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γ-LABELINGS OF COMPLETE BIPARTITE GRAPHS

Grady D. Bullington, Linda L. Eroh, Steven J. Winters

Department of Mathematics
University of Wisconsin Oshkosh
Oshkosh, Wisconsin 54901, USA

Abstract

Explicit formulae for the γ-min and γ-max labeling values of complete bipartite graphs are given, along with γ-labelings which achieve these extremes. A recursive formula for the γ-min labeling value of any complete multipartite is also presented.

Keywords: γ-labelings, bipartite graphs, multipartite graphs.

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1. Introduction

Throughout the paper, G denotes a simple, connected graph with order n and size m. We use $K_{n_1,n_2,\ldots,n_k}$ to denote a complete k-partite graph and $K_{r,s}$ in the case $k = 2$.

Definition 1 [1]**.** A γ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2 \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G.

The *value* of a γ-labeling f on G is $val(f) = \sum_{e\in E(G)} f'(e)$. The *spectrum* of G is $spec(G) = \{val(f) : f$ is a γ-labeling of $G\}$. The γ-min *and* γ-max *labeling values* of G refer to

$$val_{min}(G) = \min\{val(f) : \ f \text{ is a } \gamma\text{-labeling on } G\} \text{ and}$$

$$val_{max}(G) = \max\{val(f) : \ f \text{ is a } \gamma\text{-labeling on } G\}.$$

The γ-labelings that achieve these extrema are called γ-min *and* γ-max *labelings.*

The above definition and notation first appeared in [1] where the authors found closed formulae for γ-min and γ-max labeling values of stars, paths, cycles and complete graphs. They also derived general sharp lower bounds of $val_{min}(G)$ in terms of n and m. Focusing on trees, [2] gave formulae for γ-min and γ-max labeling values of double stars and describes all graphs G for which $val_{min}(G)$ equals n and $n + 1$. The spectrum for paths, cycles, and complete graphs was determined in [3]. An alternate definition of γ-labelings for oriented graphs (having signed edge labels) was presented in [4] where the authors determined which connected graphs have an orientation and γ-labeling with value zero.

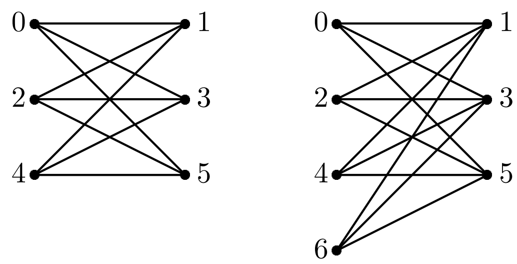
The main objective of this paper is to find explicit formulae for the γ-min and γ-max labeling values of a complete bipartite graph. The following will be shown.

Main Result 1. For any two positive integers $r \geq s$,

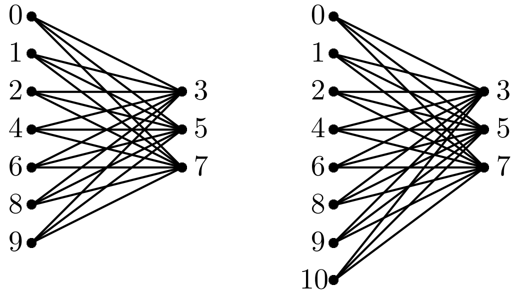
$$val_{min}(K_{r,s}) = \frac{s(2s^2+1)}{3} + s^2(r-s) + s\left\lfloor \frac{(r-s)^2}{4} \right\rfloor \text{ and}$$

$$val_{max}(K_{r,s}) = rs\left(rs - \frac{1}{2}(r+s) + 1\right).$$

Example 1. To motivate the above result, consider the γ-min labeling value of $K_{3,3}$ and $K_{4,3}$. Intuitively, one may guess that γ-min labelings on these graphs would be as follows.



Intuition coincides with reality in these cases. Larger examples are slightly more difficult to guess. For instance, γ-min labelings for $K_{7,3}$ and $K_{8,3}$, resp., are:



In the labeling of $K_{7,3}$ above, notice that both the two highest and two lowest labels are assigned to vertices in the largest partite set. If these vertices and corresponding edges are disregarded, then the graph that remains has a labeling that is equivalent to a γ-min labeling on $K_{3,3}$. The analogous observation can be made for the labelings above for $K_{8,3}$ and $K_{4,3}$.

