ■

We leave the following problem as an open question.

Problem 3.20. Charactrize graphs G for which $g_1(G \Box K_2) = g_1(G)$.

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Received 28 May 2008
Revised 11 December 2008
Accepted 21 January 2009