# FORBIDDEN-MINOR CHARACTERIZATION FOR THE CLASS OF GRAPHIC ELEMENT SPLITTING MATROIDS 

Kiran Dalvi*, Y.M. Borse** and M.M. Shikare**<br>*Department of Mathematics Government College of Engineering, Pune 411 005, India<br>e-mail: kiran_dalvi111@yahoo.com<br>** Department of Mathematics University of Pune, Pune 411 007, India<br>e-mail: ymborse@math.unipune.ernet.in<br>e-mail: mms@math.unipune.ernet.in


#### Abstract

This paper is based on the element splitting operation for binary matroids that was introduced by Azadi as a natural generalization of the corresponding operation in graphs. In this paper, we consider the problem of determining precisely which graphic matroids $M$ have the property that the element splitting operation, by every pair of elements on $M$ yields a graphic matroid. This problem is solved by proving that there is exactly one minor-minimal matroid that does not have this property.


Keywords: binary matroid, graphic matroid, minor, splitting operation, element splitting operation.
2000 Mathematics Subject Classification: 05B35.

## 1. Introduction

Let $M(G)$ and $M^{*}(G)$ denote the circuit matroid and the cocircuit matriod, respectively of a graph $G$. A matroid is Eulerian if its ground set can be expressed as a union of disjoint circuits of the matroid (see [14]). A matroid is bipartite if every circuit of it has an even number of elements. Welsh [14]
proved that a binary matroid is Eulerian if and only if its dual is bipartite. As the matroids $F_{7}$ and $M\left(K_{5}\right)$ are Eulerian, their dual matroids $F_{7}^{*}$ and $M^{*}\left(K_{5}\right)$ are bipartite. It is easy to see that a binary matroid $M$ is Eulerian iff the sum of column vectors of $A$ is zero where $A$ is a matrix over $G F(2)$ that represents $M$. For undefined notation and terminology in graphs and matroids, we refer [6] and [8].

Fleischner [3] defined the splitting operation for a graph with respect to a pair of adjacent edges as follows: Let $G$ be a connected graph and $v$ be a vertex of degree at least three in $G$. If $x=u v$ and $y=w v$ are two edges incident at $v$, then splitting away the pair $x, y$ from $v$ results in a new graph $G_{x, y}$ obtained from $G$ by deleting the edges $x$ and $y$, and adding a new vertex $v_{x, y}$ adjacent to $u$ and $w$. The transition from $G$ to $G_{x, y}$ is called the splitting operation on $G$. For practical purposes, we denote the new edges $v_{x, y} u$ and $v_{x, y} w$ in $G_{x, y}$ by $x$ and $y$, respectively (See Figure 1). Fleischner [3] characterized Eulerian graphs and developed an algorithm to find all distinct Eulerian trails in an Eulerian graph using the splitting operation.


Figure 1
In a similar way, Tutte [13] specified the point splitting operation for graphs as follows: Let $G$ be a graph and $v$ be a vertex of degree at least 4 in $G$. Let $H$ be the graph obtained from $G$ by replacing $v$ by two adjacent vertices $v_{1}, v_{2}$ such that each point formerly joined to $v$ is joined to exactly one of $v_{1}$ and $v_{2}$ so that in $H, \operatorname{deg} v_{1} \geq 3$ and $\operatorname{deg} v_{2} \geq 3$. We say that $H$ arises from $G$ by point-splitting operation. Tutte [13] characterized 3 -connected graphs using this operation. Later on, Slater [12] classified 4-connected graphs using $n$-point splitting operation which is a natural generalization of the point splitting operation.

Azadi [1] defined an operation which, in a sense, combines the splitting operation and the point splitting operation as follows: Let $v$ be a vertex of $G$ and let $x, y$ be distinct edges of $G$ incident at $v$. Let $G_{x, y}^{\prime}$ be the graph obtained from $G$ such that $G_{x, y}^{\prime}=G_{x, y}+v_{x, y} v$, where $G_{x, y}$ is the graph obtained from $G$ by splitting operation with respect to the edges $x$ and $y$.

Then we say that $G_{x, y}^{\prime}$ is obtained from $G$ by the element splitting operation with respect to the pair of edges $x$ and $y$ (see Figure 2).


Figure 2
Raghunathan et al. [7] extended the definition of Fleischner's splitting operation to binary matroids as follows: Let $A$ be a matrix over $G F(2)$ that represents the matroid $M$. Consider distinct elements $x$ and $y$ of $M$. Let $A_{x, y}$ be the matrix that is obtained by adjoining an extra row to $A$ with this row being zero everywhere except in the columns corresponding to $x$ and $y$ where it takes the value 1 . Suppose $M_{x, y}$ is the matroid represented by the matrix $A_{x, y}$. Then $M_{x, y}$ is said to be obtained from $M$ by splitting away the pair $x, y$. Various properties concerning the splitting matroid have been studied in $[2,7,9,10,11]$.

Azadi [1] further extended the operation of element splitting with respect to the pair of edges in graphs to binary matroids as follows: Let $A$ be a matrix over $G F(2)$ that represents the matroid $M$. Suppose that $x$ and $y$ are distinct elements of $M$. Let $A_{x, y}^{\prime}$ be the matrix that is obtained by adjoining an extra row to $A$ with this row being zero everywhere except in the columns corresponding to $x$ and $y$ where it takes the value 1 and then adjoining an extra column (corresponding to $a$ ) with this column being zero everywhere except in the last row where it takes the value 1 . Suppose $M_{x, y}^{\prime}$ is the matroid represented by the matrix $A_{x, y}^{\prime}$. Then $M_{x, y}^{\prime}$ is said to be obtained from $M$ by element splitting the pair of elements $x$ and $y$.

