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FRACTIONAL GLOBAL DOMINATION IN GRAPHS

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Abstract

Let $G = (V, E)$ be a graph. A function $g : V \to [0, 1]$ is called a global dominating function (GDF) of G, if for every $v \in V$, $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ and $g(\overline{N(v)}) = \sum_{u \notin N(v)} g(u) \geq 1$. A GDF g of a graph G is called minimal $(MGDF)$ if for all functions $f : V \to [0, 1]$ such that $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V$, f is not a GDF. The fractional global domination number $\gamma_{fg}(G)$ is defined as follows: $\gamma_{fg}(G) = min\{|g| : g$ is an MGDF of $G\}$ where $|g| = \sum_{v \in V} g(v)$. In this paper we initiate a study of this parameter.

Keywords: domination, global domination, dominating function, global dominating function, fractional global domination number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For basic terminology in graphs we refer to Chartrand and Lesniak [3].

The study of domination and related subset problems such as independence, covering and matching is one of the fastest growing areas within graph theory. A comprehensive treatment of fundamentals of domination in graphs is given in the book by Haynes *et al.* [6]. Survey of several advanced topics on domination are given in the book edited by Haynes *et al.* [7].

Let $G = (V, E)$ be a graph. A subset D of V is called a *dominating set* of G if every vertex in $V - D$ is adjacent to at least one vertex in D. A dominating set D is called a *minimal dominating set* if no proper subset of D is a dominating set of G. The minimum (maximum) cardinality of a minimal dominating set of G is called the *domination number* (*upper domination number*) of G and is denoted by $\gamma(G)$ $(\Gamma(G))$.

The *open neighborhood* $N(v)$ and the *closed neighborhood* $N[v]$ of a vertex $v \in V$ are defined by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = \{v\} \cup N(v)$.

Sampathkumar [9] introduced the concept of global domination.

A dominating set S of $G = (V, E)$ is a *global dominating set* of G if S is also a dominating set of the complement $\overline{G}$ of G. The minimum cardinality of a global dominating set of G is called the *global domination number* of G and is denoted by $\gamma_g(G)$ or simply γ_g. A global dominating set of cardinality γ_g is called a γ_g-set.

Brigham and Carrington has given a survey of results on global domination in Chapter 11 of Haynes *et al.* [7].

A recent trend in graph theory is to generalize integer-valued graph theoretic concepts in such a way that they take on rational values. A detailed study of fractional graph theory and fractionalization of various graph parameters are given in Scheinerman and Ullman [10].

Hedetniemi *et al.* [8] introduced the concept of fractional domination in graphs.

Let $G = (V, E)$ be a graph. Let $f : V \to R$ be any function. For any subset S of V, let $f(S) = \sum_{v \in S} f(v)$. The *weight* of f is defined by $|f| = f(V) = \sum_{v \in V} f(v)$.

A function $h : V \to [0, 1]$ is called a *dominating function* of the graph $G = (V, E)$ if $h(N[v]) = \sum_{u \in N[v]} h(u) \geq 1$ for all $v \in V$.

A dominating function h of a graph G is minimal if for all functions $f : V \to [0,1]$ such that $f \leq h$ and $f(v) \neq h(v)$ for at least one $v \in V$, f is not a dominating function of G.

The *fractional domination number* $\gamma_f(G)$ and the *upper fractional domination number* $\Gamma_f(G)$ are defined as follows:

$\gamma_f(G) = \min\{|h| : h \text{ is a dominating function of } G\}$ and
$\Gamma_f(G) = \max\{|h| : h \text{ is a minimal dominating function of } G\}$.

For a survey of various domination related functions we refer to Chapters 1, 2, 3 and 5 of Haynes *et al.* [7]. In this paper we introduce the concept of global dominating function and fractional global domination number.

We need the following definition and theorems.

