Discussiones Mathematicae
Graph Theory 30 (2010) 5–15

VALUE SETS OF GRAPHS EDGE-WEIGHTED WITH ELEMENTS OF A FINITE ABELIAN GROUP

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Abstract

Given a graph $G = (V, E)$ of order n and a finite abelian group $\mathrm{H} = (H, +)$ of order n, a bijection f of V onto H is called a *vertex H-labeling* of G. Let $g(e) \equiv (f(u) + f(v)) \mod \mathrm{H}$ for each edge $e = \{u, v\}$ in E induce an *edge H-labeling* of G. Then, the sum $\mathrm{Hval}_f(G) \equiv \sum_{e\in E} g(e) \mod \mathrm{H}$ is called the *H-value* of G relative to f and the set $\mathrm{Hval}S(G)$ of all H-values of G over all possible vertex H-labelings is called the *H-value set* of G. Theorems determining $\mathrm{Hval}S(G)$ for given H and G are obtained.

Keywords: graph labeling, edge labeling, vertex labeling, abelian group.

2010 Mathematics Subject Classification: 05C78, 05C22, 05C25.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E having *order* n and *size* $|E|$.

Definition 1.1. A *vertex H-labeling* f of G is a bijective function $f : V \to H$ where $\mathrm{H} = (H, +)$ is a finite abelian group of order n with identity 0. The *associated edge H-labeling* induced by f is the function $g : E \to H$ given by

$$g(e) \equiv (f(u) + f(v)) \mod H \text{ for each edge } e = \{u, v\} \text{ in } E.$$

Note that g is not expected to be bijective nor assumed to have any specific properties.

Definition 1.2. The *value of an H-labeled graph* G relative to f is defined

$$\mathrm{Hval}_f\ (G) \equiv \sum_{e \in E} g(e) \mod \mathrm{H}.$$

For convenience we set $\mathrm{Hval}_f(G) \equiv 0$, if E is empty and note that $\equiv$ shall mean mod H even if not stated explicitly. The *value set of G relative to* H is defined

$$\mathrm{Hval}S(G) = \{\,\mathrm{Hval}_f(G) :\ f \text{ is a vertex } H\text{-labeling }\}.$$

We consider the problem of determining the value set of a given graph G relative to a given finite abelian group H. Namely,

Problem A. Given a graph G and a finite abelian group H, determine $\mathrm{Hval}S(G)$.

In [1] this problem was studied when H is a finite cyclic group. All results in [1] are contained herein. For a comprehensive and dynamic survey of graph labeling in general see [2]. For algebraic and graph theoretic concepts see [3, 4].

2. The Fundamental Theorems

The well known structure of abelian groups is used throughout this paper (see Theorem 2.1). Theorem 2.2, Corollary 2.2.1, and Theorem 2.3 play the key roles in determining $\mathrm{Hval}S(G)$ when H is a finite abelian group.

Theorem 2.1. *The Fundamental Theorem of Finite Abelian Groups* (*see* [3]). *Every finite abelian group* H *is a direct product of cyclic groups of prime-power order. Moreover, the number of factors in the product and the orders of the cyclic groups are uniquely determined by the group* H.

Therefore, without loss of generality, assume

$$\mathrm{H} \cong \prod_{i=1}^{k} \mathrm{Z}_{n_i}$$

where $n_i = p_i^{m_i}$, p_i is prime, $2 \le p_i \le p_{i+1} \le p_k$, and $1 \le m_i$.

A *standard form* for H is obtained if the Z_{n_i} are written with respect to increasing p_i and in the case of $p_i = p_{i+1}$ are written with respect to increasing order of m_i. Each factorization of n into prime powers corresponds to a unique abelian group of order n. The *identity element*, when H is expressed in standard form, is 0_k, the k-tuple of zeros.

For $n = 360$ there are exactly six distinct prime power factorizations. Thus, there are exactly six isomorphically distinct abelian groups of order 360. These are shown in Example 2.1.

