# FRACTIONAL GLOBAL DOMINATION IN GRAPHS 

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#### Abstract

Let $G=(V, E)$ be a graph. A function $g: V \rightarrow[0,1]$ is called a global dominating function $(G D F)$ of $G$, if for every $v \in V, g(N[v])=$ $\sum_{u \in N[v]} g(u) \geq 1$ and $g(\overline{N(v)})=\sum_{u \notin N(v)} g(u) \geq 1$. A $G D F g$ of a graph $G$ is called minimal ( $M G D F$ ) if for all functions $f: V \rightarrow[0,1]$ such that $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V, f$ is not a $G D F$. The fractional global domination number $\gamma_{f g}(G)$ is defined as follows: $\gamma_{f g}(G)=\min \{|g|: g$ is an MGDF of $G\}$ where $|g|=\sum_{v \in V} g(v)$. In this paper we initiate a study of this parameter. Keywords: domination, global domination, dominating function, global dominating function, fractional global domination number.


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## 1. Introduction

By a graph $G=(V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For basic terminology in graphs we refer to Chartrand and Lesniak [3].

The study of domination and related subset problems such as independence, covering and matching is one of the fastest growing areas within graph theory. A comprehensive treatment of fundamentals of domination in graphs is given in the book by Haynes et al. [6]. Survey of several advanced topics on domination are given in the book edited by Haynes et al. [7].

Let $G=(V, E)$ be a graph. A subset $D$ of $V$ is called a dominating set of $G$ if every vertex in $V-D$ is adjacent to at least one vertex in $D$. A dominating set $D$ is called a minimal dominating set if no proper subset of $D$ is a dominating set of $G$. The minimum (maximum) cardinality of a minimal dominating set of $G$ is called the domination number (upper domination number) of $G$ and is denoted by $\gamma(G)(\Gamma(G))$.

The open neighborhood $N(v)$ and the closed neighborhood $N[v]$ of a vertex $v \in V$ are defined by $N(v)=\{u \in V: u v \in E\}$ and $N[v]=\{v\} \cup N(v)$.

Sampathkumar [9] introduced the concept of global domination.
A dominating set $S$ of $G=(V, E)$ is a global dominating set of $G$ if $S$ is also a dominating set of the complement $\bar{G}$ of $G$. The minimum cardinality of a global dominating set of $G$ is called the global domination number of $G$ and is denoted by $\gamma_{g}(G)$ or simply $\gamma_{g}$. A global dominating set of cardinality $\gamma_{g}$ is called a $\gamma_{g}$-set.

Brigham and Carrington has given a survey of results on global domination in Chapter 11 of Haynes et al. [7].

A recent trend in graph theory is to generalize integer-valued graph theoretic concepts in such a way that they take on rational values. A detailed study of fractional graph theory and fractionalization of various graph parameters are given in Scheinerman and Ullman [10].

Hedetniemi et al. [8] introduced the concept of fractional domination in graphs.

Let $G=(V, E)$ be a graph. Let $f: V \rightarrow R$ be any function. For any subset $S$ of $V$, let $f(S)=\sum_{v \in S} f(v)$. The weight of $f$ is defined by $|f|=f(V)=\sum_{v \in V} f(v)$.

A function $h: V \rightarrow[0,1]$ is called a dominating function of the graph $G=(V, E)$ if $h(N[v])=\sum_{u \in N[v]} h(u) \geq 1$ for all $v \in V$.

A dominating function $h$ of a graph $G$ is minimal if for all functions $f: V \rightarrow$ $[0,1]$ such that $f \leq h$ and $f(v) \neq h(v)$ for at least one $v \in V, f$ is not a dominating function of $G$.

The fractional domination number $\gamma_{f}(G)$ and the upper fractional domination number $\Gamma_{f}(G)$ are defined as follows:
$\gamma_{f}(G)=\min \{|h|: h$ is a dominating function of $G\}$ and $\Gamma_{f}(G)=\max \{|h|: h$ is a minimal dominating function of $G\}$.

For a survey of various domination related functions we refer to Chapters $1,2,3$ and 5 of Haynes et al. [7]. In this paper we introduce the concept of global dominating function and fractional global domination number.