Definition 1.1. The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

Theorem 1.2 [7]**.** *For any tree* T*,* $\gamma_f(T) = \gamma(T)$*.*

Theorem 1.3 [7]**.** *If* G *is a triangle-free graph, then* $\gamma \leq \gamma_g \leq \gamma + 1$*.*

Theorem 1.4 [1]**.** *Let* T *be a tree. Then* $\gamma_g(T) = \gamma + 1$ *if and and only if either* T *is a star or* T *is a tree of diameter* 4 *which is constructed from two or more stars, each having at least two pendant vertices, by connecting the centres of these stars to a common vertex.*

2. GLOBAL DOMINATING FUNCTION

Definition 2.1. A function $g : V \to [0,1]$ is called a *global dominating function* (GDF) of a graph $G = (V,E)$, if for every $v \in V$, $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ and $g(\overline{N(v)}) = \sum_{u \notin N(v)} g(u) \geq 1$. A GDF g of a graph G is called minimal ($MGDF$) if for all functions $f : V \to [0,1]$ such that $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V$, f is not a GDF. The *fractional global domination number* $\gamma_{fg}(G)$ is defined as follows: $\gamma_{fg}(G) = \min\{|g| : g \text{ is an MGDF of } G\}$.

Remark 2.2. The fractional global domination number is the optimal solution of the following linear programming problem (LPP).

Minimize $z = \sum_{i=1}^{n} f(v_i)$

Subject to $\sum_{u \in N[v]} f(u) \geq 1$ for all $v \in V$,

$\sum_{u \notin N(v)} f(u) \geq 1$ for all $v \in V$ and

$0 \leq f(v) \leq 1$ for all $v \in V$.

Remark 2.3. We observe that if u is an isolated vertex of G or $\overline{G}$ then $g(u) = 1$ for every $MGDF$ g of G. Hence it follows that $\gamma_{fg}(K_n) = n$.

Remark 2.4. Since every GDF of G is a dominating function of G and the characteristic function of a γ_g-set is a GDF of G, we have $\gamma_f \leq \gamma_{fg} \leq \gamma_g$. These inequalities can be strict. For example, for the graph G given in Figure 2.1, it can be easily verified that $\gamma_f(G) = 2$, $\gamma_{fg}(G) = 2.5$ and $\gamma_g(G) = 3$.

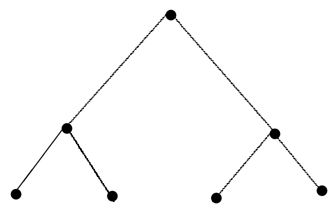

Figure 2.1

Further, for the corona $G \circ K_1$ of any graph G and for the cycle C_{3n}, we have $\gamma_f = \gamma_{fg} = \gamma_g$.

Theorem 2.5. *For any graph G of order n, $1 \leq \gamma_{fg}(G) \leq n$. Further $\gamma_{fg}(G) = n$ if and only if $G = K_n$ or $\overline{K_n}$.*

Proof. The inequalities are trivial. Suppose $\gamma_{fg}(G) = 1$. Let g be a minimum GDF of G and let $v \in V(G)$. Then $\sum_{u \in N[v]} g(u) = 1$ and $\sum_{u \notin N(v)} g(u) = 1$. Summing up these inequalities, we have $|g| + g(v) = 2$. Hence $g(v) = 1$ and consequently $G = K_1$. Now, suppose $n \geq 2, \gamma_{fg}(G) = n$, and $G \neq \overline{K_n}$. If there exist two non-isolated vertices u and v in G which are not adjacent in G, then $g : V \to [0,1]$ defined by $g(u) = 0$ and $g(w) = 1$ for all $w \neq u$, is a GDF and hence $\gamma_{fg}(G) \leq |g| = n-1$, which is a contradiction. Hence $G = K_n$. The Converse is obvious. ■

We now proceed to determine γ_{fg} for some standard graphs.