Example 2.1. If H is an abelian group of order $n = 360$, then H is isomorphic to one of the following:

$\mathrm{H}_1 = \mathrm{Z}_8 \times \mathrm{Z}_9 \times \mathrm{Z}_5$; $\mathrm{H}_2 = \mathrm{Z}_2 \times \mathrm{Z}_4 \times \mathrm{Z}_9 \times \mathrm{Z}_5$; $\mathrm{H}_3 = \mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_9 \times \mathrm{Z}_5$;

$\mathrm{H}_4 = \mathrm{Z}_8 \times \mathrm{Z}_3 \times \mathrm{Z}_3 \times \mathrm{Z}_5$; $\mathrm{H}_5 = \mathrm{Z}_2 \times \mathrm{Z}_4 \times \mathrm{Z}_3 \times \mathrm{Z}_3 \times \mathrm{Z}_5$; or

$\mathrm{H}_6 = \mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_3 \times \mathrm{Z}_3 \times \mathrm{Z}_5$.

Theorem 2.2. *Let* H *denote an abelian group expressed in standard form and* $\Sigma(H)$ *the sum* mod H *of the elements of* H*. Then,*

$$\Sigma(Z_n) \equiv \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

$$\Sigma(Z_{n_1} \times Z_{n_2}) \equiv \begin{cases} (n_1/2, 0) & \text{if } n_1 \text{ is even and } n_2 \text{ is odd,} \\ 0_2 & \text{otherwise,} \end{cases}$$

and in general

$$\Sigma\left(\prod_{i=1}^{k} Z_{n_i}\right) \equiv \begin{cases} (2^{m_1-1}, 0, 0, \ldots, 0) & \text{if } n_1 \text{ is even and } n_i \text{ is odd for } i \geq 2, \\ 0_k & \text{otherwise.} \end{cases}$$

Proof.

$$\Sigma(Z_n) = \sum_{j=0}^{n-1} j = \frac{(n-1)n}{2} \equiv \begin{cases} n/2 \mod n & \text{if } n \text{ is even,} \\ 0 \mod n & \text{if } n \text{ is odd.} \end{cases}$$

$$\Sigma(Z_{n_1} \times Z_{n_2}) = \sum_{j_2=0}^{n_2-1} \sum_{j_1=0}^{n_1-1} (j_1, j_2) = \sum_{j_2=0}^{n_2-1} \big(\Sigma(Z_{n_1}), n_1 j_2\big)$$

$$= \big(n_2\Sigma(Z_{n_1}), n_1\Sigma(Z_{n_2})\big) = \left(n_2\frac{n_1(n_1-1)}{2}, n_1\frac{n_2(n_2-1)}{2}\right),$$

which is as asserted in the statement of the theorem.

$$\Sigma\left(\prod_{i=1}^{k} Z_{n_i}\right) = \sum_{j_k=0}^{n_k-1}\sum_{j_{k-1}=0}^{n_{k-1}-1}\cdots\sum_{j_1=0}^{n_1-1}(j_1, j_2, \ldots, j_k)$$

$$= \sum_{j_k=0}^{n_k-1}\sum_{j_{k-1}=0}^{n_{k-1}-1}\cdots\sum_{j_2=0}^{n_2-1}(\Sigma(Z_{n_1}), n_1j_2, n_1j_3, ..., n_1j_k)$$

$$= \sum_{j_k=0}^{n_k-1}\sum_{j_{k-1}=0}^{n_{k-1}-1}\cdots\sum_{j_3=0}^{n_3-1}(n_2\Sigma(Z_{n_1}), n_1\Sigma(Z_{n_2}), n_1n_2j_3, \ldots, n_1n_2j_k)$$

$$\ldots$$

$$= (\bar{n}_1n_2\ldots n_k\Sigma(Z_{n_1}),\ n_1\bar{n}_2\ldots n_k\Sigma(Z_{n_2}), \ldots,$$

$$n_1n_2\ldots\bar{n}_j\ldots n_k\Sigma(Z_{n_j}), \ldots, n_1n_2\ldots\bar{n}_k\Sigma(Z_{n_k}))$$

where $\bar{n}_j$ means n_j is not a factor.