Alternatively, the element splitting operation can be defined in terms of circuits of binary matroids [1] as follows:

Let $M=(S, \mathcal{C})$ be a binary matroid, $\{x, y\} \subseteq S$, and $a \notin S$. Let $\mathcal{C}_{0}=\{C \in \mathcal{C}: x, y \in C$ or $x, y \notin C\}$,
$\mathcal{C}_{1}=$ set of minimal members of $\left\{C_{1} \cup C_{2}: C_{1}, C_{2} \in \mathcal{C}, C_{1} \cap C_{2}=\phi\right.$ and $x \in C_{1}, y \in C_{2}$ such that $C_{1} \cup C_{2}$ does not contain any member of $\left.\mathcal{C}_{0}\right\}$, and $\mathcal{C}_{2}=\{C \cup\{a\}: C \in \mathcal{C}$ and contains exactly one of $x$ and $y\}$.
Let $\mathcal{C}^{\prime}=\mathcal{C}_{0} \cup \mathcal{C}_{1} \cup \mathcal{C}_{2}$. Then $M_{x, y}^{\prime}=\left(S \cup\{a\}, \mathcal{C}^{\prime}\right)$.
If $x$ and $y$ are non-adjacent edges of a graph $G$, then $M(G)_{x, y}$ may not
be graphic. Shikare and Waphare [11] characterized graphic matroids whose splitting matroids are also graphic in the following theorem.

Theorem 1.1 [11]. The splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if the circuit matroid of the corresponding graph has no minor isomorphic to the circuit matroid of any of the following four graphs.


Figure 3
The element splitting operation on a graphic matroid may not yield a graphic matroid. In this paper, we obtain the forbidden-minor characterization for graphic matroids whose element splitting matroid is graphic. The main result in this paper is the following theorem.

Theorem 1.2. The element splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if it has no minor isomorphic to $M\left(K_{4}\right)$, where $K_{4}$ is the complete graph on 4 vertices.

## 2. The Element Splitting Operation and its Properties

In this section we provide necessary lemmas. We assume that $M$ is a binary matroid and $x, y$ are distinct elements of $M$.

Lemma 2.1. Let $x$ and $y$ be elements of a binary matroid $M$ and let $r(M)$ denote the rank of $M$. Then, using the notations introduced in Section 1,
(i) $M_{x, y}=M_{x, y}^{\prime} \backslash\{a\}$;
(ii) $M=M_{x, y}^{\prime} /\{a\}$;
(iii) $r\left(M_{x, y}^{\prime}\right)=r(M)+1$;
(iv) every cocircuit of $M$ is a cocircuit of the matroid $M_{x, y}^{\prime}$;
(v) if $\{x, y\}$ is a cocircuit of $M$ then $\{a\}$ and $\{x, y\}$ are cocircuits of $M_{x, y}^{\prime}$;
(vi) if $\{x, y\}$ does not contain a cocircuit, then $\{x, y, a\}$ is a cocircuit of $M_{x, y}^{\prime}$
(vii) $M_{x, y}^{\prime} \backslash x / y \cong M \backslash x$;
(viii) if $M$ is graphic and $x, y$ are adjacent edges in a corresponding graph, then $M_{x, y}^{\prime}$ is graphic;
(ix) $M_{x, y}^{\prime}$ is not eulerian.

Proof. (i), (ii), (iii), (v), (vi), (vii) and (viii) are straightforward. The proof of (iv) follows from Lemma 2.4.1 of [4]. If $A_{x, y}^{\prime}$ represents the matroid $M_{x, y}^{\prime}$, then the number of one's in the last row of $A_{x, y}^{\prime}$ is odd. Hence $M_{x, y}^{\prime}$ is not eulerian. This proves (ix).

The following result is well known.
Lemma 2.2 [6]. A binary matroid is graphic if and only if it has no minor isomorphic to $F_{7}, F_{7}^{*}, M^{*}\left(K_{5}\right)$, or $M^{*}\left(K_{3,3}\right)$.

Notation. For convenience, let $\mathcal{F}=\left\{F_{7}, F_{7}^{*}, M^{*}\left(K_{5}\right), M^{*}\left(K_{3,3}\right)\right\}$.
Lemma 2.3. Let $M$ be a graphic matroid and let $x, y \in E(M)$ such that $M_{x, y}^{\prime}$ is not graphic. Then there is a minor $N$ of $M$ such that no two elements of $N$ are in series and $N_{x, y}^{\prime} \backslash\{a\} /\{x\} \cong F$ or $N_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong F$ or $N_{x, y}^{\prime} \cong F$ or $N_{x, y}^{\prime} /\{x\} \cong F$ or $N_{x, y}^{\prime} /\{y\} \cong F$ or $N_{x, y}^{\prime} /\{x, y\} \cong F$ for some $F \in \mathcal{F}$.

Proof. Since $M_{x, y}^{\prime}$ is not graphic, $M_{x, y}^{\prime} \backslash T_{1} / T_{2} \cong F$ for some $T_{1}, T_{2} \subseteq$ $E\left(M_{x, y}^{\prime}\right)$. Let $T_{i}^{\prime}=T_{i}-\{a, x, y\}$ for $i=1,2$. Then $T_{i}^{\prime} \subseteq E(M)$ for each $i$. Let $N=M \backslash T_{1}^{\prime} / T_{2}^{\prime}$. Then $N_{x, y}^{\prime}=M_{x, y}^{\prime} \backslash T_{1}^{\prime} / T_{2}^{\prime}$. Let $T_{i}^{\prime \prime}=T_{i}-T_{i}^{\prime}$ for $i=1,2$. Then $N_{x, y}^{\prime} \backslash T_{1}^{\prime \prime} / T_{2}^{\prime \prime} \cong F$. If $a \in T_{2}^{\prime \prime}$, then $F$ is a minor of $M_{x, y}^{\prime} / a$ and hence, by Lemma $2.1(\mathrm{i}), F$ is a minor of $M$, which is a contradiction. Suppose $a \in T_{1}^{\prime \prime}$. By Lemma 2.1(i), $M_{x, y}=M_{x, y}^{\prime} \backslash a$. Hence $F$ is a minor of $M_{x, y}$. It follows from Theorem 2.3 of [11] that $N$ does not contain a 2 -cocircuit and further, $N_{x, y} / x \cong F$ or $N_{x, y} /\{x, y\} \cong F$. This implies that $N_{x, y}^{\prime} \backslash\{a\} / x \cong F$ or $N_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong F$. Suppose that $a \notin T_{1}^{\prime \prime} \cup T_{2}^{\prime \prime}$. Hence $a \notin T_{1} \cup T_{2}$. If $T_{1}^{\prime \prime} \cup T_{2}^{\prime \prime}=\phi$, then $N_{x, y}^{\prime} \cong F$. If $T_{2}^{\prime \prime}=\phi$, then $N_{x, y} \backslash x \cong F$ or $N_{x, y} \backslash y \cong F$ or $N_{x, y}^{\prime} \backslash\{x, y\} \cong F$. In the first case, $a$ forms a 2 -cocircuit with $x$ or $y$ whichever is remained, and in later case, $a$ is a coloop. It is a contradiction.