We need the following definition and theorems.
Definition 1.1. The corona of two graphs $G_{1}$ and $G_{2}$ is the graph $G=$ $G_{1} \circ G_{2}$ formed from one copy of $G_{1}$ and $\left|V\left(G_{1}\right)\right|$ copies of $G_{2}$ where the $i^{\text {th }}$ vertex of $G_{1}$ is adjacent to every vertex in the $i^{\text {th }}$ copy of $G_{2}$.

Theorem 1.2 [7]. For any tree $T, \gamma_{f}(T)=\gamma(T)$.
Theorem 1.3 [7]. If $G$ is a triangle-free graph, then $\gamma \leq \gamma_{g} \leq \gamma+1$.
Theorem 1.4 [1]. Let $T$ be a tree. Then $\gamma_{g}(T)=\gamma+1$ if and and only if either $T$ is a star or $T$ is a tree of diameter 4 which is constructed from two or more stars, each having at least two pendant vertices, by connecting the centres of these stars to a common vertex.

## 2. Global Dominating Function

Definition 2.1. A function $g: V \rightarrow[0,1]$ is called a global dominating function $(G D F)$ of a graph $G=(V, E)$, if for every $v \in V, g(N[v])=$ $\sum_{u \in N[v]} g(u) \geq 1$ and $g(\overline{N(v)})=\sum_{u \notin N(v)} g(u) \geq 1$. A GDF $g$ of a graph $G$ is called minimal (MGDF) if for all functions $f: V \rightarrow[0,1]$ such that $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V, f$ is not a GDF. The fractional global domination number $\gamma_{f g}(G)$ is defined as follows: $\gamma_{f g}(G)=\min \{|g|$ : $g$ is an MGDF of $G\}$.

Remark 2.2. The fractional global domination number is the optimal solution of the following linear programming problem (LPP).

$$
\begin{aligned}
& \text { Minimize } z=\sum_{i=1}^{n} f\left(v_{i}\right) \\
& \text { Subject to } \sum_{u \in N[v]} f(u) \geq 1 \text { for all } v \in V, \\
& \\
& \sum_{u \notin N(v)} f(u) \geq 1 \text { for all } v \in V \text { and } \\
& \\
& 0 \leq f(v) \leq 1 \text { for all } v \in V .
\end{aligned}
$$

Remark 2.3. We observe that if $u$ is an isolated vertex of $G$ or $\bar{G}$ then $g(u)=1$ for every MGDF $g$ of $G$. Hence it follows that $\gamma_{f g}\left(K_{n}\right)=n$.

Remark 2.4. Since every $G D F$ of $G$ is a dominating function of $G$ and the characteristic function of a $\gamma_{g}$-set is a $G D F$ of $G$, we have $\gamma_{f} \leq \gamma_{f g} \leq \gamma_{g}$. These inequalities can be strict. For example, for the graph $G$ given in Figure 2.1, it can be easily verified that $\gamma_{f}(G)=2, \gamma_{f g}(G)=2.5$ and $\gamma_{g}(G)=3$.


Figure 2.1
Further, for the corona $G \circ K_{1}$ of any graph $G$ and for the cycle $C_{3 n}$, we have $\gamma_{f}=\gamma_{f g}=\gamma_{g}$.

Theorem 2.5. For any graph $G$ of order $n, 1 \leq \gamma_{f g}(G) \leq n$. Further $\gamma_{f g}(G)=n$ if and only if $G=K_{n}$ or $\overline{K_{n}}$.

Proof. The inequalities are trivial. Suppose $\gamma_{f g}(G)=1$. Let $g$ be a minimum $G D F$ of $G$ and let $v \in V(G)$. Then $\sum_{u \in N[v]} g(u)=1$ and $\sum_{u \notin N(v)} g(u)=1$. Summing up these inequalities, we have $|g|+g(v)=2$. Hence $g(v)=1$ and consequently $G=K_{1}$. Now, suppose $n \geq 2, \gamma_{f g}(G)=n$, and $G \neq \overline{K_{n}}$. If there exist two non-isolated vertices $u$ and $v$ in $G$ which are not adjacent in $G$, then $g: V \rightarrow[0,1]$ defined by $g(u)=0$ and $g(w)=1$ for all $w \neq u$, is a $G D F$ and hence $\gamma_{f g}(G) \leq|g|=n-1$, which is a contradiction. Hence $G=K_{n}$. The Converse is obvious.