Noting that $n_1n_2\ldots\bar{n}_j\ldots n_k\Sigma(Z_{n_j}) = n_1n_2\ldots\bar{n}_j\ldots n_k\frac{n_j(n_j-1)}{2}$ is equal to $\lambda n_j \equiv 0$, if any n_i with $i \neq j$ is even. If all n_i with $i \neq j$ are odd, there are two cases:

(a) if n_j is odd, one gets $\lambda n_j \equiv 0$.

(b) if n_j is even, one gets $\lambda\frac{n_j}{2} \equiv \frac{n_j}{2}$.

However since H is in standard form, j must equal to 1, so that $n_1 = 2^{m_1}$. Thus,

$$\Sigma\Big(\prod_{i=1}^{k} Z_{n_i}\Big) \equiv \begin{cases} (2^{m_1-1}, 0, 0, \ldots, 0) & \text{if } n_1 \text{ is even and } n_i \text{ is odd for } i \geq 2, \\ (0, 0, 0, \ldots, 0) & \text{otherwise.} \end{cases}$$ ■

Corollary 2.2.1. *Let w be a nonnegative integer. Then*

$$w\Sigma(H) \equiv \begin{cases} (2^{m_1-1}, 0, 0, \ldots, 0) & \text{if } n_1 \text{ is even, } n_i \text{ is odd for } i \geq 2, \text{ and } w \text{ is odd,} \\ 0_k & \text{otherwise.} \end{cases}$$

Proof. $\Sigma(H) \equiv 0_k$ except when n_1 is even and n_i is odd for $i \geq 2$. Thus, $w\Sigma(H) = (w2^{m_1-1}, 0, \ldots, 0)$ with $w2^{m_1-1} \neq 0$ only when w is odd. ■

Let $\deg(v)$ denote the *degree of vertex* v, the number of edges incident to v.

Theorem 2.3. *For any graph G,* $\mathrm{Hval}_f(G) \equiv \sum_{v\in V} \deg(v)f(v) \bmod \mathrm{H}$.

Proof. The edge weight function g is defined $g(e) = (f(u) + f(v)) \bmod \mathrm{H}$ for each edge $e = \{u, v\}$ in E. Thus, each vertex x in G contributes $\deg(x)f(x)$ to the sum $\sum_{e\in E} g(e) \bmod \mathrm{H} = \mathrm{Hval}_f(G)$. ■

3. RESULTS FOR SPECIFIC CLASSES OF GRAPHS

3.1. Regular graphs

A graph G is *regular* if each vertex of G has the same degree.

Theorem 3.1.1. *If G is regular of even degree r and* $\mathrm{H} \cong \prod_{i=1}^{k} \mathrm{Z}_{n_i}$ *then* $\mathrm{Hval}_f(G) \equiv 0_k$ *for any vertex H-labeling f and so* $\mathrm{Hval}S(G) = \{0_k\}$.

Proof. $\mathrm{Hval}_f(G) = \sum_{v\in V(G)} \deg(v)f(v) = r\sum_{v\in V(G)} f(v) = r\Sigma(H)$. Since r is even, by Corollary 2.2.1, $r\Sigma(H) \equiv (0, 0, 0, \ldots, 0) \bmod \mathrm{H}$ for any vertex H-labeling f. ■

Example 3.1.1. If G is regular of even degree r and H is an abelian group of order 360, that is, H is any one of the six groups listed in Example 2.1, $\mathrm{Hval}S(G) = \{0_t\}$, where $t = 3, 4, 5, 4, 5,$ or 6 corresponding to the identity element of H_1, H_2, H_3, H_4, H_5, or H_6, respectively.