Hence $T_{2}^{\prime \prime} \neq \phi$. If $T_{1}^{\prime \prime} \neq \phi$ then, by Lemma $2.1(\mathrm{vi}), F$ is minor of $M$, which is a contradiction. Hence $T_{1}^{\prime \prime}=\phi$. Hence $N_{x, y}^{\prime} / x \cong F$ or $N_{x, y}^{\prime} / y \cong F$ or $N_{x, y}^{\prime} /\{x, y\} \cong F$.

Assume that $N$ contains a 2 -cocircuit $Q$. By Lemma $2.1(i v), Q$ is 2 cocircuit in $N_{x, y}^{\prime}$. Since $F$ is 3-connected, it does not contain a 2-cocircuit. It follows that $N_{x, y}^{\prime}$ is not isomorphic to $F$. Hence $N_{x, y}^{\prime} \backslash\{a\} / x \cong F$ or $N_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong F$ or $N_{x, y}^{\prime} /\{x\} \cong F$ or $N_{x, y}^{\prime} /\{y\} \cong F$ or $N_{x, y}^{\prime} /\{x, y\} \cong F$. If $Q \cap\{x, y\}=\phi$, then it is retained in all these cases and thus $F$ has a 2 -cocircuit, which is a contradiction. If $Q=\{x, y\}$, a contradiction follows from Lemma 2.1(v). Hence $Q$ contains exactly one of $x, y$. Suppose that $x \in Q$. Then $N_{x, y}^{\prime} / y \not \approx F$. Let $x_{1}$ be the other element of $Q$. Let $L=N / x_{1}$. Then $L$ is a minor of $M$ in which no pair of elements is in series. Further, $L_{x, y}^{\prime}=N_{x, y}^{\prime} / x_{1} \cong N_{x, y}^{\prime} / x$. Thus we have $L_{x, y}^{\prime} \backslash\{a\} \cong F$ or $L_{x, y}^{\prime} \backslash\{a\} / y \cong F$ or $L_{x, y}^{\prime} \cong F$ or $L_{x, y}^{\prime} / y \cong F$. Since $L_{x, y} \cong L_{x, y}^{\prime} \backslash\{a\}$, and $x, y$ are in series in $L_{x, y}$, it follows that $L_{x, y}^{\prime} \backslash\{a\} \not \equiv F$ and also $L_{x, y}^{\prime} \backslash\{a\} / y \cong L_{x, y}^{\prime} \backslash\{a\} / x$. If $y \in Q$, then $N_{x, y}^{\prime} / x \nsupseteq F$. Also, $L_{x, y}^{\prime} \cong N_{x, y}^{\prime} / y$. In this case we get $L_{x, y}^{\prime} \backslash\{a\} / x \cong F$ or $L_{x, y}^{\prime} \cong F$ or $L_{x, y}^{\prime} / x \cong F$.

Definition 2.4. Let $M$ be a graphic matroid in which no two elements are in series and let $F \in \mathcal{F}$. We say that $M$ is minimal with respect to $F$ if there exist two elements $x$ and $y$ of $M$ such that $M_{x, y}^{\prime} \backslash\{a\} /\{x\} \cong F$ or $M_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong F$ or $M_{x, y}^{\prime} \cong F$ or $M_{x, y}^{\prime} /\{x\} \cong F$ or $M_{x, y}^{\prime} /\{x, y\} \cong F$.

Corollary 2.5. Let $M$ be a graphic matroid. For any $x, y \in E(M)$, the matroid $M_{x, y}^{\prime}$ is graphic if and only if $M$ has no minor isomorphic to a minimal matroid with respect to any $F \in \mathcal{F}$.

Proof. If $M_{x, y}^{\prime}$ is not graphic for some $x, y$, then by Lemma 2.3, $M$ has a minor $N$ in which no two elements are in series and $N_{x, y}^{\prime} \backslash\{a\} /\{x\} \cong F$ or $N_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong F$ or $N_{x, y}^{\prime} \cong F$ or $N_{x, y}^{\prime} /\{x\} \cong F$ or $N_{x, y}^{\prime} /\{y\} \cong F$ or $N_{x, y}^{\prime} /\{x, y\} \cong F$ for some $F \in \mathcal{F}$. If $N_{x, y}^{\prime} / y \cong F$ but $N_{x, y}^{\prime} / x \not \approx F$, then interchange roles of $x$ and $y$. Conversely, suppose that $M$ has a minor $N$ isomorphic to a minimal matroid with respect to some $F \in \mathcal{F}$. Then $N_{x, y}^{\prime} \backslash\{a\}$ or $N_{x, y}^{\prime} /\{x\}$ or $N_{x, y}^{\prime} /\{x, y\}$ or $N_{x, y}^{\prime} \cong F$, for some $x, y \in E(M)$. Then $M_{x, y}^{\prime}$ has a minor isomorphic to $F$ and hence it is not graphic.