We now proceed to determine $\gamma_{f g}$ for some standard graphs.
Theorem 2.6. For the complete $k$-partite graph $G=K_{n_{1}, n_{2}, \ldots, n_{k}}$, we have $\gamma_{f g}(G)=k$.

Proof. Let $X_{1}, X_{2}, \ldots, X_{k}$ be the $k$-partition of $G$ and let $X_{i}=\left\{x_{i j}: 1 \leq\right.$ $\left.j \leq n_{i}\right\}$. Then $g: V \rightarrow[0,1]$ defined by

$$
g\left(x_{i j}\right)= \begin{cases}1 & \text { if } j=1 \\ 0 & \text { otherwise }\end{cases}
$$

for all $i=1,2, \ldots, k$, is a $G D F$ and hence $\gamma_{f g}(G) \leq|g|=k$.
Now, let $g$ be any $G D F$ of $G$. Since $\bar{G}=K_{n_{1}} \cup K_{n_{2}} \cup \cdots \cup K_{n_{k}}$, it follows that $|g| \geq k$ and hence $\gamma_{f g}(G) \geq k$. Thus $\gamma_{f g}(G)=k$.

Theorem 2.7. For any $r$-regular graph $G$ of order $n, \gamma_{f g}(G)=\frac{n}{k+1}$, where $k=\min \{r, n-r-1\}$.

Proof. The constant function $g: V \rightarrow[0,1]$ defined by $g(v)=\frac{1}{k+1}$ is a $G D F$ of $G$ and hence $\gamma_{f g}(G) \leq|g|=\frac{n}{k+1}$.

Now, let $g$ be a $G D F$ of $G$. Then for every $v \in V$, we have

$$
\begin{equation*}
\sum_{u \in N[v]} g(u) \geq 1 \text { and } \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{u \notin N(v)} g(u) \geq 1 . \tag{2}
\end{equation*}
$$

Adding the $n$ inequalities in (1), we get $(r+1)|g| \geq n$ and hence $|g| \geq \frac{n}{r+1}$. Similarly $|g| \geq \frac{n}{(n-r-1)+1}$, so that $|g| \geq \frac{n}{k+1}$, where $k=\min \{r, n-r-1\}$. Thus $\gamma_{f g}(G) \geq \frac{n}{k+1}$ and hence $\gamma_{f g}(G)=\frac{n}{k+1}$.

Corollary 2.8. For the cycle $C_{n}$ on $n$-vertices, we have

$$
\gamma_{f g}\left(C_{n}\right)= \begin{cases}3 & \text { if } n=3, \\ 2 & \text { if } n=4, \\ \frac{n}{3} & \text { if } n \geq 5 .\end{cases}
$$

Theorem 2.9. For the wheel $W_{n}=K_{1}+C_{n-1}$, we have $\gamma_{f g}\left(W_{n}\right)=\frac{2 n-4}{n-3}$.

Proof. Let $V\left(W_{n}\right)=\left\{v_{0}, v_{1}, v_{2}, \ldots, v_{n-1}\right\}$ and $E\left(W_{n}\right)=\left\{v_{0} v_{i}: 1 \leq i \leq\right.$ $n-1\} \cup\left\{v_{i} v_{i+1}: 1 \leq i \leq n-2\right\} \cup\left\{v_{n-1} v_{1}\right\}$. Then $g: V \rightarrow[0,1]$ defined by $g\left(v_{0}\right)=1$ and $g\left(v_{i}\right)=\frac{1}{n-3}$ for $i=1,2, \ldots, n-1$, is a $G D F$ of $W_{n}$. Hence $\gamma_{f g}\left(W_{n}\right) \leq|g|=\frac{2 n-4}{n-3}$. Now, let $g$ be any $G D F$ of $W_{n}$. Since $v_{0}$ is an isolated vertex in $\bar{W}_{n}$, we have $g\left(v_{0}\right)=1$. Also $\sum_{u \notin N\left(v_{i}\right)} g(u) \geq 1,1 \leq i \leq n-1$. Adding these $(n-1)$ inequalities, we get $(n-3) \sum_{i=1}^{n-1} g\left(v_{i}\right) \geq(n-1)$. Hence $(n-3)[|g|-1] \geq(n-1)$, so that $|g| \geq \frac{2 n-4}{n-3}$. Thus $\gamma_{f g}\left(W_{n}\right) \geq \frac{2 n-4}{n-3}$ and hence $\gamma_{f g}\left(W_{n}\right)=\frac{2 n-4}{n-3}$.