Theorem 3.1.2. *If G is regular of odd degree r and* $\mathrm{H} \cong \prod_{i=1}^{k} \mathrm{Z}_{n_i}$, *then*

$$\mathrm{Hval}_f(G) \equiv \begin{cases} (2^{m_1-1}, 0, 0, \ldots, 0) & \text{if } n_1 \text{ is even and } n_i \text{ is odd for } i \geq 2, \\ 0_k & \text{otherwise.} \end{cases}$$

so that

$$\mathrm{Hval}S(G) = \begin{cases} \{(2^{m_1-1}, 0, 0, \ldots, 0)\} & \text{if } n_1 \text{ is even and } n_i \text{ is odd for } i \geq 2, \\ \{0_k\} & \text{otherwise.} \end{cases}$$

Proof. As in the proof of Theorem 3.1.1, $\mathrm{Hval}_f(G) = r\Sigma(H)$. However, since r is odd, by Corollary 2.2.1, $\Sigma(H)$ is not changed by multiplication by an odd number. Thus, $\mathrm{Hval}_f(G)$ and $\mathrm{Hval}S(G)$ are as in the statement of the theorem. ■

Example 3.1.2. If G is regular of odd degree r and H is an abelian group of order 360 (see Example 2.1), then

if $\mathrm{H} = \mathrm{H}_1 \cong \mathrm{Z}_8 \times \mathrm{Z}_9 \times \mathrm{Z}_5$, then $\mathrm{Hval}S(G) = \{(4, 0, 0)\}$,

if $\mathrm{H} = \mathrm{H}_4 \cong \mathrm{Z}_8 \times \mathrm{Z}_3 \times \mathrm{Z}_3 \times \mathrm{Z}_5$, then $\mathrm{Hval}S(G) = \{(4, 0, 0, 0)\}$, and

if $\mathrm{H} = \mathrm{H}_2, \mathrm{H}_3, \mathrm{H}_5$, or H_6, then $\mathrm{Hval}S(G) = \{0_t\}$ with $t = 4, 5, 5$, or 6, respectively.

3.2. Graphs with exactly two vertex degrees

Theorem 3.2.1. *If G is a graph of order n with x vertices $u_1, u_2, \ldots, u_x$ of degree d_1 and y vertices $v_1, v_2, \ldots, v_y$ of degree $d_2 > d_1$, such that $x + y = n$, then for any H-labeling f of its vertices*

$$\mathrm{Hval}_f(G) \equiv d_1\Sigma(H) + (d_2 - d_1)\sum_{j=1}^{y} f(v_j) \equiv d_2\Sigma(H) - (d_2 - d_1)\sum_{i=1}^{x} f(u_i).$$

Proof. By Theorem 2.3, for any H-labeling f of the vertices of G

$$\begin{aligned}\mathrm{Hval}_f(G) &\equiv d_1\sum_{i=1}^{x} f(u_i) + d_2\sum_{j=1}^{y} f(v_j) \equiv d_1\sum_{i=1}^{x} f(u_i)\\ &+ d_2\sum_{j=1}^{y} f(v_j) + d_1\sum_{j=1}^{y} f(v_j) - d_1\sum_{j=1}^{y} f(v_j)\\ &\equiv d_1\Sigma(H) + (d_2 - d_1)\sum_{j=1}^{y} f(v_j).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{Hval}_f(G) &\equiv d_1\sum_{i=1}^{x} f(u_i) + d_2\sum_{j=1}^{y} f(v_j) + d_2\sum_{i=1}^{x} f(u_i) - d_2\sum_{i=1}^{x} f(u_i)\\ &\equiv d_2\Sigma(H) - (d_2 - d_1)\sum_{i=1}^{x} f(u_i).\end{aligned}$$

Thus,

$$\mathrm{Hval}_f(G) \equiv d_1\Sigma(H) + (d_2 - d_1)\sum_{j=1}^{y} f(v_j) \equiv d_2\Sigma(H) - (d_2 - d_1)\sum_{i=1}^{x} f(u_i).$$

■

Corollary 3.2.1.1.

(a) If $y = 1$, then $\text{Hval}_f(G) \equiv d_1\Sigma(H) + (d_2 - d_1)f(v_1)$.

(b) If $x = 1$, then $\text{Hval}_f(G) \equiv d_2\Sigma(H) - (d_2 - d_1)f(u_1)$.