Lemma 2.6. Let $M$ be a graphic matroid corresponding to a graph $G$. If $M$ is minimal with respect to some $F \in \mathcal{F}$, then
(i) $M$ has neither loops nor coloops;
(ii) $x$ and $y$ are non-adjacent edges of $G$ and the minimum degree of $G$ is at least 3;
(iii) $x$ and $y$ cannot be parallel in $G$;
(iv) every pair of parallel edges of $G$ must contain either $x$ or $y$;
(v) if $M_{x, y}^{\prime}$ or $M_{x, y}^{\prime} /\{x\} \cong F_{7}^{*}$ or $M^{*}\left(K_{5}\right)$, then $G$ is simple;
(vi) if $M_{x, y}^{\prime} /\{x\} \cong F_{7}$ or $M^{*}\left(K_{3,3}\right)$, then $G$ is simple or has exactly one pair of parallel edges and one of these two edges must be $y$, and further there is no 3 -circuit in $G$ containing both $x$ and $y$;
(vii) if $M_{x, y}^{\prime} /\{x, y\} \cong F$ then $G$ is simple and there is no 3 -circuit or 4circuit in $G$ containing both $x$ and $y$.

Proof. (i) On the contrary, suppose $M$ has a loop, say $z$. If $z$ is different from $x$ and $y$, then it is a loop in $M_{x, y}^{\prime}$ and hence in $F$, a contradiction. If $z$ is one of the two elements $x$ and $y$, say $x$, then $M_{x, y}^{\prime} \backslash\{a\} /\{x\} \cong M \backslash\{x\}$ and $M_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong M \backslash\{x\} /\{y\}$. This implies that $F$ is a minor of $M$, a contradiction. Also, $M_{x, y}^{\prime}$ contains a 2 -circuit, so it cannot be isomorphic to $F$. Further $M_{x, y}^{\prime} /\{x\}$ and $M_{x, y}^{\prime} /\{x, y\}$ contains a loop, a contradiction. Thus, $M$ cannot have loops.

Suppose that $M$ has a coloop, say $w$. If $w$ is different from $x$ and $y$ then it is preserved in $M_{x, y}^{\prime}$ and hence in $F$, a contradiction. If $w$ is one of the two elements $x$ and $y$, say $x$, then $\{y, a\}$ is a 2 -cocircuit or $\{y\}$ is a coloop of $M_{x, y}^{\prime}$. Now, in $M_{x, y}^{\prime} \backslash\{a\} /\{x\}, y$ becomes a coloop, a contradiction. Also, $M_{x, y}^{\prime} \backslash\{a\} /\{x, y\} \cong M /\{x\} \backslash\{y\}$. This means that $F$ is a minor of $M$, a contradiction. In $M_{x, y}^{\prime},\{x\}$ remains a coloop and hence $M_{x, y}^{\prime}$ cannot be isomorphic to $F$. Moreover, in $M_{x, y}^{\prime} /\{x\},\{y, a\}$ remains a 2 -cocircuit or $\{y\}$ remains a cocircuit and in $F$, a contradiction. Also, $M_{x, y}^{\prime} /\{x, y\} \cong$ $M_{x, y}^{\prime} \backslash\{x\} /\{y\} \cong M \backslash\{x\}$, that is $F$ is a minor of $M$, a contradiction. Hence $M$ cannot have coloops.
(ii) Follows from Lemma 2.1(viii) and Lemma 2.3.
(iii) If $x$ and $y$ are parallel in $G$, then $x$ and $y$ remain parallel in $M_{x, y}^{\prime}$. So, we get a loop in $M_{x, y}^{\prime} \backslash\{a\} /\{x\}, M_{x, y}^{\prime} /\{x\}$ and a 2 -circuit in $M_{x, y}^{\prime}$, a contradiction. Also, $M_{x, y}^{\prime} \backslash\{a\} /\{x, y\}=M_{x, y} /\{x, y\}=M \backslash\{x, y\}$, a contradiction. Now, $M_{x, y}^{\prime} /\{x, y\}=M_{x, y}^{\prime} / y \backslash x \cong M \backslash x$, a contradiction to Lemma 2.1(vii). Hence these matroids are not isomorphic to $F$, a contradiction.
(iv) Suppose that the edges $x_{1}$ and $x_{2}$ are in a parallel class of $G$ that does not contain $x$ or $y$, then $x_{1}$ and $x_{2}$ remain in parallel in each of the ma-
troids $M_{x, y}^{\prime} \backslash\{a\} /\{x\}, M_{x, y}^{\prime} \backslash\{a\} /\{x, y\}, M_{x, y}^{\prime}, M_{x, y}^{\prime} /\{x\}$ and $M_{x, y}^{\prime} /\{x, y\}$, a contradiction. If $x_{1}$ and $x_{2}$ are in a parallel class containing $x$ or $y$, then we get a loop in $M_{x, y}^{\prime} \backslash\{a\} /\{x\}, M_{x, y}^{\prime} \backslash\{a\} /\{x, y\}, M_{x, y}^{\prime} /\{x\}, M_{x, y}^{\prime} /\{x, y\}$ and a 2 -circuit in $M_{x, y}^{\prime}$. Hence these matroids are not isomorphic to $F$, a contradiction.
(v) As $F_{7}^{*}$ and $M^{*}\left(K_{5}\right)$ are bipartite, if $G$ contains a pair of parallel edges then by (iv) above, it must contain $x$ or $y$. So, we get a 3 -circuit in $M_{x, y}^{\prime}$ containing $a$. Therefore $M_{x, y}^{\prime} \neq F_{7}^{*}$ or $M^{*}\left(K_{5}\right)$. Also, we get a 2 -circuit in $M_{x, y}^{\prime} /\{x\}$ and $M_{x, y}^{\prime} /\{x, y\}$, a contradiction.
(vi) Suppose that $G$ is not simple. Then by (iv) above, each pair of parallel edges must contain $x$ or $y$. If $\left\{x, x_{1}\right\}$ is a 2 -circuit for some edge $x_{1}$ of $G$, then $\left\{x, x_{1}, a\right\}$ is a 3 -circuit in $M_{x, y}^{\prime}$ and hence, $\left\{x_{1}, a\right\}$ is a 2 -circuit in $M_{x, y}^{\prime} /\{x\}$, a contradiction. Hence $G$ has exactly one pair of parallel edges and one of these two edges must be $y$.
(vii) If $G$ contains a pair of parallel edges, it must contain $x$ or $y$, say $x$. Then $M_{x, y}^{\prime}$ contains a 3 -circuit containing $x$ and $a$. Consequently, $M_{x, y}^{\prime} /\{x, y\}$ contains a 2 -circuit and hence it is in $F$, a contradiction. Now, if $G$ contains 3 or a 4 -circuit containing both $x$ and $y$, then $M_{x, y}^{\prime} /\{x, y\}$ contains a loop or 2-circuit respectively and hence it is in $F$, a contradiction.