Theorem 2.10. For any graph $G$ on $n$ vertices $\gamma_{f g}\left(G \circ K_{1}\right)=n$.
Proof. Let $V(G)=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and let $u_{1}, u_{2}, \ldots, u_{n}$ be the pendant vertices of $G \circ K_{1}$ adjacent to $v_{1}, v_{2}, \ldots, v_{n}$, respectively. Then $g: V(G \circ$ $\left.K_{1}\right) \rightarrow[0,1]$ defined by $g\left(v_{i}\right)=1$ and $g\left(u_{i}\right)=0,1 \leq i \leq n$, is a $G D F$ of $G \circ K_{1}$ and hence $\gamma_{f g}\left(G \circ K_{1}\right) \leq|g|=n$. Also if $g$ is any $G D F$ of $G \circ K_{1}$, we have $g\left(u_{i}\right)+g\left(v_{i}\right) \geq 1$ for all $i=1,2, \ldots, n$. Hence $|g| \geq n$ so that $\gamma_{f g}\left(G \circ K_{1}\right) \geq n$. Thus $\gamma_{f g}\left(G \circ K_{1}\right)=n$.

Theorem 2.11. For any bipartite graph $G$, we have $\gamma_{f} \leq \gamma_{f g} \leq \gamma_{f}+1$.
Proof. Let $(X, Y)$ be the bipartition of $G$ with $|X| \leq|Y|$. Obviously $\gamma_{f} \leq \gamma_{f g}$. Now let $h$ be a $\gamma_{f}$-function of $G$. Suppose $\sum_{u \in X} h(u) \geq 1$. Let $y \in Y$. Then the function $g: V \rightarrow[0,1]$ defined by $g(y)=1$ and $g(v)=h(v)$ for $v \neq y$ is a $G D F$ of $G$ and hence $\gamma_{f g}(G) \leq|g| \leq|h|+1=$ $\gamma_{f}+1$. The proof is similar if $\sum_{u \in Y} h(u) \geq 1$. Suppose $\sum_{x \in X} h(x)<1$ and $\sum_{y \in Y} h(y)<1$. Let $\sum_{x \in X} h(x)=1-\alpha$ and $\sum_{y \in Y} h(y)=1-\beta$ where $0<\alpha, \beta<1$. Clearly $\gamma_{f}(G)=|h|=2-\alpha-\beta$ and since $\gamma_{f} \geq 1$ it follows that $\alpha+\beta \leq 1$. Now let $x \in X$ and $y \in Y$. Then the function $g: V \rightarrow[0,1]$ defined by

$$
g(v)= \begin{cases}h(v)+\alpha & \text { if } v=x \\ h(v)+\beta & \text { if } v=y \\ h(v) & \text { otherwise }\end{cases}
$$

is a $G D F$ of $G$, so that $\gamma_{f g}(G) \leq|g|=|h|+\alpha+\beta \leq \gamma_{f}+1$.
Corollary 2.12. For any tree $T$, we have $\gamma \leq \gamma_{f g} \leq \gamma+1$.
Proof. It follows from Theorem 1.2 that $\gamma_{f}(T)=\gamma(T)$ and hence the result follows.

Theorem 2.13. Let $\mathcal{F}$ denote the family of trees obtained from two or more stars each having at least two pendant vertices by joining the centres of these stars to a common vertex. Let $T$ be any tree and let $s=\min \{d e g u-1$ : $u$ is a support of $T\}$. Then,

$$
\gamma_{f g}(T)= \begin{cases}\gamma+1 & \text { if } T \text { is a star } \\ \gamma+1-\frac{1}{s} & \text { if } T \in \mathcal{F} \\ \gamma & \text { otherwise }\end{cases}
$$

Proof. If $T$ is neither a star nor a member of $\mathcal{F}$, then by Theorem 1.2 and Remark 2.4 we have $\gamma \leq \gamma_{f g} \leq \gamma_{g}$. Also, by Theorem 1.3 and Theorem 1.4, we have $\gamma_{g}=\gamma$ and hence $\gamma_{f g}=\gamma$.