Theorem 2.6. *For the complete k-partite graph $G = K_{n_1,n_2,\ldots,n_k}$, we have $\gamma_{fg}(G) = k$.*

Proof. Let $X_1, X_2, \ldots, X_k$ be the k-partition of G and let $X_i = \{x_{ij} : 1 \leq j \leq n_i\}$. Then $g : V \to [0,1]$ defined by

$$g(x_{ij}) = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, 2, \ldots, k$, is a GDF and hence $\gamma_{fg}(G) \leq |g| = k$.

Now, let g be any GDF of G. Since $\overline{G} = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$, it follows that $|g| \geq k$ and hence $\gamma_{fg}(G) \geq k$. Thus $\gamma_{fg}(G) = k$. ■

Theorem 2.7. *For any r-regular graph G of order n, $\gamma_{fg}(G) = \frac{n}{k+1}$, where $k = \min\{r, n-r-1\}$.*

Proof. The constant function $g : V \to [0,1]$ defined by $g(v) = \frac{1}{k+1}$ is a GDF of G and hence $\gamma_{fg}(G) \leq |g| = \frac{n}{k+1}$.

Now, let g be a GDF of G. Then for every $v \in V$, we have

$$\sum_{u \in N[v]} g(u) \geq 1 \text{ and} \tag{1}$$

$$\sum_{u \notin N(v)} g(u) \geq 1. \tag{2}$$

Adding the n inequalities in (1), we get $(r+1)|g| \geq n$ and hence $|g| \geq \frac{n}{r+1}$. Similarly $|g| \geq \frac{n}{(n-r-1)+1}$, so that $|g| \geq \frac{n}{k+1}$, where $k = \min\{r, n-r-1\}$. Thus $\gamma_{fg}(G) \geq \frac{n}{k+1}$ and hence $\gamma_{fg}(G) = \frac{n}{k+1}$. ■

Corollary 2.8. *For the cycle C_n on n-vertices , we have*

$$\gamma_{fg}(C_n) = \begin{cases} 3 & \text{if } n = 3, \\ 2 & \text{if } n = 4, \\ \frac{n}{3} & \text{if } n \geq 5. \end{cases}$$

Theorem 2.9. *For the wheel $W_n = K_1 + C_{n-1}$, we have $\gamma_{fg}(W_n) = \frac{2n-4}{n-3}$.*

Proof. Let $V(W_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0v_i : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-2\} \cup \{v_{n-1}v_1\}$. Then $g : V \to [0,1]$ defined by $g(v_0) = 1$ and $g(v_i) = \frac{1}{n-3}$ for $i = 1, 2, \ldots, n-1$, is a *GDF* of W_n. Hence $\gamma_{fg}(W_n) \leq |g| = \frac{2n-4}{n-3}$. Now, let g be any *GDF* of W_n. Since v_0 is an isolated vertex in $\overline{W}_n$, we have $g(v_0) = 1$. Also $\sum_{u \notin N(v_i)} g(u) \geq 1$, $1 \leq i \leq n-1$. Adding these $(n-1)$ inequalities, we get $(n-3)\sum_{i=1}^{n-1} g(v_i) \geq (n-1)$. Hence $(n-3)[|g|-1] \geq (n-1)$, so that $|g| \geq \frac{2n-4}{n-3}$. Thus $\gamma_{fg}(W_n) \geq \frac{2n-4}{n-3}$ and hence $\gamma_{fg}(W_n) = \frac{2n-4}{n-3}$. ■

Theorem 2.10. *For any graph G on n vertices $\gamma_{fg}(G \circ K_1) = n$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $u_1, u_2, \ldots, u_n$ be the pendant vertices of $G \circ K_1$ adjacent to $v_1, v_2, \ldots, v_n$, respectively. Then $g : V(G \circ K_1) \to [0,1]$ defined by $g(v_i) = 1$ and $g(u_i) = 0, 1 \leq i \leq n$, is a *GDF* of $G \circ K_1$ and hence $\gamma_{fg}(G \circ K_1) \leq |g| = n$. Also if g is any *GDF* of $G \circ K_1$, we have $g(u_i) + g(v_i) \geq 1$ for all $i = 1, 2, \ldots, n$. Hence $|g| \geq n$ so that $\gamma_{fg}(G \circ K_1) \geq n$. Thus $\gamma_{fg}(G \circ K_1) = n$. ■

