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# BOUNDS ON THE GLOBAL OFFENSIVE *k*-ALLIANCE NUMBER IN GRAPHS

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## Abstract

Let G = (V(G), E(G)) be a graph, and let  $k \geq 1$  be an integer. A set  $S \subseteq V(G)$  is called a *global offensive k-alliance* if  $|N(v) \cap S| \geq |N(v) - S| + k$  for every  $v \in V(G) - S$ , where N(v) is the neighborhood of v. The global offensive k-alliance number  $\gamma_o^k(G)$  is the minimum cardinality of a global offensive k-alliance in G. We present different bounds on  $\gamma_o^k(G)$  in terms of order, maximum degree, independence number, chromatic number and minimum degree.

**Keywords:** global offensive k-alliance number, independence number, chromatic number.

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## 1. TERMINOLGY

Let G = (V, E) = (V(G), E(G)) be a finite and simple graph. The open neighborhood of a vertex  $v \in V$  is  $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is  $N_G[v] = N[v] = N(v) \cup \{v\}$ . The degree of v, denoted by  $d_G(v)$ , is |N(v)|. By n(G) = n,  $\Delta(G) = \Delta$  and  $\delta(G) = \delta$ we denote the order, the maximum degree and the minimum degree of the graph G, respectively. If  $A \subseteq V(G)$ , then G[A] is the graph induced by the vertex set A. We denote by  $K_n$  the complete graph of order n, and by  $K_{r,s}$ the complete bipartite graph with partite sets X and Y such that |X| = rand |Y| = s. A set  $D \subseteq V(G)$  is a k-dominating set of G if every vertex of V(G) - D has at least  $k \ge 1$  neighbors in D. The k-domination number  $\gamma_k(G)$  is the cardinality of a minimum k-dominating set. The case k = 1leads to the classical domination number  $\gamma(G) = \gamma_1(G)$ .

In [11], Kristiansen, Hedetniemi and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive k-alliances, given by Shafique and Dutton [14, 15]. A set S of vertices of a graph G is called a global offensive k-alliance if  $|N(v) \cap S| \ge |N(v) - S| + k$  for every  $v \in V(G) - S$ , where  $k \ge 1$  is an integer. The global offensive k-alliance in G. If S is a global k-offensive alliance of G and  $|S| = \gamma_o^k(G)$ , then we say that S is a  $\gamma_o^k(G)$ -set. A global offensive 1-alliance is a global offensive alliance. In [7], Fernau, Rodríguez and Sigarreta show that the problem of finding optimal global offensive k-alliances is NP-complete.

If  $k \geq 1$  is an integer, then let  $L_k(G) = \{x \in V(G) : d_G(x) \leq k-1\}$ . Denote by  $\alpha(G)$  the *independence number*, by  $\chi(G)$  the *chromatic number*, and by  $\omega(G)$  the *clique number* of G, respectively. The corona graph  $G \circ K_1$ of a graph G is the graph constructed from a copy of G, where for each vertex  $v \in V(G)$ , a new vertex v' and a pendant edge vv' are added. Next assume that  $G_1$  and  $G_2$  are two graphs with disjoint vertex sets. The *union*  $G = G_1 \cup G_2$  has  $V(G) = V(G_1) \cup V(G_2)$  and  $E(G) = E(G_1) \cup E(G_2)$ . The *join*  $G = G_1 + G_2$  has  $V(G) = V(G_1) \cup V(G_2)$  and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

## 2. Upper Bounds

We begin this section by giving an upper bound on the global offensive k-alliance number for an *r*-partite graph *G* in terms of its order and  $|L_k(G)|$ .