If $T$ is a star, then obviously $\gamma_{f g}=\gamma+1$.
Now let $T \in \mathcal{F}$. We claim that $\gamma_{f g}(T)=\gamma+1-\frac{1}{s}$. Let $u$ be the centre of $T$. Let $v_{1}, v_{2}, \ldots, v_{r}$ be the support vertices of $T$. Let $w_{i 1}, w_{i 2}, \ldots, w_{i t_{i}}$ be the pendant vertices of $T$ adjacent to $v_{i}$, where $1 \leq i \leq r$ and $t_{i} \geq 2$. Then $s=\min t_{i}$. Without loss of generality, we assume $s=t_{1}$. Define $g: V(T) \rightarrow[0,1]$ by

$$
g(x)= \begin{cases}1-\frac{1}{s} & \text { if } x=v_{1} \\ \frac{1}{s} & \text { if } x=w_{1 i}, 1 \leq i \leq t_{1}(=s) \\ 1 & \text { if } x=v_{i}, 2 \leq i \leq r \\ 0 & \text { otherwise }\end{cases}
$$

It is easy to see that $g$ is a $G D F$ of $T$ and $|g|=\gamma+1-\frac{1}{s}$, so that $\gamma_{f g} \leq$ $\gamma+1-\frac{1}{s}$.

Now let $g$ be any $M G D F$ of $T$. We claim that $|g| \geq \gamma+1-\frac{1}{s}$. If $g\left(v_{i}\right)=1$ for all $i, 1 \leq i \leq r$, then $|g| \geq r+1>\gamma+1-\frac{1}{s}$. Suppose $g\left(v_{i}\right)<1$ for at least one $i$. Let $g\left(v_{1}\right)=1-k$, where $k>0$. Then $g\left(w_{1 j}\right) \geq k$, where $1 \leq j \leq t_{1}$ and hence $|g| \geq r-1+(1-k)+t_{1} k=\gamma+\left(t_{1}-1\right) k$. If $k \geq \frac{1}{t_{1}}$, then $|g| \geq \gamma+\left(t_{1}-1\right) \frac{1}{t_{1}} \geq \gamma+1-\frac{1}{s}$. If $k<\frac{1}{t_{1}}$, let $k=\frac{1}{t_{1}}-x, x>0$. Now, since $g(u)+\sum_{i=1}^{r} \sum_{j=1}^{t_{i}} g\left(w_{i j}\right) \geq 1$ and $\sum_{j=1}^{t_{1}} g\left(w_{1 j}\right) \geq t_{1} k=t_{1}\left(\frac{1}{t_{1}}-x\right)=1-t_{1} x$, it follows that

$$
\begin{aligned}
|g| & \geq\left(\gamma+1-\frac{1}{t_{1}}\right)-x\left(t_{1}-1\right)+t_{1} x \\
& =\gamma+1-\frac{1}{t_{1}}+x>\gamma+1-\frac{1}{t_{1}} \geq \gamma+1-\frac{1}{s}
\end{aligned}
$$

Thus $|g| \geq \gamma+1-\frac{1}{s}$ and the result follows.

Corollary 2.14. Let $a, b$ and $c$ be three positive integers such that $1<a<$ $\frac{b}{c}<a+1$ and $\frac{c}{c(1+a)-b}$ is an integer. Then there exists a tree $T$ such that $\gamma(T)=a$ and $\gamma_{f g}(T)=\frac{b}{c}$.

Proof. Let $k=\frac{c}{c(1+a)-b}$. Clearly $k \geq 2$. Let $T$ be a tree obtained from $a$ stars, each having at least $k$ pendant vertices, by joining the centres of these stars to a common vertex. Clearly $\gamma(T)=a$. Further by Theorem 2.13, we have $\gamma_{f g}(T)=\gamma+1-\frac{1}{k}=a+1-\frac{c(1+a)-b}{c}=\frac{b}{c}$.

Corollary 2.15. For any integer $n \geq 2$, there exists a tree $T$ such that $1+\gamma(T)-\gamma_{f g}(T)=\frac{1}{n}$.

Proof. Take $a=n, b=n^{2}+n-1$ and $c=n$ in Corollary 2.14.
We now proceed to obtain bounds for $\gamma_{f g}$.
Theorem 2.16. For any graph $G$ of order $n, \gamma_{f g}(G) \geq \frac{2 n}{n+1}$.
Proof. Let $V(G)=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and let $g$ be any $G D F$ of $G$. Let $v \in V$. Then $\sum_{u \in N[v]} g(u) \geq 1$ and $\sum_{u \notin N(v)} g(u) \geq 1$. Adding these inequalities, we get $g(v)+|g| \geq 2$ for all $v \in V$. Hence $\sum_{v \in V}(g(v)+|g|) \geq 2 n$, so that $(n+1)|g| \geq 2 n$. Thus $|g| \geq \frac{2 n}{n+1}$, so that $\gamma_{f g}(G) \geq \frac{2 n}{n+1}$.