Generally, if $K_{r,s}$ where $r = s$ or $r = s + 1$, then the labeling given by assigning consecutive labels from 0 back and forth between vertices in "opposite" partite sets achieves $val_{min}(K_{r,s})$. When $r > s + 1$, the labels in $\{0, \ldots, \lfloor \frac{r-s}{2} \rfloor\} \cup \{r + s - \lfloor \frac{r-s}{2} \rfloor, \ldots, r + s - 1\}$ can be assigned to the partite set of order r, and labels in $\{\lfloor \frac{r-s}{2} \rfloor + 1, \ldots, r + s - \lfloor \frac{r-s}{2} \rfloor - 1\}$ can be assigned consecutively back and forth from the largest partite set to the smallest to achieve a γ-min labeling. Though these labelings are not a ramification of the proofs that follow, it is a straightforward exercise to show that their values agree with our formula for $val_{min}(K_{r,s})$.

To find a labeling that achieves $val_{max}(K_{r,s})$, give labels $0, \ldots, r - 1$ to the vertices in the partite set of order r and labels $rs - (s - 1), rs - (s - 2), \ldots, rs$ to vertices in the partite set of order s. We prove that this is an appropriate labeling in Section 3, leaving the given formula for $val_{max}(K_{r,s})$ as an elementary calculation.

2. γ-Min Labeling Value of $K_{n_1,n_2,\ldots,n_k}$

Given a γ-labeling $\mathcal{L}$ on G, we use italics and interval notation in the superscript to denote the set of vertex labels used in $\mathcal{L}$. For example, $\mathcal{A}^{(j,\infty)}$ (resp., $\mathcal{A}^{[0,j]}$), denotes the set of labels in $\mathcal{L}$ that are strictly greater than j (resp, in $[0, j]$). Without italics, corresponding capital letters denote underlying vertex subsets. So, $A^{[0,j)}$ denotes the set of vertices having labels in $\mathcal{A}^{[0,j)}$. The interval notation is used on superscripts of n to denote the cardinalities of these sets (e.g., $n^{[0,j)} = |\mathcal{A}^{[0,j)}|$). When $G = K_{n_1,\ldots,n_k}$, a subscript is used to denote the restriction of the above definitions to a particular partite set. For instance, $\mathcal{A}_i^{[0,j)}$ represents the set of labels from $[0,j)$ assigned to vertices in the i-th partite set—its cardinality is denoted $n_i^{[0,j)}$. In the case $[0,0) = \phi$, $\mathcal{A}_i^{[0,0)} = \phi$ and $n_i^{[0,0)} = 0$.

The following lemma will be used to find a recursive formula for the γ-min labeling value of a complete multipartite graph in Theorem 1.

Lemma 1. *Consider a γ-labeling on $G = K_{n_1,n_2,\ldots,n_k}$. Let A_α and A_β be two partite sets in G with orders n_α and n_β, respectively. Let j be a label on a vertex in A_β, and let $i = \min\{l : l$ is a label on vertex in A_α and $j < l\}$. If $n_\alpha^{[0,j)} - n_\beta^{[0,j)} \leq \frac{n_\alpha - n_\beta}{2}$, then the value of the γ-labeling is not increased (and may be decreased) if label j is swapped with label i.*

Proof. Let $K_{n_1,n_2,\ldots,n_k}$ have partite set A_r of order n_r, $1 \leq r \leq k$. Assume without loss of generality that $n_1 \geq n_2 \geq \cdots \geq n_k$.

Using i and j as defined in the assertion, let v_α and v_β be the vertex in A_α and A_β labeled i and j, respectively. Notice, $|A_\alpha^{[0,j)}| = n_\alpha^{[0,j)}$, $|A_\beta^{[0,j)}| = n_\beta^{[0,j)}$, $|A_\alpha^{(j,\infty)}| = |A_\alpha^{(i,\infty)}| = n_\alpha - n_\alpha^{[0,j)} - 1$, and $|A_\beta^{(j,\infty)}| = n_\beta - n_\beta^{[0,j)} - 1$.