## 3. The Element Splitting operation on Graphic Matroids

In this section, we obtain the minimal matroids corresponding to each of the four matroids $F_{7}, F_{7}^{*}, M^{*}\left(K_{3,3}\right)$ and $M^{*}\left(K_{5}\right)$ and use them to give a proof of Theorem 1.2.

In the following lemma, we characterize minimal matroids corresponding to the matroid $F_{7}$.


Figure 4

Lemma 3.1. Let $M$ be a graphic matroid. Then $M$ is minimal with respect to the matroid $F_{7}$ if and only if $M$ is isomorphic to one of the three circuit matroids $M\left(G_{1}\right), M\left(G_{2}\right)$ and $M\left(G_{3}\right)$, where $G_{1}, G_{2}$ and $G_{3}$ are the graphs of Figure 4.

Proof. Firstly, we consider the graph $G_{3}$ and prove that $M^{\prime}\left(G_{3}\right)_{x, y} /$ $\{x\} \cong F_{7}$.

Let matrices $A$ and $A_{x, y}^{\prime}$ represent the matroids $M\left(G_{3}\right)$ and $M^{\prime}\left(G_{3}\right)_{x, y}$ respectively. Then

$$
A=\left[\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & x & y \\
1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right]
$$

So, we have

$$
A_{x, y}^{\prime}=\left[\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & x & y & a \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]
$$

Therefore

$$
A_{x, y}^{\prime} /\{x\}=\left[\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & y & a \\
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1
\end{array}\right] .
$$

Hence $M^{\prime}\left(G_{3}\right)_{x, y} /\{x\} \cong F_{7}$.
One can check similarly that $M^{\prime}\left(G_{1}\right)_{x, y} \backslash\{a\} /\{x\} \cong F_{7} ; M^{\prime}\left(G_{2}\right)_{x, y} \backslash$ $\{a\} /\{x, y\} \cong F_{7}$. Thus the matroids $M\left(G_{1}\right), M\left(G_{2}\right)$ and $M\left(G_{3}\right)$ are minimal with respect to $F_{7}$.

Conversely, let $M$ be a minimal matroid with respect to $F_{7}$. Let $G$ be a graph corresponding to $M$. Let the edges $x$ and $y$ of $G$ are such that $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x\} \cong F_{7}$ or $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x, y\} \cong F_{7}$ or $M^{\prime}(G)_{x, y} \cong F_{7}$ or $M^{\prime}(G)_{x, y} /\{x\} \cong F_{7}$ or $M^{\prime}(G)_{x, y} /\{x, y\} \cong F_{7}$.

By Lemma 2.1(i), $M(G)_{x, y} /\{x\} \cong F_{7}$. If $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x\} \cong F_{7}$, then by Lemma 3.1 of [11], $G$ is isomorphic to the graph $G_{1}$ of Figure 4 . Similarly, if $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x, y\} \cong F_{7}$, then by Lemma 3.1 of [11], $G$ is isomorphic to the graph $G_{2}$ of Figure 4. Further, $M^{\prime}(G)_{x, y} \not \neq F_{7}$ because $M_{x, y}^{\prime}$ is not eulerian by Lemma 2.1( $x$ ).

Suppose that $M^{\prime}(G)_{x, y} /\{x\} \cong F_{7}$. Since $r\left(F_{7}\right)=3, r\left(M^{\prime}(G)_{x, y}\right)=4$. Further $\left|E\left(M^{\prime}(G)_{x, y}\right)\right|=8$. Consequently, $r(M(G))=3$ and $|E(M(G))|=7$. Thus, $G$ is a graph with 4 vertices and 7 edges. This implies that $G$ is nonsimple. Also, by Lemma $2.6(\mathrm{vi}), G$ has exactly one pair of parallel edges. Hence $G$ can be obtained from a simple graph with 4 vertices and 6 edges by adding an edge in parallel. Since the complete graph $K_{4}$ is the only simple graph with 4 vertices and 6 edges (see [5]), $G$ must be isomorphic to the graph $G_{3}$ of Figure 4.

Suppose that $M^{\prime}(G)_{x, y} /\{x, y\} \cong F_{7}$. Then $r\left(M^{\prime}(G)_{x, y}\right)=5$ and $\left|E\left(M^{\prime}(G)_{x, y}\right)\right|=9$. This implies that $r(M(G))=4$ and $|E(M(G))|=8$. Thus, $G$ is a graph with 5 vertices and 8 edges. Hence, by Lemma 2.6(ii), $G$ has degree sequence ( $4,3,3,3,3$ ). By Lemma $2.6(\mathrm{vii}), G$ is simple and does not have a 3 -circuit or a 4 -circuit containing both $x$ and $y$. There is only one simple graph with 5 vertices and 8 edges (see [5]) as shown in Figure 5. In this graph, any two edges are either in a 3 -circuit or in a 4 -circuit. Hence we discard this graph.


Figure 5
We characterize minimal matroids corresponding to the matroid $F_{7}^{*}$ in the following lemma.