Remark 2.17. The bound given in Theorem 2.16 is sharp. Let $n$ be any integer with $n \equiv 1(\bmod 4)$. Then it follows from Theorem 2.7 , that for any $\frac{n-1}{2}$-regular graph $G$ on $n$ vertices, $\gamma_{f g}(G)=\frac{2 n}{n+1}$.

Theorem 2.18. For any non-regular graph $G$ with $\Delta \leq \frac{n}{2}$, we have $\gamma_{f g}(G) \leq$ $\frac{n}{\delta+1}$.

Proof. Define $g: V \rightarrow[0,1]$ by $f(g)=\frac{1}{\delta+1}$ for all $v \in V$.
Let $v \in V$. Then $\sum_{u \in N[v]} g(u)=|N[v]| \frac{1}{\delta+1} \geq(\delta+1) \frac{1}{\delta+1}=1$. Also $\sum_{u \notin N(v)} g(u)=(n-|N(v)|)\left(\frac{1}{\delta+1}\right) \geq \frac{n-\Delta}{\delta+1}$. Since $\Delta \leq \frac{n}{2}$ and $\delta<\Delta$ it follows that $\frac{n-\Delta}{\delta+1} \geq 1$ and hence $\sum_{u \notin N(v)} g(u) \geq 1$. Thus $g$ is a $G D F$ of $G$ and hence $\gamma_{f g}(G) \leq|g|=\frac{n}{\delta+1}$.

Remark 2.19. The bound given in Theorem 2.18 is sharp. For any graph $G$ on $n$-vertices, it follows from Theorem 2.10 that $\gamma\left(G \circ K_{1}\right)=n=\frac{2 n}{\delta+1}$.

## 3. Minimal Global Dominating Functions

We recall that, a $G D F g$ of a graph $G$ is minimal if $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V$, then $f$ is not a $G D F$ of $G$.

Definition 3.1. Let $g$ be a $G D F$ of a graph $G$. The boundary set $\mathcal{B}_{g}$ and the positive set $\mathcal{P}_{g}$ of $g$ are defined by $\mathcal{B}_{g}=N_{g} \cup \overline{N_{g}}$ where $N_{g}=$ $\left\{v \in V: \sum_{w \in N[v]} g(w)=1\right\}, \overline{N_{g}}=\left\{v \in V: \sum_{w \notin N(v)} g(w)=1\right\}$ and $\mathcal{P}_{g}=\{v \in V: g(v)>0\}$.

Example 3.2. Consider the graph $G$ given in Figure 3.1. Define $g\left(v_{1}\right)=$ $g\left(v_{2}\right)=g\left(v_{3}\right)=\frac{1}{2}, g\left(v_{4}\right)=g\left(v_{6}\right)=0$ and $g\left(v_{5}\right)=1$. Then $\mathcal{P}_{g}=$ $\left\{v_{1}, v_{2}, v_{3}, v_{5}\right\}, N_{g}=\left\{v_{1}, v_{2}, v_{5}, v_{6}\right\}$ and $\overline{N_{g}}=\left\{v_{4}\right\}$. Hence $\mathcal{B}_{g}=N_{g} \cup \overline{N_{g}}=$ $\left\{v_{1}, v_{2}, v_{4}, v_{5}, v_{6}\right\}$.


Figure 3.1
Definition 3.3. Let $g$ be a $G D F$ of a graph with positive set $\mathcal{P}_{g}$ and boundary set $\mathcal{B}_{g}=N_{f} \cup \bar{N}_{g}$. We say that $\mathcal{B}_{g}$ globally dominates $\mathcal{P}_{g}$ if for every vertex $v \in \mathcal{P}_{g}-\mathcal{B}_{g}$, there exists a vertex $u \in N_{g}$ such that $u$ is adjacent to $v$ or there exists a vertex $u \in \bar{N}_{g}$ such that $u$ is not adjacent to $v$ and write $\mathcal{B}_{g} \rightarrow \mathcal{P}_{g}$.