(c) If $y = 1$ and $d_2 - d_1 = 1$, with d_2 even, then $\text{Hval}_f(G) \equiv d_1\Sigma(H) + f(v_1)$.

(d) If $x = 1$ and $d_2 - d_1 = 1$, with d_1 even, then $\text{Hval}_f(G) \equiv d_2\Sigma(H) - f(u_1)$.

Proof. Direct application of Theorem 3.2.1. ■

Remark 3.2.1. With respect to (c) and (d) in Corollary 3.2.1.1, note that if $y = 1$ and d_2 is odd or if $x = 1$ and d_1 is odd no graph will exist, since the sum of the degrees would be odd. Also note that evaluation of $d_1\Sigma(H)$ and $d_2\Sigma(H)$ will depend on Corollary 2.2.1.

Theorem 3.2.2. *For the path P_n of order n,* $\text{Hval}S(P_n) = H$*, except when $n = 2$ or when* H *is the direct product of* $k\text{Z}_2$*'s, here* $\text{Hval}S(P_n) = H - \{0_k\}$.

Proof. If $n = 2$, $\text{H} \cong Z_2 = \{0, 1\}$ so that any H-labeling of P_2 produces exactly one $\text{Hval}_f(P_2) = 1$.

For $n > 2$, $x = 2$, $d_1 = 1$, $y = n - 2$ and $d_2 = 2$, apply Theorem 3.2.1 to obtain $\text{Hval}_f(P_n) \equiv d_2\Sigma(H) - (d_2 - d_1)(f(u_1) + f(u_2))$, where u_1 and u_2 are the two vertices of degree 1 in P_n. By Corollary 2.2.1, $d_2\Sigma(H) \equiv 0_k$. Thus,

$$\text{Hval}_f(P_n) \equiv -(f(u_1) + f(u_2)).$$

Note that $f(u_1) + f(u_2) \equiv h$ for any $h \in H$ can be obtained by assigning f as follows,

(a) Let $f(u_1) \equiv h$ and $f(u_2) \equiv 0_k$ to get h, when $h \neq 0_k$.

(b) Let $f(u_1) \equiv h$ and $f(u_2) \equiv -h$ to get 0_k by using any $h \neq -h$.

Note that (b) cannot be satisfied if H is the direct product of $k\text{Z}_2$'s, since here $h \equiv -h$ for all h in H. Thus, with the preceding exception, by the appropriate choice of f, $\text{Hval}_f(P_n)$ can take on any value h in H. Therefore, $\text{Hval}S(P_n) = H$ and by definition $\text{Hval}_f(P_1) \equiv 0$. ■

Let $G = G(C(1), C(2), \ldots, C(s))$ denote the union of $s \geq 2$ cycles C_i of order n_i having exactly one vertex in common. Then, G has order $\sum_{i=1}^{s} n_i - (s-1)$ and size $\sum_{i=1}^{s} n_i$. Graph G has exactly one vertex of degree $2s$ and all other vertices of degree 2.

Theorem 3.2.3. *The graph* $G = G(C(1), C(2), \ldots, C(s))$ *has* $\mathrm{Hval}_f(G) \equiv 2(s-1)f(v_1)$ *for any* f *and* $\mathrm{Hval}S(G) = \{2(s-1)h : h \in H\}$.

Proof. Apply Corollary 3.2.1.1 (a) with $d_1 = 2, d_2 = 2s$, and $y = 1$ to obtain, $\mathrm{Hval}_f(G) \equiv 2(s-1)f(v_1)$, where v_1 is the vertex of degree $2s$. Since $f(v_1)$ can be assigned any value of H, we obtain $\mathrm{Hval}S(G) = \{2(s-1)h : h \in H\}$. ■

Remark 3.2.2. Note how $\mathrm{Hval}S(G(C(1), C(2), \ldots, C(s))) = \mathrm{Hval}S(G)$ depends on the group H and the value of s. For example,