Theorem 2.11. *For any bipartite graph G, we have $\gamma_f \leq \gamma_{fg} \leq \gamma_f + 1$.*

Proof. Let (X, Y) be the bipartition of G with $|X| \leq |Y|$. Obviously $\gamma_f \leq \gamma_{fg}$. Now let h be a γ_f-function of G. Suppose $\sum_{u \in X} h(u) \geq 1$. Let $y \in Y$. Then the function $g : V \to [0,1]$ defined by $g(y) = 1$ and $g(v) = h(v)$ for $v \neq y$ is a *GDF* of G and hence $\gamma_{fg}(G) \leq |g| \leq |h| + 1 = \gamma_f + 1$. The proof is similar if $\sum_{u \in Y} h(u) \geq 1$. Suppose $\sum_{x \in X} h(x) < 1$ and $\sum_{y \in Y} h(y) < 1$. Let $\sum_{x \in X} h(x) = 1 - \alpha$ and $\sum_{y \in Y} h(y) = 1 - \beta$ where $0 < \alpha, \beta < 1$. Clearly $\gamma_f(G) = |h| = 2 - \alpha - \beta$ and since $\gamma_f \geq 1$ it follows that $\alpha + \beta \leq 1$. Now let $x \in X$ and $y \in Y$. Then the function $g : V \to [0,1]$ defined by

$$g(v) = \begin{cases} h(v) + \alpha & \text{if } v = x, \\ h(v) + \beta & \text{if } v = y, \\ h(v) & \text{otherwise} \end{cases}$$

is a *GDF* of G, so that $\gamma_{fg}(G) \leq |g| = |h| + \alpha + \beta \leq \gamma_f + 1$. ■

Corollary 2.12. *For any tree T, we have $\gamma \leq \gamma_{fg} \leq \gamma + 1$.*

Proof. It follows from Theorem 1.2 that $\gamma_f(T) = \gamma(T)$ and hence the result follows. ■

Theorem 2.13. *Let $\mathcal{F}$ denote the family of trees obtained from two or more stars each having at least two pendant vertices by joining the centres of these stars to a common vertex. Let T be any tree and let $s = \min\{deg\ u - 1 : u \text{ is a support of } T\}$. Then,*

$$\gamma_{fg}(T) = \begin{cases} \gamma + 1 & \textit{if } T \textit{ is a star,} \\ \gamma + 1 - \frac{1}{s} & \textit{if } T \in \mathcal{F}, \\ \gamma & \textit{otherwise.} \end{cases}$$

Proof. If T is neither a star nor a member of $\mathcal{F}$, then by Theorem 1.2 and Remark 2.4 we have $\gamma \le \gamma_{fg} \le \gamma_g$. Also, by Theorem 1.3 and Theorem 1.4, we have $\gamma_g = \gamma$ and hence $\gamma_{fg} = \gamma$.

If T is a star, then obviously $\gamma_{fg} = \gamma + 1$.

Now let $T \in \mathcal{F}$. We claim that $\gamma_{fg}(T) = \gamma + 1 - \frac{1}{s}$. Let u be the centre of T. Let $v_1, v_2, \ldots, v_r$ be the support vertices of T. Let $w_{i1}, w_{i2}, \ldots, w_{it_i}$ be the pendant vertices of T adjacent to v_i, where $1 \le i \le r$ and $t_i \ge 2$. Then $s = \min t_i$. Without loss of generality, we assume $s = t_1$. Define $g : V(T) \to [0,1]$ by

$$g(x) = \begin{cases} 1 - \frac{1}{s} & \text{if } x = v_1, \\ \frac{1}{s} & \text{if } x = w_{1i},\ 1 \le i \le t_1\ (= s), \\ 1 & \text{if } x = v_i,\ 2 \le i \le r, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that g is a GDF of T and $|g| = \gamma + 1 - \frac{1}{s}$, so that $\gamma_{fg} \le \gamma + 1 - \frac{1}{s}$.