Theorem 3.4. A GDF $g$ of a graph $G$ is an MGDF if and only if $\mathcal{B}_{g} \rightarrow \mathcal{P}_{g}$.
Proof. Suppose $\mathcal{B}_{g} \rightarrow \mathcal{P}_{g}$. Let $v \in \mathcal{P}_{g}$, suppose $h: V \rightarrow[0,1]$ be such that $h(v)<g(v)$ and $h \leq g$. We claim that $h$ is not a GDF.

If $v \in \mathcal{B}_{g}$, then $v \in N_{g}$ or $v \in \overline{N_{g}}$.
Hence $\sum_{w \in N[v]} g(w)=1$ or $\sum_{w \notin N(v)} g(w)=1$. Since $h(v)<g(v)$, it follows that $\sum_{w \in N[v]} h(w)<1$ or $\sum_{w \notin N(v)}^{w \notin N(v)} h(w)<1$.

If $v \notin \mathcal{B}_{g}$ then $v \in \mathcal{P}_{g}-\mathcal{B}_{g}$. Since $\mathcal{B}_{g} \rightarrow \mathcal{P}_{g}$, there exists $u \in N_{g}$ such that $v$ is adjacent to $u$ or there exists $w \in N_{g}$ such that $v$ is not adjacent to $w$. Hence $\sum_{x \in N[u]} g(x)=1$ or $\sum_{x \notin N(w)} g(x)=1$ and hence $\sum_{x \in N[u]} h(x)<1$ or $\sum_{x \notin N(w)} h(x)<1$.

Thus $h$ is not a $G D F$ of $G$, so that $g$ is an MGDF of $G$.

Conversely, suppose $g$ is an MGDF of $G$. Let $v \in \mathcal{P}_{g}-\mathcal{B}_{g}$. Suppose that for every $u \in N[v], \sum_{x \in N[u]} g(x)>1$ and for every $w \notin N(v), \sum_{x \notin N(w)} g(x)>1$.

Let $\sum_{x \in N[u]} g(x)=1+\epsilon_{u}$ and $\sum_{x \notin N(w)} g(x)=1+\epsilon_{w}$.
Let $\epsilon_{1}=\min \left\{\epsilon_{u}: u \in N[v]\right\}, \epsilon_{2}=\min \left\{\epsilon_{w}: w \notin N(v)\right\}$ and $\epsilon=$ $\min \left\{\epsilon_{1}, \epsilon_{2}\right\}$. Define $h: V \rightarrow[0,1]$ by $h(v)=g(v)-\frac{\epsilon}{2}$ and $h(u)=g(u)$ for all $u \in V-\{v\}$. Clearly $h$ is a GDF of $G$ and $h<g$, which is a contradiction. Hence $\mathcal{B}_{g} \rightarrow \mathcal{P}_{g}$.

Definition 3.5. Let $f$ and $g$ be GDFs of $G$ and let $0<\lambda<1$. Then $h_{\lambda}=\lambda f+(1-\lambda) g$ is called a convex combination of $f$ and $g$.

Theorem 3.6. A convex combination of two GDFs of $G$ is again a GDF of $G$.

Proof. Let $f$ and $g$ be two $G D F s$ of $G$. Let $h_{\lambda}=\lambda f+(1-\lambda) g$, where $0<\lambda<1$. Let $v \in V$. Then $\sum_{w \in N[v]} h_{\lambda}(w)=\sum_{w \in N[v]}(\lambda f(w)+(1-\lambda) g(w))$ $=\lambda \sum_{w \in N[v]} f(w)+(1-\lambda) \sum_{w \in N[v]} g(w) \geq \lambda \cdot 1+(1-\lambda) \cdot 1=1$.

Similarly $\sum_{w \notin N(v)} h_{\lambda}(w) \geq 1$. Hence $h_{\lambda}$ is a $G D F$ of $G$.
Remark 3.7. A convex combination of two MGDFs of $G$ need not be an $M G D F$ of $G$. For example, consider the graph given in Figure 3.2.