Lemma 3.2. Let $M$ be a graphic matroid. Then $M$ is minimal with respect to the matroid $F_{7}^{*}$ if and only if $M$ is isomorphic to one of the two circuit matroids $M\left(G_{4}\right)$ and $M\left(G_{5}\right)$, where $G_{4}$ and $G_{5}$ are the graphs of Figure 6 .


Figure 6

Proof. Observe that $M^{\prime}\left(G_{4}\right)_{x, y} \backslash\{a\} /\{x\} \cong F_{7}^{*}$ and $M^{\prime}\left(G_{5}\right)_{x, y} \cong F_{7}^{*}$. Therefore $M\left(G_{4}\right)$ and $M\left(G_{5}\right)$ are minimal with respect to $F_{7}^{*}$.

Conversely, let $M(G)$ be a minimal graph with respect to $F_{7}^{*}$ and let $x$ and $y$ be edges of $G$ such that $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x\} \cong F_{7}^{*}$ or $M^{\prime}(G)_{x, y} \backslash$ $\{a\} /\{x, y\} \cong F_{7}^{*}$ or $M^{\prime}(G)_{x, y} \cong F_{7}^{*}$ or $M^{\prime}(G)_{x, y} /\{x\} \cong F_{7}^{*}$ or $M^{\prime}(G)_{x, y} /\{x, y\} \cong F_{7}^{*}$. If $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x\} \cong F_{7}^{*}$, then by Lemma 2.1(i), $M(G)_{x, y} /\{x\} \cong F_{7}^{*}$. Hence, by Lemma 3.2 of [11], $G$ is isomorphic to the graph $G_{4}$ of Figure 6. Similarly, if $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x, y\} \cong F_{7}^{*}$, then $M(G)_{x, y} /\{x, y\} \cong F_{7}^{*}$. By Lemma 3.2 of [11], there is no minimal graphic matroid such that $M(G)_{x, y} /\{x, y\} \cong F_{7}^{*}$. In each of the remaining three cases, $G$ is simple by Lemma $2.6(\mathrm{v})$.

Suppose that $M^{\prime}(G)_{x, y} \cong F_{7}^{*}$. Then $r(M(G))=r\left(M^{\prime}(G)_{x, y}\right)-1=3$. Further, $|E(M)|=7$. Since $r\left(F_{7}^{*}\right)=4, r(M(G))=3$. Consequently, $G$ is a simple graph with 4 vertices and 6 edges. Hence $G$ is isomorphic to $K_{4}$, which is the graph $G_{5}$ of Figure 6. Suppose that $M^{\prime}(G)_{x, y} /\{x\} \cong F_{7}^{*}$. Then $r(M(G))=4$ and $|E(M(G))|=7$. Hence $G$ is a graph with 5 vertices and 7 edges and has a vertex of degree less than 3, a contradiction to Lemma 2.6(ii). Finally, if $M^{\prime}(G)_{x, y} /\{x, y\} \cong F_{7}^{*}$, then $G$ has 6 vertices and 8 edges and hence a vertex of degree less than 3 , a contradiction.

The minimal matroids corresponding to the matroid $M^{*}\left(K_{3,3}\right)$ are characterized as follows.

Lemma 3.3. Let $M$ be a graphic matroid. Then $M$ is minimal with respect to the matroid $M^{*}\left(K_{3,3}\right)$ if and only if $M$ is isomorphic to one of the five circuit matroids $M\left(G_{6}\right), M\left(G_{7}\right), M\left(G_{8}\right), M\left(G_{9}\right)$ and $M\left(G_{10}\right)$, where $G_{6}, G_{7}, G_{8}, G_{9}$ and $G_{10}$ are the graphs of Figure 7.


Figure 7
Proof. Observe that $M^{\prime}\left(G_{6}\right)_{x, y} \backslash\{a\} /\{x\} \cong M^{*}\left(K_{3,3}\right) ; M^{\prime}\left(G_{7}\right)_{x, y} \backslash\{a\} /\{x\}$ $\cong M^{*}\left(K_{3,3}\right) ; M^{\prime}\left(G_{8}\right)_{x, y} \backslash\{a\} /\{x, y\} \cong M^{*}\left(K_{3,3}\right) ; M^{\prime}\left(G_{9}\right)_{x, y} \backslash\{a\} /\{x, y\} \cong$
$M^{*}\left(K_{3,3}\right)$ and $M^{\prime}\left(G_{10}\right)_{x, y} /\{x\} \cong M^{*}\left(K_{3,3}\right)$. This implies that $M\left(G_{6}\right)$, $M\left(G_{7}\right), M\left(G_{8}\right), M\left(G_{9}\right)$ and $M\left(G_{10}\right)$ are minimal matroids with respect to the matroid $M^{*}\left(K_{3,3}\right)$.

Conversely, let $M(G)$ be a minimal matroid with respect to $M^{*}\left(K_{3,3}\right)$. Let $x$ and $y$ be edges of $G$ such that $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x\} \cong M^{*}\left(K_{3,3}\right)$ or $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x, y\} \cong M^{*}\left(K_{3,3}\right)$ or $M^{\prime}(G)_{x, y} \cong M^{*}\left(K_{3,3}\right)$ or $M^{\prime}(G)_{x, y} /$ $\{x\} \cong M^{*}\left(K_{3,3}\right)$ or $M^{\prime}(G)_{x, y} /\{x, y\} \cong M^{*}\left(K_{3,3}\right)$. If $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x\} \cong$ $M^{*}\left(K_{3,3}\right)$, then by Lemma 2.1(i), $M(G)_{x, y} /\{x\} \cong M^{*}\left(K_{3,3}\right)$. Hence, by Lemma 3.3 of [11], $G$ is isomorphic to one of the two graphs $G_{6}$ and $G_{7}$ of Figure 7. If $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x, y\} \cong M^{*}\left(K_{3,3}\right)$, then by Lemma 2.1(i), $M(G)_{x, y} /\{x, y\} \cong M^{*}\left(K_{3,3}\right)$. Hence by Lemma 3.3 of [11], $G$ is isomorphic to one of the two graphs $G_{8}$ and $G_{9}$ of Figure 7. $M_{x, y}^{\prime}$ is not eulerian, by Lemma 2.1 $(x)$. Therefore $M^{\prime}(G)_{x, y} \not \neq M^{*}\left(K_{3,3}\right)$.