Now let g be any $MGDF$ of T. We claim that $|g| \ge \gamma + 1 - \frac{1}{s}$. If $g(v_i) = 1$ for all $i, 1 \le i \le r$, then $|g| \ge r + 1 > \gamma + 1 - \frac{1}{s}$. Suppose $g(v_i) < 1$ for at least one i. Let $g(v_1) = 1 - k$, where $k > 0$. Then $g(w_{1j}) \ge k$, where $1 \le j \le t_1$ and hence $|g| \ge r - 1 + (1-k) + t_1 k = \gamma + (t_1 - 1)k$. If $k \ge \frac{1}{t_1}$, then $|g| \ge \gamma + (t_1 - 1)\frac{1}{t_1} \ge \gamma + 1 - \frac{1}{s}$. If $k < \frac{1}{t_1}$, let $k = \frac{1}{t_1} - x$, $x > 0$. Now, since $g(u) + \sum_{i=1}^{r} \sum_{j=1}^{t_i} g(w_{ij}) \ge 1$ and $\sum_{j=1}^{t_1} g(w_{1j}) \ge t_1 k = t_1(\frac{1}{t_1} - x) = 1 - t_1 x$, it follows that

$$\begin{aligned} |g| &\ge \left(\gamma + 1 - \frac{1}{t_1}\right) - x(t_1 - 1) + t_1 x \\ &= \gamma + 1 - \frac{1}{t_1} + x > \gamma + 1 - \frac{1}{t_1} \ge \gamma + 1 - \frac{1}{s}. \end{aligned}$$

Thus $|g| \ge \gamma + 1 - \frac{1}{s}$ and the result follows. ∎

Corollary 2.14. *Let a, b and c be three positive integers such that $1 < a < \frac{b}{c} < a+1$ and $\frac{c}{c(1+a)-b}$ is an integer. Then there exists a tree T such that $\gamma(T) = a$ and $\gamma_{fg}(T) = \frac{b}{c}$.*

Proof. Let $k = \frac{c}{c(1+a)-b}$. Clearly $k \geq 2$. Let T be a tree obtained from a stars, each having at least k pendant vertices, by joining the centres of these stars to a common vertex. Clearly $\gamma(T) = a$. Further by Theorem 2.13, we have $\gamma_{fg}(T) = \gamma + 1 - \frac{1}{k} = a + 1 - \frac{c(1+a)-b}{c} = \frac{b}{c}$. ■

Corollary 2.15. *For any integer $n \geq 2$, there exists a tree T such that $1 + \gamma(T) - \gamma_{fg}(T) = \frac{1}{n}$.*

Proof. Take $a = n$, $b = n^2 + n - 1$ and $c = n$ in Corollary 2.14. ■

We now proceed to obtain bounds for γ_{fg}.

Theorem 2.16. *For any graph G of order $n, \gamma_{fg}(G) \geq \frac{2n}{n+1}$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let g be any GDF of G. Let $v \in V$. Then $\sum_{u \in N[v]} g(u) \geq 1$ and $\sum_{u \notin N(v)} g(u) \geq 1$. Adding these inequalities, we get $g(v) + |g| \geq 2$ for all $v \in V$. Hence $\sum_{v \in V}(g(v) + |g|) \geq 2n$, so that $(n+1)|g| \geq 2n$. Thus $|g| \geq \frac{2n}{n+1}$, so that $\gamma_{fg}(G) \geq \frac{2n}{n+1}$. ■

Remark 2.17. The bound given in Theorem 2.16 is sharp. Let n be any integer with $n \equiv 1 (\bmod\ 4)$. Then it follows from Theorem 2.7, that for any $\frac{n-1}{2}$-regular graph G on n vertices, $\gamma_{fg}(G) = \frac{2n}{n+1}$.