Define $f: V \rightarrow[0,1]$ by $f\left(v_{1}\right)=f\left(v_{2}\right)=f\left(v_{3}\right)=f\left(v_{4}\right)=\frac{1}{2}, f\left(v_{5}\right)=0$ and $f\left(v_{6}\right)=1$, and $g: V \rightarrow[0,1]$ by $g\left(v_{1}\right)=g\left(v_{2}\right)=g\left(v_{3}\right)=\frac{1}{2}, g\left(v_{4}\right)=$ $g\left(v_{6}\right)=0$ and $g\left(v_{5}\right)=1$. It is easy to verify that $f$ and $g$ are GDFs of G. Also $\mathcal{B}_{f}=\left\{v_{1}, v_{2}, v_{4}, v_{6}\right\}, \mathcal{P}_{f}=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{6}\right\}, \mathcal{B}_{g}=\left\{v_{1}, v_{2}, v_{4}, v_{5}, v_{6}\right\}$ and $\mathcal{P}_{g}=\left\{v_{1}, v_{2}, v_{3}, v_{5}\right\}$. Clearly $\mathcal{B}_{f} \rightarrow \mathcal{P}_{f}$ and $\mathcal{B}_{g} \rightarrow \mathcal{P}_{g}$ and hence $f$ and $g$ are MGDFs of $G$. But $h_{\lambda}$ is not an $M G D F$ for $\lambda=\frac{1}{2}$, since the boundary set of $h_{\lambda}$ does not globally dominate the positive set of $h_{\lambda}$.

Theorem 3.8. Let $f$ and $g$ be two minimal GDFs of $G$ and let $0<\lambda<1$. Then $h_{\lambda}=\lambda f+(1-\lambda) g$ is a minimal GDF of $G$ if and only if $\left(N_{f} \cap N_{g}\right) \cup$ $\left(\overline{N_{f}} \cap \overline{N_{g}}\right) \rightarrow \mathcal{P}_{f} \cup \mathcal{P}_{g}$.

Proof. We prove that $\mathcal{B}_{h_{\lambda}}=\left(N_{f} \cap N_{g}\right) \cup\left(\overline{N_{f}} \cap \overline{N_{g}}\right)$ and $\mathcal{P}_{h_{\lambda}}=\mathcal{P}_{f} \cup \mathcal{P}_{g}$. The result is then immediate from Theorem 3.4. If $v \notin \mathcal{P}_{f} \cup \mathcal{P}_{g}$, then $f(v)=g(v)=h_{\lambda}(v)=0$. Also if $v \in \mathcal{P}_{f}$, then $h_{\lambda}(v) \geq \lambda f(v)>0$. Thus $\mathcal{P}_{h_{\lambda}}=\mathcal{P}_{f} \cup \mathcal{P}_{g}$.

Now, suppose $v \in\left(N_{f} \cap N_{g}\right) \cup\left(\overline{N_{f}} \cap \overline{N_{g}}\right)$. If $v \in\left(N_{f} \cap N_{g}\right)$, then $\sum_{w \in N[v]} h_{\lambda}(v)=\lambda \sum_{w \in N[v]} f(v)+(1-\lambda) \sum_{w \in N[v]} g(v)=\lambda+(1-\lambda)=1$. Also if $v \in \overline{N_{f}} \cap \overline{N_{g}}$, then $\sum_{w \notin N(v)} h_{\lambda}(v)=\lambda \sum_{w \notin N(v)} f(v)+(1-\lambda)$ $\sum_{w \notin N(v)} g(v)=\lambda+(1-\lambda)=1$. A similar calculation shows that if $v \notin$ $\left(N_{f} \cap N_{g}\right) \cup\left(\overline{N_{f}} \cap \overline{N_{g}}\right)$, then $h_{\lambda}(v)>1$. Hence $\mathcal{B}_{h_{\lambda}}=\left(N_{f} \cap N_{g}\right) \cup\left(\overline{N_{f}} \cap \overline{N_{g}}\right)$.

Remark 3.9. The above theorem shows that if $f$ and $g$ are MGDFs of $G$ then either all convex combinations of $f$ and $g$ are MGDFs or no convex combination of $f$ and $g$ is an MGDF.

Conclusion and Scope. As in the case of minimal dominating functions, it follows from Theorem 3.8 that $f$ and $g$ are MGDFs of $G$, then either all convex combinations $f$ and $g$ are MGDFs or no convex combinations of $f$ and $g$ is an MGDF. Hence one can introduce and study the concepts analogus to universal minimal dominating functions [5], basic minimal dominating functions [2] and convexity graphs [4] with respect to global dominating functions. Results in these directions will be reported in subsequent papers.

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