Figure 8
Suppose that $M^{\prime}(G)_{x, y} /\{x\} \cong M^{*}\left(K_{3,3}\right)$. Then $r(M(G))=4$ and $|E(M(G))|=9$. Consequently, $G$ is a graph with 5 vertices and 9 edges. Suppose that $G$ is simple. By [5], any simple graph with 5 vertices and 9 edges is isomorphic to the graph of Figure 8. Suppose $G$ is isomorphic to this graph. Then $G$ has two edge-disjoint 3 -cocircuits. Out of which, by Lemma 2.1(iv), at least one 3-cocircuit is preserved in $M^{\prime}(G)_{x, y} /\{x\}$ and hence it is preserved in $M^{*}\left(K_{3,3}\right)$, a contradiction. Thus $G$ is non-simple. By Lemma 2.6(vi), $G$ has exactly one pair of parallel edges. Since the degree of a vertex in $G$ is at least 3, the degree sequence of $G$ is $(6,3,3,3,3)$, $(5,4,3,3,3)$ or $(4,4,4,3,3)$. Therefore $G$ can be obtained from a simple graph with 5 vertices and 8 edges by adding an edge in parallel. There are in all 2 non-isomorphic simple graphs with 5 vertices and 8 edges (see [5]). So, there are in all 3 possibilities for $G$ as shown in Figure 9.

If $G$ is isomorphic to one of the two graphs (i) and (ii) of Figure 9, then it has two edge-disjoint 3 -cocircuits, and hence at least one of them is survived in $M^{\prime}(G)_{x, y} /\{x\} \cong M^{*}\left(K_{3,3}\right)$, a contradiction. Hence $G$ is isomorphic to third graph which is nothing but the graph $G_{10}$ of Figure 7.


Figure 9
Finally, suppose that $M^{\prime}(G)_{x, y} /\{x, y\} \cong M^{*}\left(K_{3,3}\right)$. Since $r\left(M^{*}\left(K_{3,3}\right)\right)=$ $4, r\left(M^{\prime}(G)_{x, y}\right)=6$. This shows that $r(M(G))=5$ and $|E(M(G))|=10$. Consequently, $G$ is a graph with 6 vertices and 10 edges with minimum degree at least 3 . So, the degree sequence of $G$ is $(4,4,3,3,3,3)$ or $(5,3,3,3,3,3)$. By Lemma 2.6 (vii), $G$ is simple. There are in all 4 non-isomorphic simple graphs with 6 vertices and 10 edges having the said degree sequences as shown in Figure 10 (see [5]). By Lemma 2.6(vii), $G$ does not have a 3circuit or a 4 -circuit containing both $x$ and $y$. As there are no 3 -cocircuits and 5 -cocircuits in $M^{*}\left(K_{3,3}\right)$, every such cocircuit in $G$ contains $x$ or $y$. Suppose $G$ is the graph (i) or graph (ii) of Figure 10. Then there is only one choice for $x, y$, as shown in the figure. For these choices $M^{\prime}(G)_{x, y} /\{x, y\}$ is not Eulerian, a contradiction. If $G$ is the graph (iii) or graph (iv) of Figure 10 , then we get a 3 -cocircuit or a 5 -cocircuit in $M^{\prime}(G)_{x, y} /\{x, y\}$ and hence it is in $M^{*}\left(K_{3,3}\right)$, a contradiction.


Figure 10
Finally, we characterize minimal matroids corresponding to the matroid $M^{*}\left(K_{5}\right)$ in the following lemma.

Lemma 3.4. Let $M$ be a graphic matroid. Then $M$ is minimal with respect to the matroid $M^{*}\left(K_{5}\right)$ if and only if $M$ is isomorphic to one of the three circuit matroids $M\left(G_{11}\right), M\left(G_{12}\right)$ and $M\left(G_{13}\right)$, where $G_{11}, G_{12}$ and $G_{13}$ are the graphs of Figure 11.


Figure 11
Proof. Observe that $M^{\prime}\left(G_{11}\right)_{x, y} \backslash\{a\} /\{x\} \cong M^{*}\left(K_{5}\right) ; M^{\prime}\left(G_{12}\right)_{x, y} \backslash\{a\} /$ $\{x\} \cong M^{*}\left(K_{5}\right)$ and $M^{\prime}\left(G_{13}\right)_{x, y} \cong M^{*}\left(K_{5}\right)$. Therefore $M\left(G_{11}\right), M\left(G_{12}\right)$ and $M\left(G_{13}\right)$ are minimal matroids with respect to the matroid $M^{*}\left(K_{5}\right)$.

Conversely, let $M(G)$ be a minimal matroid with respect to $M^{*}\left(K_{5}\right)$ and let $x$ and $y$ be edges of $G$ such that $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x\} \cong M^{*}\left(K_{5}\right)$ or $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x, y\} \cong M^{*}\left(K_{5}\right)$ or $M^{\prime}(G)_{x, y} \cong M^{*}\left(K_{5}\right)$ or $M^{\prime}(G)_{x, y} /\{x\} \cong$ $M^{*}\left(K_{5}\right)$ or $M^{\prime}(G)_{x, y} /\{x, y\} \cong M^{*}\left(K_{5}\right)$. If $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x\} \cong M^{*}\left(K_{5}\right)$, then by Lemma 2.1(i), $M(G)_{x, y} /\{x\} \cong M^{*}\left(K_{5}\right)$. Therefore, by Lemma 3.4 of [11], $G$ is isomorphic to one of the two graphs $G_{11}$ and $G_{12}$ of Figure 11. If $M^{\prime}(G)_{x, y} \backslash\{a\} /\{x, y\} \cong M^{*}\left(K_{5}\right)$, then $M(G)_{x, y} /\{x, y\} \cong M^{*}\left(K_{5}\right)$. By Lemma 3.4 of [11], there is no minimal graphic matroid $M(G)$ such that $M(G)_{x, y} /\{x, y\} \cong M^{*}\left(K_{5}\right)$. By Lemma 2.6(v), $G$ is simple in the remaining three cases. Suppose that $M^{\prime}(G)_{x, y} \cong M^{*}\left(K_{5}\right)$. Then $r(M(G))=5$ and $|E(M(G))|=9$. Hence, $G$ is a graph with 6 vertices and 9 edges having degree sequence ( $3,3,3,3,3,3$ ). There are only two such non-isomorphic simple graphs, (see [5]) as shown in Figure 12. In graph (i) of Figure 12,