Suppose the labels on vertices v_α and v_β are swapped. The only edge labels that are affected are edges from v_α to A_β and the edges from v_β to A_α. Of course, the label on edge $v_\alpha v_\beta$ will be the same.

If $w \in A_\alpha^{[0,j)}$, then the label on edge wv_β will increase by $i - j$. If $w \in A_\alpha^{(i,\infty)}$, then the label on wv_β will decrease by $i - j$. If $w \in A_\beta^{[0,j)}$, then the label on wv_α will decrease by $i - j$. If $w \in A_\beta^{(j,\infty)}$, it is more difficult to tell how the label on wv_α will change, since the label on w might be between j and i or greater than i. However, at most it could increase by $i - j$.

The change in the sum of the γ-labeling is at most $n_\alpha^{[0,j)}(i-j) - (n_\alpha - n_\alpha^{[0,j)} - 1)(i-j) - n_\beta^{[0,j)}(i-j) + (n_\beta - n_\beta^{[0,j)} - 1)(i-j) = (i-j)(2n_\alpha^{[0,j)} - 2n_\beta^{[0,j)} + n_\beta - n_\alpha)$, which will be zero or negative provided $n_\alpha^{[0,j)} - n_\beta^{[0,j)} \leq \frac{n_\alpha - n_\beta}{2}$. ■

The next theorem gives a recursive formula for $val_{min}(K_{n_1,n_2,\ldots,n_k})$ provided at least one partite set contains two or more vertices. (If every partite set contains only one vertex, then $K_{n_1,n_2,\ldots,n_k}$ is a complete graph. It is shown in [1] that $val_{min}(K_n) = \binom{n+1}{3}$.)

Theorem 1. *For any positive integers $n_1, n_2, \ldots, n_k$, with $n_1 = \max(n_1, n_2, \ldots, n_k) \geq 2$,*

$$val_{min}\left(K_{n_1,n_2,\ldots,n_k}\right) = val_{min}(K_{n_1-2,n_2,n_3,\ldots,n_k}) + \left(\left(\sum_{i=1}^{k} n_i\right) - 1\right)\left(\sum_{i=2}^{k} n_i\right).$$

Proof. Suppose $K_{n_1,n_2,\ldots,n_k}$ is labeled with a γ-min labeling. By [1, Corollary 2.3], we may assume that the labels used are $0, 1, 2, 3, \ldots, (\sum_{i=1}^{k} n_i) - 1$. Let A_1 be the partite set of order n_1. We may further assume that label 0 is used in set A_1 by Lemma 1 since $0 = n_1^{[0,0)} - n_\beta^{[0,0)} \leq \frac{n_1 - n_\beta}{2}$ for any n_β.

Let j be the largest label used on set A_1. If $j \neq (\sum_{i=1}^{r} n_i) - 1$, then label $j+1$ appears on a vertex in some other partite set A_α. In the notation of Lemma 1, $n_\alpha^{[0,j)} \leq n_\alpha - 1$ and $n_1^{[0,j)} = n_1 - 1$. It follows that $2n_\alpha^{[0,j)} - n_\alpha + 2 \leq n_\alpha \leq n_1$. This is equivalent to $n_\alpha^{[0,j)} - (n_1 - 1) < \frac{n_\alpha - n_1}{2}$. By Lemma 1, we may interchange the labels j and $j+1$ without increasing the value of the labeling. This process may be repeated until label $(\sum_{i=1}^{k} n_i) - 1$ is in set A_1.

Thus, we may assume that the largest and smallest labels, 0 and $(\sum_{i=1}^{k} n_i) - 1$, are used in partite set A_1. Say label 0 is used on vertex v and label $(\sum_{i=1}^{k} n_i) - 1$ is used on vertex w. Every other label lies in between these two labels. Thus, for each vertex $x \notin A_1$, the sum of the labels on the edges xv and xw is $(\sum_{i=1}^{k} n_i) - 1$, regardless of the label on vertex x.