**Theorem 1.** Let  $k \ge 1$  be an integer. If G is an r-partite graph, then

$$\gamma_o^k(G) \le \frac{(r-1)n(G) + |L_k(G)|}{r}.$$

**Proof.** Clearly, the set  $L_k(G)$  is contained in every  $\gamma_o^k(G)$ -set. In the case that  $|L_k(G)| = |V(G)|$ , we are finished. In the remaining case that  $|L_k(G)| < |V(G)|$ , let  $V_1, V_2, \ldots, V_r$  be a partition of the *r*-partite graph  $G - L_k(G)$  such that  $|V_1| \ge |V_2| \ge \cdots \ge |V_r|$ , where  $V_i = \emptyset$  is possible for  $i \ge 2$ . Then every vertex of  $V_1$  has degree at least k in G, and all its neighbors are in  $V(G) - V_1$ . Thus  $V(G) - V_1$  is a global offensive k-alliance of G. Since

$$|V_1| \ge \frac{|V_1| + |V_2| + \dots + |V_r|}{r} = \frac{n(G) - |L_k(G)|}{r},$$

we obtain

$$\gamma_o^k(G) \le n(G) - |V_1| \le n(G) - \frac{n(G) - |L_k(G)|}{r} = \frac{(r-1)n(G) + |L_k(G)|}{r},$$

and the proof is complete.

The case k = r = 2 in Theorem 1 leads to the next result.

**Corollary 2** (Chellali [4]). If G is a bipartite graph, then

$$\gamma_o^2(G) \le \frac{n(G) + |L_2(G)|}{2}$$

**Observation 3.** If  $k \ge 1$  is an integer, then  $\gamma_o^k(G) \ge \gamma_k(G)$  for any graph G.

**Proof.** If S is any  $\gamma_o^k(G)$ -set, then every vertex of V(G) - S has at least k neighbors in S. Thus S is a k-dominating set of G and so  $\gamma_k(G) \leq |S| = \gamma_o^k(G)$ .

Using Theorem 1 for r = 2 and Observation 3, we obtain the known theorem by Blidia, Chellali and Volkmann [2].

**Corollary 4** (Blidia, Chellali, Volkmann [2] 2006). Let k be a positive integer. If G is a bipartite graph, then

$$\gamma_k(G) \le \frac{n(G) + |L_k(G)|}{2}.$$

Since every graph G is  $\chi(G)$ -partite and  $n(G) \leq \chi(G)\alpha(G)$ , we obtain also the following corollaries from Theorem 1.

**Corollary 5.** If G is a graph and k a positive integer, then

$$\gamma_o^k(G) \le \frac{(\chi(G) - 1)n(G) + |L_k(G)|}{\chi(G)}.$$

**Corollary 6.** Let  $k \ge 1$  be an integer. If G is a graph with  $\delta(G) \ge k$ , then

$$\gamma_o^k(G) \le (\chi(G) - 1)\alpha(G).$$

**Theorem 7** (Brooks [3] 1941). If G is a connected graph different from the complete graph and from a cycle of odd length, then  $\chi(G) \leq \Delta(G)$ .

Combining Brooks' Theorem and Corollary 6, we can prove the following result.

**Theorem 8.** Let  $k \ge 1$  be an integer, and let G be a connected graph with  $\delta(G) \ge k$ . Then

(1) 
$$\gamma_{\alpha}^{k}(G) \leq (\Delta(G) - 1)\alpha(G)$$

if and only if G is neither isomorphic to the complete graphs  $K_{k+1}$  or  $K_{k+2}$ nor to a cycle of odd length when  $1 \le k \le 2$ .

**Proof.** If G is the complete graph  $K_n$ , then  $\Delta(G) = \delta(G) = n - 1 \ge k \ge 1$ and  $\alpha(G) = 1$ . Since  $\gamma_o^k(K_{k+1}) = k$  and  $\gamma_o^k(K_{k+2}) = k + 1$ , inequality (1) is not true for these two complete graphs. However, in the remaining case that  $n \ge k+3$ , we observe that  $\gamma_o^k(G) \le n-2$ , and we arrive at the desired bound

$$\gamma_o^k(G) \le n - 2 = \Delta(G) - 1 = (\Delta(G) - 1)\alpha(G).$$

Assume next that  $1 \le k \le 2$ . If G is a cycle of odd length, then  $\Delta(G) = 2$ ,  $\gamma_o^1(G) = \gamma_o^2(G) = \lceil n(G)/2 \rceil$  and  $\alpha(G) = \lfloor n(G)/2 \rfloor$  and thus (1) is not valid in these cases.