Theorem 2.18. *For any non-regular graph G with $\Delta \leq \frac{n}{2}$, we have $\gamma_{fg}(G) \leq \frac{n}{\delta+1}$.*

Proof. Define $g : V \to [0,1]$ by $f(g) = \frac{1}{\delta+1}$ for all $v \in V$.

Let $v \in V$. Then $\sum_{u \in N[v]} g(u) = |N[v]| \frac{1}{\delta+1} \geq (\delta+1)\frac{1}{\delta+1} = 1$. Also $\sum_{u \notin N(v)} g(u) = (n - |N(v)|)(\frac{1}{\delta+1}) \geq \frac{n-\Delta}{\delta+1}$. Since $\Delta \leq \frac{n}{2}$ and $\delta < \Delta$ it follows that $\frac{n-\Delta}{\delta+1} \geq 1$ and hence $\sum_{u \notin N(v)} g(u) \geq 1$. Thus g is a GDF of G and hence $\gamma_{fg}(G) \leq |g| = \frac{n}{\delta+1}$. ■

Remark 2.19. The bound given in Theorem 2.18 is sharp. For any graph G on n-vertices, it follows from Theorem 2.10 that $\gamma(G \circ K_1) = n = \frac{2n}{\delta+1}$.

3. MINIMAL GLOBAL DOMINATING FUNCTIONS

We recall that, a *GDF* g of a graph G is minimal if $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V$, then f is not a *GDF* of G.

Definition 3.1. Let g be a *GDF* of a graph G. The *boundary set* $\mathcal{B}_g$ and the positive set $\mathcal{P}_g$ of g are defined by $\mathcal{B}_g = N_g \cup \overline{N_g}$ where $N_g = \big\{v \in V : \sum_{w \in N[v]} g(w) = 1\big\}$, $\overline{N_g} = \big\{v \in V : \sum_{w \notin N(v)} g(w) = 1\big\}$ and $\mathcal{P}_g = \big\{v \in V : g(v) > 0\big\}$.

Example 3.2. Consider the graph G given in Figure 3.1. Define $g(v_1) = g(v_2) = g(v_3) = \frac{1}{2}$, $g(v_4) = g(v_6) = 0$ and $g(v_5) = 1$. Then $\mathcal{P}_g = \{v_1, v_2, v_3, v_5\}$, $N_g = \{v_1, v_2, v_5, v_6\}$ and $\overline{N_g} = \{v_4\}$. Hence $\mathcal{B}_g = N_g \cup \overline{N_g} = \{v_1, v_2, v_4, v_5, v_6\}$.

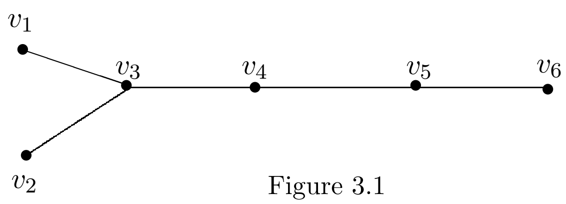

Figure 3.1

Definition 3.3. Let g be a *GDF* of a graph with positive set $\mathcal{P}_g$ and boundary set $\mathcal{B}_g = N_f \cup \overline{N}_g$. We say that $\mathcal{B}_g$ globally dominates $\mathcal{P}_g$ if for every vertex $v \in \mathcal{P}_g - \mathcal{B}_g$, there exists a vertex $u \in N_g$ such that u is adjacent to v or there exists a vertex $u \in \overline{N}_g$ such that u is not adjacent to v and write $\mathcal{B}_g \to \mathcal{P}_g$.