The minimum labeling on the remainder of the graph must be a γ-min labeling of the graph $K_{n_1-2,n_2,n_3,\ldots,n_k}$ using the labels $1, 2, 3, \ldots, (\sum_{i=1}^{k} n_i) - 2$. Since the value of the labeling is not changed if we add one to every label, or if we subtract one from every label, the value of the labeling on $K_{n_1,n_2,n_3,\ldots,n_k} - \{v, w\}$ is $val_{min}(K_{n_1-2,n_2,\ldots,n_k})$. As noted above, the edges incident with vertices v and w add $(\sum_{i=1}^{k} n_i) - 1$ to the value for each vertex $x \notin A_1$. The result follows. ■

Corollary (complete bipartite graph). For $r \geq s \geq 1$,

$$val_{min}(K_{r,s}) = \frac{s(2s^2+1)}{3} + s^2(r-s) + s\left\lfloor \frac{(r-s)^2}{4} \right\rfloor.$$

Proof. Formulae for stars and cycles are given in [1], handling the assertion when $s = 1$ (where $K_{r,s} = K_{r,1}$) and $r = s = 2$ (where $K_{2,2} = C_4$). Suppose $r \geq s \geq 2$ where $r \geq 3$, and assume that the desired formula holds for $K_{r-2,s}$.

If $r = s$, then

$$\begin{aligned} val_{min}(K_{r,s}) &= val_{min}(K_{s-2,s}) + (2s-1)s \\ &= \frac{2s^3+s}{3}. \end{aligned}$$

If $r = s+1$, then

$$\begin{aligned} val_{min}(K_{r,s}) &= val_{min}(K_{r-2,r-1}) + (r+r-1-1)(r-1) \\ &= \frac{(r-1)(2(r-1)^2+1)}{3} + (r-1)^2(1). \end{aligned}$$

If $r \geq s+2$, then by Theorem 1,

$$\begin{aligned} val_{min}(K_{r,s}) &= val_{min}(K_{r-2,s}) + (r+s-1)s \\ &= \frac{s(2s^2+1)}{3} + s^2(r-s) + s\left\lfloor \frac{(r-s)^2}{4} \right\rfloor. \end{aligned}$$

■

3. γ-Max Labeling Value of $K_{r,s}$

In this section, we prove the following theorem from which the closed formula for $val_{max}(K_{r,s})$ in the Main Result follows.

Theorem 2. *The γ-labeling of $K_{r,s}$ where the first partite set is labeled $\{0, \ldots, r-1\}$ and the second partite set is labeled $\{rs-(s-1), rs-(s-2), \ldots, rs\}$ is a γ-max labeling.*

The following lemma will be used to deduce a γ-labeling that achieves $val_{max}(K_{r,s})$.

Lemma 2. *Let $\mathcal{A}$ be the set of vertex labels in a γ-labeling $\mathcal{L}$ on $K_{\alpha,\beta}$ with partite sets A_α and A_β. Suppose $j \in \mathcal{A}$ is a label on vertex $v \in A_\alpha$.*

(1) *If $j-1 \in \{0, \ldots, \alpha\beta\} \backslash \mathcal{A}$ and vertex label j is replaced with $j-1$ on v, then the value of the γ-labeling is changed by $n_\beta^{(j,\infty)} - n_\beta^{[0,j)}$.*

(2) *If $j+1 \in \{0, \ldots, \alpha\beta\} \backslash \mathcal{A}$ and j is replaced with $j+1$ on v, then the value of the γ-labeling is changed by $n_\beta^{[0,j)} - n_\beta^{(j,\infty)}$.*

Proof of **(1).** Suppose that $\mathcal{L}$, j, $n_\beta^{[0,j)}$ and $n_\beta^{(j,\infty)}$ are as above. Let $\sum |j-x|$ denote the sum of terms in the computation of the value of $\mathcal{L}$ involving j and vertex labels on A_β. If j is changed to $j-1$, then

$$\sum |j-1-x| = \sum_{x<j}(j-x-1)+\sum_{x>j}(x-j+1) = \sum |j-x|+n_\beta^{(j,\infty)}-n_\beta^{[0,j)}.$$

So, the change in value is $n_\beta^{(j,\infty)}-n_\beta^{[0,j)}$. The assertion of (2) follows similarly. ∎

Throughout the rest of the section, let $\mathcal{L}$ be a γ-max labeling on $K_{r,s}$. Notice that $\mathcal{L}$ contains the vertex labels 0 and rs. (To see this, apply Lemma 2 on the extreme labels of $\mathcal{L}$.)