For all other graphs inequality (1) follows directly from Brooks' Theorem and Corollary 6.

**Lemma 9** (Hansberg, Meierling, Volkmann [10]). Let  $k \ge 1$  be an integer. If G is a connected graph with  $\delta(G) \le k - 1$  and  $\Delta(G) \le k$ , then

$$k\alpha(G) \ge n(G).$$

**Theorem 10.** Let  $k \ge 1$  be an integer. If G is a connected r-partite graph with  $\Delta(G) \ge k$ , then

$$\gamma_o^k(G) \le \frac{\alpha(G)}{r}((r-1)r+k-1).$$

**Proof.** Assume that k = 1. Since G is connected and  $\Delta(G) \ge 1$ , we note that  $|L_1(G)| = 0$ . Applying Theorem 1, and using the fact that  $r\alpha(G) \ge n(G)$ , we receive the desired inequality immediately.

Assume next that  $k \geq 2$ . Since G is connected and  $G - L_k(G)$  is not empty, every component Q of  $G[L_k(G)]$  fufills  $\delta(Q) \leq k-2$  and  $\Delta(Q) \leq k-1$ . Hence Lemma 9 implies  $(k-1)\alpha(Q) \geq n(Q)$ . If  $Q_1, Q_2, \ldots, Q_t$  are the components of  $G[L_k(G)]$ , we therefore deduce that

$$\alpha(G) \ge \alpha(G[L_k(G)]) = \sum_{i=1}^t \alpha(Q_i) \ge \frac{|L_k(G)|}{k-1}.$$

Combining  $n(G) \leq r\alpha(G)$  with Theorem 1, we receive the desired inequality as follows:

$$\gamma_o^k(G) \leq \frac{(r-1)n(G) + |L_k(G)|}{r}$$
$$\leq \frac{(r-1)r\alpha(G) + (k-1)\alpha(G)}{r}$$
$$= \frac{\alpha(G)}{r}((r-1)r + k - 1).$$

The case r = 2 in Theorem 10 leads to the next result.

**Corollary 11.** Let  $k \ge 1$  be an integer. If G is a connected bipartite graph with  $\Delta(G) \ge k$ , then

$$\gamma_o^k(G) \le \frac{(k+1)\alpha(G)}{2}.$$

Using Observation 3, we obtain the following known bounds on the 2domination number.

**Corollary 12** (Fujisawa, Hansberg, Kubo, Saito, Sugita, Volkmann [9] 2008). If G is a connected bipartite graph of order at least 3, then

$$\gamma_2(G) \le \frac{3\alpha(G)}{2}.$$

**Corollary 13** (Blidia, Chellali, Favaron [1] 2005). If T is a tree of order at least 3, then

$$\gamma_2(T) \le \frac{3\alpha(T)}{2}.$$

**Theorem 14** (Favaron, Hansberg, Volkmann [6] 2008). Let G be a graph. If  $r \geq 1$  is an integer, then there is a partition  $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$  of V(G) such that

(2) 
$$|N_G(u) \cap V_i| \le \frac{d_G(u)}{r}$$

for each  $i \in \{1, 2, \ldots, r\}$  and each  $u \in V_i$ .

**Theorem 15.** Let  $k \ge 1$  be an integer. If G is a graph of order n with minimum degree  $\delta \ge k$ , then

(3) 
$$\gamma_o^k(G) \le \frac{k+1}{k+2}n,$$

and the bound given in (3) is best possible.

**Proof.** Choose r = k + 2 in Theorem 14, and let  $V_1, V_2, \ldots, V_r$  be a partition of V(G) as in Theorem 14 such that  $|V_1| \ge |V_2| \ge \cdots \ge |V_r|$ . If  $D = V_2 \cup V_3 \cup \cdots \cup V_r$ , then it follows from (2) and the hypothesis that  $\delta \ge k$  for each  $v \in V_1 = V(G) - D$  that

$$|N_G(v) \cap D| \ge \left\lceil \frac{k+1}{k+2} d_G(v) \right\rceil \ge \left\lfloor \frac{d_G(v)}{k+2} \right\rfloor + k$$
$$\ge |N_G(v) \cap V_1| + k = |N_G(v) - D| + k.$$

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Thus D is a global offensive k-alliance of G such that  $|D| \le (k+1)n/(k+2)$ , and (3) is proved.