Theorem 3.4. *A GDF g of a graph G is an MGDF if and only if $\mathcal{B}_g \to \mathcal{P}_g$.*

Proof. Suppose $\mathcal{B}_g \to \mathcal{P}_g$. Let $v \in \mathcal{P}_g$, suppose $h : V \to [0,1]$ be such that $h(v) < g(v)$ and $h \leq g$. We claim that h is not a *GDF*.

If $v \in \mathcal{B}_g$, then $v \in N_g$ or $v \in \overline{N_g}$.

Hence $\sum_{w \in N[v]} g(w) = 1$ or $\sum_{w \notin N(v)} g(w) = 1$. Since $h(v) < g(v)$, it follows that $\sum_{w \in N[v]} h(w) < 1$ or $\sum_{w \notin N(v)} h(w) < 1$.

If $v \notin \mathcal{B}_g$ then $v \in \mathcal{P}_g - \mathcal{B}_g$. Since $\mathcal{B}_g \to \mathcal{P}_g$, there exists $u \in N_g$ such that v is adjacent to u or there exists $w \in \overline{N}_g$ such that v is not adjacent to w. Hence $\sum_{x \in N[u]} g(x) = 1$ or $\sum_{x \notin N(w)} g(x) = 1$ and hence $\sum_{x \in N[u]} h(x) < 1$ or $\sum_{x \notin N(w)} h(x) < 1$.

Thus h is not a *GDF* of G, so that g is an *MGDF* of G.

Conversely, suppose g is an *MGDF* of G. Let $v \in \mathcal{P}_g - \mathcal{B}_g$. Suppose that for every $u \in N[v]$, $\sum_{x \in N[u]} g(x) > 1$ and for every $w \notin N(v)$, $\sum_{x \notin N(w)} g(x) > 1$.

Let $\sum_{x \in N[u]} g(x) = 1 + \epsilon_u$ and $\sum_{x \notin N(w)} g(x) = 1 + \epsilon_w$.

Let $\epsilon_1 = \min\{\epsilon_u : u \in N[v]\}$, $\epsilon_2 = \min\{\epsilon_w : w \notin N(v)\}$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Define $h : V \to [0,1]$ by $h(v) = g(v) - \frac{\epsilon}{2}$ and $h(u) = g(u)$ for all $u \in V - \{v\}$. Clearly h is a GDF of G and $h < g$, which is a contradiction. Hence $\mathcal{B}_g \to \mathcal{P}_g$. ■

Definition 3.5. Let f and g be *GDFs* of G and let $0 < \lambda < 1$. Then $h_\lambda = \lambda f + (1 - \lambda) g$ is called a *convex combination* of f and g.

Theorem 3.6. *A convex combination of two GDFs of G is again a GDF of G.*

Proof. Let f and g be two *GDFs* of G. Let $h_\lambda = \lambda f + (1 - \lambda) g$, where $0 < \lambda < 1$. Let $v \in V$. Then $\sum_{w \in N[v]} h_\lambda(w) = \sum_{w \in N[v]} (\lambda f(w) + (1-\lambda) g(w))$ $= \lambda \sum_{w \in N[v]} f(w) + (1 - \lambda) \sum_{w \in N[v]} g(w) \geq \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1$.

Similarly $\sum_{w \notin N(v)} h_\lambda(w) \geq 1$. Hence h_λ is a *GDF* of G. ■

Remark 3.7. A convex combination of two *MGDFs* of G need not be an *MGDF* of G. For example, consider the graph given in Figure 3.2.