Standard form of a γ-max labeling on $K_{r,s}$. By Lemma 2, the labels of $\mathcal{L}$ can be arranged in the following *standard form.* The vertices in the partite set of $K_{r,s}$ of order r can be partitioned into subsets $R_1, R_2, \dots, R_\alpha$ so that the labels on vertices in R_i are consecutive. Using $\mathcal{R}_i$ to denote the vertex labels on R_i, all labels in $\mathcal{R}_i$ are taken to be less than those in $\mathcal{R}_j$ whenever $i<j$; as shorthand, we write $\mathcal{R}_i < \mathcal{R}_j$. In addition, the labels in $\mathcal{R}_i \cup \mathcal{R}_j$ are nonconsecutive when $i \neq j$; for brevity we say $\mathcal{R}_i$ and $\mathcal{R}_j$ are nonconsecutive. Defining $S_1, S_2, \dots, S_\beta$ and $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\beta$ analogously for the partite set of $K_{r,s}$ of order s, one can assume by Lemma 2 that the labeling $\mathcal{L}$ is of the form

$$\mathcal{R}_1 < \mathcal{S}_1 < \mathcal{R}_2 < \mathcal{S}_2 < \cdots \text{ or } \mathcal{S}_1 < \mathcal{R}_1 < \mathcal{S}_2 < \mathcal{R}_2 < \cdots$$

where 0 (resp., rs) is a label in the first (resp., last) subset of the chain, $|\alpha-\beta| \leq 1$, and the labels in each "intermediate" $\mathcal{R}_i$ (resp., $\mathcal{S}_i$) are consecutive with the labels in some $\mathcal{S}_j$ (resp., $\mathcal{R}_j$). Let r_i (resp., s_i) denote the cardinality of $\mathcal{R}_i$ (resp., $\mathcal{S}_i$).

For convenience, we let r be the cardinality of the partition set of $K_{r,s}$ that has vertex label 0 in $\mathcal{L}$. This forces $\mathcal{L}$ to be of the form $0 \in \mathcal{R}_1 < \mathcal{S}_1 < \mathcal{R}_2 < \cdots$; it is not necessary that $r \geq s$.

Notice that if $\mathcal{R}_1 < \mathcal{S}_1$ is the standard form of $\mathcal{L}$, then the labeling, by Lemma 2, is

$$\{0, 1, \dots, r-1\} < \{rs-(s-1), rs-(s-2), \dots, rs-1, rs\}$$

and Theorem 2 is shown. Therefore throughout the rest of the section, there may be an implicit assumption that $\mathcal{R}_2 \neq \emptyset$. If needed, we can also assume that $r, s \geq 2$ and r and s are not both two since the "$\min\{r, s\} = 1$" case follows from [1, Cor. 1.3] and it is easy to check that $\{0,1\} < \{3,4\}$ is a maximum γ-labeling on $K_{2,2}$.

Lemma 3. *If* γ-max *labeling* $\mathcal{L}$ *is in standard form* $0 \in \mathcal{R}_1 < \mathcal{S}_1 < \mathcal{R}_2 < \cdots$ *on* $K_{r,s}$ *with* $\mathcal{R}_2 \neq \emptyset$, *then* $\mathcal{S}_1$ *and* $\mathcal{R}_2$ *are not consecutive label sets.*

Proof. Let i be the least label in $\mathcal{S}_1$, and let j be the least label in $\mathcal{R}_2$. If i and j are switched, the γ-labeling value (of $\mathcal{L}$) changes by:

$$d := (j-i)(r_1 + (s - s_1) - (r - r_1 - 1) - (s_1 - 1)) + s_1(s_1 - 1).$$

Since $\mathcal{S}_1$ and $\mathcal{R}_2$ are consecutive, $j - i = s_1$ and

$$d = s_1(r_1 + 1 - (r - r_1) + (s - s_1)).$$

Therefore, since $s_1 \neq 0$ and $\mathcal{L}$ is a γ-max labeling,

$$r_1 + 1 \leq (r - r_1) - (s - s_1).$$

Letting i be the greatest label in $\mathcal{R}_1$ and j be the greatest label in $\mathcal{S}_1$, swapping i and j changes the γ-labeling value by: $(j-i)(-(r_1-1)-(s-s_1) +(r-r_1)) = s_1((r-r_1)-(s-s_1)-r_1+1) \geq 2s_1 > 0$, a contradiction. ■