(a) Let $\mathrm{H} \cong \mathrm{Z}_2 \times \cdots \times \mathrm{Z}_2$ with $k \geq 2$ factors. Then, $\mathrm{Hval}S(G) = \{0_k\}$, if and only if k is the integral solution to $\sum_{i=1}^{s} n_i - (s-1) = 2^k$.
This follows from, if k exists, then G has the same order as H and an H-labeling of G exists. By Theorem 3.2.3, $\mathrm{Hval}S(G) = \{2(s-1)h : h \in H\}$. Since every element of H has order 2, $\mathrm{Hval}S(G) = \{0_k\}$. If no k exists, then the order of G is not equal to the order of H. Thus, no H-labeling exists and $\mathrm{Hval}S(G) \neq \{0_k\}$.

(b) If $\mathrm{H} \cong \mathrm{Z}_2 \times \mathrm{Z}_9$ and $s = 2$, then
$\mathrm{Hval}S(G) = \{0_2, (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8)\}$.

(c) If $\mathrm{H} \cong \mathrm{Z}_2 \times \mathrm{Z}_9$ and $s = 4$, then $\mathrm{Hval}S(G) = \{0_2, (0,3), (0,6)\}$.

Note that when $s = 2$, then $2(s-1) = 2$ which is relatively prime to 9. But when $s = 4$, $2(s-1) = 6$ which is not relatively prime to 9. These are the conditions that determine the second coordinate in the elements of $\mathrm{Hval}S(G)$. These conditions can be extended to more general examples.

3.3. Graphs with exactly three vertex degrees

Theorem 3.3.1. *If* G *is a graph of order* n *with* x *vertices* $u_1, u_2, \ldots, u_x$ *of degree* d_1, y *vertices* $v_1, v_2, \ldots, v_y$ *of degree* d_2, *and* z *vertices* $w_1, w_2, \ldots, w_z$ *of degree* d_3, *with* $d_3 > d_2 > d_1$, *such that* $x + y + z = n$, *then for any* H*-labeling* f *of its vertices*

$$\begin{aligned}
\mathrm{Hval}_f(G) &\equiv d_1\Sigma(H) + (d_2 - d_1)\sum_{j=1}^{y} f(v_j) + (d_3 - d_1)\sum_{k=1}^{z} f(w_k) \\
&\equiv d_2\Sigma(H) - (d_2 - d_1)\sum_{i=1}^{x} f(u_i) + (d_3 - d_2)\sum_{k=1}^{z} f(w_k) \\
&\equiv d_3\Sigma(H) - (d_3 - d_1)\sum_{i=1}^{x} f(u_i) - (d_3 - d_2)\sum_{j=1}^{y} f(v_j).
\end{aligned}$$

Proof. Apply Theorem 2.3 to get

$\mathrm{Hval}_f(G) \equiv d_1 \sum_{i=1}^{x} f(u_i) + d_2 \sum_{j=1}^{y} f(v_j) + d_3 \sum_{k=1}^{z} f(w_k)$. Then, introduce

$d_1 \sum_{j=1}^{y} f(v_j) - d_1 \sum_{j=1}^{y} f(v_j) + d_1 \sum_{k=1}^{z} f(w_k) - d_1 \sum_{k=1}^{z} f(w_k)$ and rearrange to obtain

$d_1\Sigma(H) + (d_2 - d_1) \sum_{j=1}^{y} f(v_j) + (d_3 - d_1) \sum_{k=1}^{z} f(w_k)$. The expressions

$d_2\Sigma(H) - (d_2 - d_1) \sum_{i=1}^{x} f(u_i) + (d_3 - d_2) \sum_{k=1}^{z} f(w_k)$ and

$d_3\Sigma(H) - (d_3 - d_1) \sum_{i=1}^{x} f(u_i) - (d_3 - d_2) \sum_{j=1}^{y} f(v_j)$ are obtained analogously. ■

For specific values of $d_3 > d_2 > d_1$ and x, y, and z, a variety of special cases of Theorem 3.3.1 can be derived. For example, the Theorem can be applied to obtain the value set of a complete binary tree with n levels.