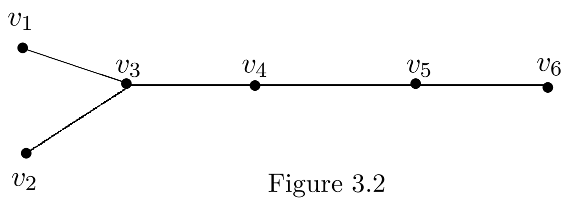

Figure 3.2

Define $f : V \to [0,1]$ by $f(v_1) = f(v_2) = f(v_3) = f(v_4) = \frac{1}{2}, f(v_5) = 0$ and $f(v_6) = 1$, and $g : V \to [0,1]$ by $g(v_1) = g(v_2) = g(v_3) = \frac{1}{2}$, $g(v_4) = g(v_6) = 0$ and $g(v_5) = 1$. It is easy to verify that f and g are *GDFs* of G. Also $\mathcal{B}_f = \{v_1, v_2, v_4, v_6\}$, $\mathcal{P}_f = \{v_1, v_2, v_3, v_4, v_6\}, \mathcal{B}_g = \{v_1, v_2, v_4, v_5, v_6\}$ and $\mathcal{P}_g = \{v_1, v_2, v_3, v_5\}$. Clearly $\mathcal{B}_f \to \mathcal{P}_f$ and $\mathcal{B}_g \to \mathcal{P}_g$ and hence f and g are *MGDFs* of G. But h_λ is not an *MGDF* for $\lambda = \frac{1}{2}$, since the boundary set of h_λ does not globally dominate the positive set of h_λ.

Theorem 3.8. *Let f and g be two minimal GDFs of G and let $0 < \lambda < 1$. Then $h_\lambda = \lambda f + (1 - \lambda) g$ is a minimal GDF of G if and only if* $(N_f \cap N_g) \cup (\overline{N_f} \cap \overline{N_g}) \to \mathcal{P}_f \cup \mathcal{P}_g$.

Proof. We prove that $\mathcal{B}_{h_\lambda} = (N_f \cap N_g) \cup (\overline{N_f} \cap \overline{N_g})$ and $\mathcal{P}_{h_\lambda} = \mathcal{P}_f \cup \mathcal{P}_g$. The result is then immediate from Theorem 3.4. If $v \notin \mathcal{P}_f \cup \mathcal{P}_g$, then $f(v) = g(v) = h_\lambda(v) = 0$. Also if $v \in \mathcal{P}_f$, then $h_\lambda(v) \geq \lambda f(v) > 0$. Thus $\mathcal{P}_{h_\lambda} = \mathcal{P}_f \cup \mathcal{P}_g$.

Now, suppose $v \in (N_f \cap N_g) \cup (\overline{N_f} \cap \overline{N_g})$. If $v \in (N_f \cap N_g)$, then $\sum_{w\in N[v]} h_\lambda(v) = \lambda \sum_{w\in N[v]} f(v) + (1-\lambda) \sum_{w\in N[v]} g(v) = \lambda + (1-\lambda) = 1$. Also if $v \in \overline{N_f} \cap \overline{N_g}$, then $\sum_{w\notin N(v)} h_\lambda(v) = \lambda \sum_{w\notin N(v)} f(v) + (1-\lambda)$ $\sum_{w\notin N(v)} g(v) = \lambda + (1-\lambda) = 1$. A similar calculation shows that if $v \notin (N_f \cap N_g) \cup (\overline{N_f} \cap \overline{N_g})$, then $h_\lambda(v) > 1$. Hence $\mathcal{B}_{h_\lambda} = (N_f \cap N_g) \cup (\overline{N_f} \cap \overline{N_g})$. ■

Remark 3.9. The above theorem shows that if f and g are *MGDFs* of G then either all convex combinations of f and g are *MGDFs* or no convex combination of f and g is an *MGDF*.

Conclusion and Scope. As in the case of minimal dominating functions, it follows from Theorem 3.8 that f and g are *MGDFs* of G, then either all convex combinations f and g are *MGDFs* or no convex combinations of f and g is an *MGDF*. Hence one can introduce and study the concepts analogus to universal minimal dominating functions [5], basic minimal dominating functions [2] and convexity graphs [4] with respect to global dominating functions. Results in these directions will be reported in subsequent papers.

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