Standard form revisited. If γ-max labeling $\mathcal{L}$ is in standard form $0 \in \mathcal{R}_1 < \mathcal{S}_1 < \mathcal{R}_2 < \cdots$ on $K_{r,s}$ with $r_1 := |\mathcal{R}_1| = r$ and $\mathcal{R}_2 \neq \emptyset$, then $r_1 < \frac{r}{2}$ and the label sets $\mathcal{R}_1$ and $\mathcal{S}_1$ are consecutive. [To see this, recall that $\mathcal{S}_1$ is either consecutive with $\mathcal{R}_1$ or $\mathcal{R}_2$. If $r_1 \geq \frac{r}{2}$, then by Lemma 2 we can consider $\mathcal{S}_1$ and $\mathcal{S}_2$ to be consecutive, but this cannot happen by Lemma 3.]

We can also deduce that the largest and smallest labels, 0 and rs, in $\mathcal{L}$ are not assigned to vertices in the same partite set. That is, the standard form of $\mathcal{L}$ on $K_{r,s}$ is not of the form

$$0 \in \mathcal{R}_1 < \mathcal{S}_1 < \mathcal{R}_2 < \cdots < \mathcal{R}_{\alpha-1} < \mathcal{S}_{\alpha-1} < \mathcal{R}_\alpha \text{ with } \alpha \geq 2 \text{ and } rs \in \mathcal{R}_\alpha.$$

Proof. Consider the complementary labeling $\mathcal{L}^*$ of $\mathcal{L}$ by replacing each label l with $rs - l$ (cf., [1]). On this labeling with the same value, we can deduce similarly that $r_\alpha < \frac{r}{2}$ and $\mathcal{S}_{\alpha-1}$ and $\mathcal{R}_\alpha$ are consecutive. Thus, $\alpha > 2$ since $r = r_1 + \cdots + r_\alpha$.

If $s_1 \leq \frac{s}{2}$, then Lemma 2 yields that $\mathcal{R}_2$ can be considered consecutive with $\mathcal{S}_1$ (resp., $\mathcal{S}_{\alpha-1}$), a contradiction of Lemma 3. If $s_1 > \frac{s}{2}$, then

Lemma 2 applied to the complementary labeling $\mathcal{L}^*$ yields that $\mathcal{R}_{\alpha-1}$ can be considered to be consecutive with $\mathcal{S}_{\alpha-1}$, a contradiction of Lemma 3. ■

The standard form of $\mathcal{L}$ achieving $val_{max}(K_{r,s})$ must assign labels 0 and rs to vertices in different partite sets. That is, the standard form of $\mathcal{L}$ must be

$$0 \in \mathcal{R}_1 < \mathcal{S}_1 < \mathcal{R}_2 < \cdots < \mathcal{R}_\beta < \mathcal{S}_\beta \text{ with } \beta \geq 1.$$

We are now ready to show that the above standard form must have $\beta = 1$.

***Proof of Theorem* 2.** Suppose that the standard form of $\mathcal{L}$ on $K_{r,s}$ is of the form

$$0 \in \mathcal{R}_1 < \mathcal{S}_1 < \mathcal{R}_2 < \cdots < \mathcal{R}_\beta < \mathcal{S}_\beta \text{ where } \beta \geq 2.$$

By Lemma 2 and the complementary labeling, $r_1 < \frac{r}{2}$, $s_k < \frac{s}{2}$, $\mathcal{R}_1$ and $\mathcal{S}_1$ are consecutive, and $\mathcal{R}_\beta$ and $\mathcal{S}_\beta$ are consecutive. There are two cases.

Case 1. $\beta = 2$. Note that vertex set S_1 is labeled with $\{r_1, \ldots, r_1+s_1-1\}$ and R_2 is labeled with $\{rs-(r_2+s_2-1), rs-(r_2+s_2-2), \ldots, rs-s_2\}$.