Corollary 3.3.1.1. *If* $d_1 = 1, d_2 = 2, d_3 = 3, x = 1, y = n - 2,$ *and* $z = 1$, *then* $\mathrm{Hval}_f(G) \equiv f(w_1) - f(u_i)$.

Proof. Apply $d_2\Sigma(H) - (d_2 - d_1) \sum_{i=1}^{x} f(u_i) + (d_3 - d_2) \sum_{k=1}^{z} f(w_k)$ in Theorem 3.3.1 and note that by Corollary 2.2.1, $2\Sigma(H) \equiv 0_k$. ■

Definition 3.3.1. A graph G is a *tadpole* (also called a *kite*) means G consists of a cycle (the *body* of G) of order at least three with a pendant path (the *tail* of G) of order at least two.

Theorem 3.3.2. *If a graph G is a tadpole* (*kite*), *then* $\mathrm{Hval}S(G) = H - \{0_k\}$.

Proof. A tadpole (kite) has degree sequence $12^{n-2}3$. Thus, by Corollary 3.3.1.1, $\mathrm{Hval}_f(G) \equiv f(w_1) - f(u_i)$. Since f is bijective, $f(w_1) - f(u_i)$ can take on any value h in H except 0_k. Thus, $\mathrm{Hval}_f(G) \equiv h$, with $h \neq 0_k$ and $\mathrm{Hval}S(G) = H - \{0_k\}$. ■

3.4. Complementary graphs

Theorem 3.4.1. *Let G be a graph of order $n = \prod_{i=1}^{k} n_i$, G^c the complement of G, and f an H-vertex labeling of G and G^c. Then,*

$$\mathrm{Hval}_f(G^c)$$

$$\equiv \begin{cases} -\mathrm{Hval}_f(G) + (2^{m_1-1}, 0, 0, \ldots, 0) & \text{if } n_1 \text{ is even and } n_i \text{ is odd for } i \geq 2, \\ -\mathrm{Hval}_f(G) & \text{otherwise.} \end{cases}$$

$$\mathrm{Hval}S(G^c)$$

$$\equiv \begin{cases} \{-h + (2^{m_1-1}, 0, 0, \ldots, 0) : h \in \mathrm{Hval}_f(G)\} & \text{if } n_1 \text{ is even and } n_i \\ & \text{is odd for } i \geq 2, \\ -\mathrm{Hval}S(G) & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.3,

$$\begin{aligned} \mathrm{Hval}_f(G^c) &\equiv \sum_{v \in V} (n-1-d(v))f(v) = (n-1) \sum_{v \in V(G)} f(v) - \sum_{v \in V(G)} d(v)f(v) \\ &\equiv (n-1)\Sigma(H) - \mathrm{Hval}_f(G). \end{aligned}$$

Then, by Corollary 2.2.1, $(n-1)\Sigma(H) \equiv 0_k$ when $n-1$ is even and by Theorem 2.2, we have the value of $\mathrm{Hval}_f(G^c)$ is as stated in the theorem. This in turn gives the value set $\mathrm{Hval}S(G^c)$ as asserted in the theorem. ■

4. Comments

The vertex/edge labeling considered here is similar to that used in studying mod sum* graphs (see p. 113 of [2]) in that every graph in this paper is a mod sum* graph, but not every mod sum* graph labeling is of the type studied here. More to the point, the problems studied here are not the same as those studied in the mod sum* graph context.

5. Open problem

In the preceding we kept both the group and the graph involved fixed.

Problem B. Given a finite abelian group H of order n. Which of the $2n-1$ non-empty subsets of H can be realized as an H-value set of some graph of order n? Equivalently, which subsets cannot be realized in this way?

It is an easy exercise to solve this problem for $1 \leq n \leq 4$ and computable for small n. It is anticipated that some interesting theorems will be obtainable when n is large.

Acknowledgements

EGD and MLG acknowledge the partial support of this work by research grants from Pace University's Dyson School of Arts and Sciences and Seidenberg School of Computer Science and Information Systems.

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Received 23 May 2008
Revised 5 January 2009
Accepted 5 January 2009