Figure 12
for every choice of non-adjacent edges $x, y$, there is a 4 -circuit containing exactly one of $x$ and $y$. Such circuit becomes a 5 -circuit in $M_{x, y}^{\prime}$, a contradiction. Hence, the circuit matroid of this graph is not minimal with respect
to $M^{*}\left(K_{5}\right)$. Hence $G$ is isomorphic to graph (ii) of Figure 12 which is in fact the graph $G_{13}$ of Figure 11.

Suppose that $M^{\prime}(G)_{x, y} /\{x\} \cong M^{*}\left(K_{5}\right)$. Then $G$ has 7 vertices and 10 edges. Hence $G$ has a vertex of degree less than 3, a contradiction. Suppose that $M^{\prime}(G)_{x, y} /\{x, y\} \cong M^{*}\left(K_{5}\right)$. Then $G$ is a graph with 8 vertices and 11 edges. Hence $G$ has a vertex of degree less than 3, a contradiction.

Now, we prove Theorem 1.2.
Proof of Theorem 1.2. Let $M$ be a graphic matroid and let $G$ be a graph such that $M=M(G)$. On combining Corollary 2.5 and Lemmas 3.1, 3.2, 3.3 and 3.4, it follows that $M^{\prime}(G)_{x, y}$ is graphic for every pair $\{x, y\}$ of edges of $G$ if and only if $M(G)$ has no minor isomorphic to any of the matroids $M\left(G_{i}\right), i=1,2, \ldots, 13$, where the graphs $G_{i}$ are as shown in Figures 4, 6, 7 and 11. However, we have $M\left(G_{5}\right) \cong M\left(G_{1}\right) \backslash\left\{e_{2}, e_{3}\right\} \cong M\left(G_{2}\right) /\{z\} \backslash$ $\left\{e_{1}, e_{2}\right\} \cong M\left(G_{3}\right) \backslash\{5\} \cong M\left(G_{4}\right) /\{w\} \backslash\left\{e_{1}\right\} \cong M\left(G_{6}\right) /\left\{e_{6}\right\} \backslash\left\{w, e_{1}, e_{2}\right\} \cong$ $M\left(G_{7}\right) /\{z\} \backslash\left\{y, e_{4}, e_{5}\right\} \cong M\left(G_{8}\right) /\left\{z, e_{2}\right\} \backslash\left\{e_{1}, e_{3}, e_{5}\right\} \cong M\left(G_{9}\right) /\left\{x, e_{1}\right\} \backslash$ $\left\{w, e_{4}, e_{5}\right\} \cong M\left(G_{10}\right) /\{3\} \backslash\{5,7\} \cong M\left(G_{11}\right) /\left\{x, e_{4}, e_{5}\right\} \backslash\left\{w, e_{1}\right\} \cong M\left(G_{12}\right) /$ $\left\{v, z, e_{3}\right\} \backslash\left\{x, e_{1}\right\} \cong M\left(G_{13}\right) /\{1,5\} \backslash\{x\}$. Thus, $M^{\prime}(G)_{x, y}$ is graphic if and only if $M(G)$ has no minor isomorphic to the matroid $M\left(G_{5}\right)$. Observe that the graph $G_{5}$ is isomorphic to the complete graph $K_{4}$. This completes the proof of the theorem.

## References

[1] G. Azadi, Generalized splitting operation for binary matroids and related results (Ph.D. Thesis, University of Pune, 2001).
[2] Y.M. Borse, M.M. Shikare and Kiran Dalvi, Excluded-Minor characterization for the class of Cographic Splitting Matroids, Ars Combin., to appear.
[3] H. Fleischner, Eulerian Graphs and Related Topics, Part 1, Vol. 1 (North Holland, Amsterdam, 1990).
[4] A. Habib, Some new operations on matroids and related results (Ph.D. Thesis, University of Pune, 2005).
[5] F. Harary, Graph Theory (Addison-Wesley, Reading, 1969).
[6] J.G. Oxley, Matroid Theory (Oxford University Press, Oxford, 1992).
[7] T.T. Raghunathan, M.M. Shikare and B.N. Waphare, Splitting in a binary matroid, Discrete Math. 184 (1998) 267-271.
[8] A. Recski, Matroid Theory and Its Applications (Springer Verlag, Berlin, 1989).
[9] M.M. Shikare and G. Azadi, Determination of the bases of a splitting matroid, European J. Combin. 24 (2003) 45-52.
[10] M.M. Shikare, Splitting lemma for binary matroids, Southeast Asian Bull. Math. 32 (2007) 151-159.
[11] M.M. Shikare and B.N. Waphare, Excluded-Minors for the class of graphic splitting matroids, Ars Combin., to appear.
[12] P.J. Slater, A classification of 4-connected graphs, J. Combin. Theory 17 (1974) 281-298.
[13] W.T. Tutte, A theory of 3-connected graphs, Indag. Math. 23 (1961) 441-455.
[14] D.J.A. Welsh, Matroid Theory (Academic Press, London, 1976).
Received 15 October 2008
Revised 17 December 2008
Accepted 17 December 2008