If $r_2 \geq s_1$, swap labels $\{r_1, \ldots, r_1+s_1-1\}$ with $\{rs-s_2-(s_1-1), \ldots, rs-s_2-1, rs-s_2\}$ and (on R_2) relabel $\{rs-(r_2+s_2-1), \ldots, rs-(s_2+s_1)\}$ with $\{r_1+s_1, \ldots, r_1+s_1+(r_2-s_1)\}$. Notice that this relabeling gives a positive change to the γ-labeling value since there is no change of labels on edges between R_2 and S_1 while there is positive change of labels on edges between R_1 and S_1 (resp., R_2 and S_2), giving the contradiction.

If $r_2 < s_1$, swap labels $\{r_1, r_1+1, \ldots, r_1+r_2-1\}$ on S_1 with $\{rs-(r_2+s_2-1), rs-(r_2+s_2-2), \ldots, rs-s_2\}$ and (on S_1) relabel $\{r_1+r_2, \ldots, r_1+s_1-1\}$ as $\{rs-(s_1+s_2+1), \ldots, rs-(r_2+s_2)\}$. As before, the γ-labeling value change between labels on S_1 and R_2 stay the same, but there is positive change of labels on edges between S_1 and R_1 (resp., R_2 and S_2), giving the contradiction.

Case 2. $\beta > 2$. If $s_1 \leq \frac{s}{2}$ (resp., $r_\beta \leq \frac{r}{2}$), then by Lemma 2 (resp., and the complementary labeling), we can consider $\mathcal{R}_1, \mathcal{S}_1$, and $\mathcal{R}_2$ (resp., $\mathcal{S}_{\beta-1}, \mathcal{R}_\beta, \mathcal{S}_\beta$) consecutive, contrary to Lemma 3 (resp., using the complementary labeling). Therefore, $s_1 > \frac{s}{2}$ and $r_\beta > \frac{r}{2}$. Let i be the highest label in $\mathcal{R}_2$ and let j be the lowest label in $\mathcal{S}_2$. The contradiction comes by swapping i and j, since the γ-labeling value changes by

$$
\begin{aligned}
&(j-i)(s_1-(s-s_1-1)+(r-(r_1+r_2))-(r_1+r_2-1))\\
&>(j-i)(\tfrac{s}{2}-(s-s_1)+1+\tfrac{r}{2}-(r_1+r_2)+1)\\
&\geq(j-i)(\tfrac{s}{2}-(s-s_1)+1+\tfrac{r}{2}-(r-r_\beta)+1)\\
&\geq(j-i)((s_1-\tfrac{s}{2})+(r_\beta-\tfrac{r}{2})+2)>0.
\end{aligned}
$$

It follows that the standard form of $\mathcal{L}$ is $0 \in \mathcal{R}_1 < \mathcal{S}_1$. ■

4. Open Question

For any graph G of order n and size m, [1, Section 2.3] gives that $val_{min}(G) = val(f)$ for some γ-labeling f with consecutive vertex labels $\{0, 1, 2, \ldots, n-1\}$.

Analogously, in wondering if more can be said about the vertex label set of a γ-max labeling, we notice that 0 and m are vertex labels in any γ-max labeling of G. The path P_2 and the cycle C_3 have γ-max labelings with vertex label sets $\{0, 1\}$ and $\{0, 1, 3\} = \{0, 1\} \cup \{3\}$ respectively; $K_{r,s}$ has a γ-max labeling with vertex label set $\{0, 1, \ldots, r-1\} \cup \{rs-(s-1), rs-(s-2), \ldots, rs\}$. All such examples lead us to ask:

For *any* connected graph, does there exist a γ-max labeling of G with a vertex label set that is the union of no more than two sets of consecutive numbers?

References

[1] G. Chartrand, D. Erwin, D.W. VanderJagt and P. Zhang, *γ-labelings of graphs*, Bull. Inst. Combin. Appl. **44** (2005) 51–68.

[2] G. Chartrand, D. Erwin, D. VanderJagt and P. Zhang, *γ-labelings of trees*, Discuss. Math. Graph Theory **25** (2005) 363–383.

[3] C.M. da Fonseca, V. Saenpholphat and P. Zhang, *The γ-spectrum of a graph*, Ars. Combin., to appear.

[4] V. Saenpholphat, F. Okamoto and P. Zhang, *On γ-labelings on oriented graphs*, Math. Bohem. **132** (2007) 185–203.

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