Let *H* be a connected graph, and let  $G_k = H \circ K_{k+1}$ . Then it is easy to see that  $\gamma_o^k(G_k) = (k+1)n(G_k)/(k+2)$ , and therefore (3) is the best possible.

**Corollary 16** (Favaron, Fricke, Goddard, Hedetniemi, Hedetniemi, Kristiansen, Laskar, Skaggs [5] 2004). Let G be graph of order n and minimum degree  $\delta$ .

If  $\delta \geq 1$ , then  $\gamma_o^1(G) \leq 2n/3$ . If  $\delta \geq 2$ , then  $\gamma_o^2(G) \leq 3n/4$ .

In the case that  $\delta \ge k+2$ , we obtain the following bound, improving the bound of Theorem 15.

**Theorem 17.** Let  $k \ge 2$  be an integer, and let G be a graph of order n with minimum degree  $\delta \ge k+2$ . Then

(4) 
$$\gamma_o^k(G) \le \frac{k}{k+1}n.$$

**Proof.** Choose r = k + 1 in Theorem 14, and let  $V_1, V_2, \ldots, V_r$  be a partition of V(G) as in Theorem 14 such that  $|V_1| \ge |V_2| \ge \cdots \ge |V_r|$ . If  $D = V_2 \cup V_3 \cup \cdots \cup V_r$ , then it follows from (2) and the hypothesis  $\delta \ge k + 2$  for each  $v \in V_1 = V(G) - D$  that

$$|N_G(v) \cap D| \ge \left\lceil \frac{k}{k+1} d_G(v) \right\rceil \ge \left\lfloor \frac{d_G(v)}{k+1} \right\rfloor + k$$
$$\ge |N_G(v) \cap V_1| + k = |N_G(v) - D| + k.$$

Thus D is a global offensive k-alliance of G such that  $|D| \le kn/(k+1)$ , and (4) is proved.

**Theorem 18.** Let  $k \geq 1$  be an integer, and let G be a connected noncomplete graph such that  $\delta(G) \geq k$  and  $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$ . Then  $\Delta(G) \leq k + 2$ ,  $\Delta(G) - \delta(G) \leq 1$  and if  $k \geq 2$ , then  $\delta(G) \leq k + 1$ .

**Proof.** Because of  $\chi(G)\alpha(G) \ge n(G)$ , Corollary 5 and the hypothesis imply that

$$(\Delta(G)-1)\alpha(G) = \gamma_o^k(G) \le \frac{(\chi(G)-1)n(G)}{\chi(G)} \le (\chi(G)-1)\alpha(G).$$

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Since G is neither a complete graph nor a cycle of odd length, it follows from Brooks' Theorem that  $\Delta(G) = \chi(G), \chi(G)\alpha(G) = n(G)$  and

(5) 
$$\gamma_o^k(G) = \frac{(\chi(G) - 1)n(G)}{\chi(G)} = \frac{(\Delta(G) - 1)n(G)}{\Delta(G)}.$$

If we suppose on the contrary that  $\Delta(G) \ge k+3$ , then it follows from (5) and Theorem 15 that

$$\frac{\Delta(G)-1}{\Delta(G)}n(G) = \gamma_o^k(G) \le \frac{k+1}{k+2}n(G) \le \frac{\Delta(G)-2}{\Delta(G)-1}n(G).$$

This contradiction shows that  $\Delta(G) \leq k+2$ .

If we suppose on the contrary that  $\Delta(G) - \delta(G) \geq 2$ , then we deduce that  $\delta(G) = k$  and  $\Delta(G) = k + 2 = \chi(G)$ . Since  $\chi(G)\alpha(G) = n(G)$ , there exists a partition of V(G) in  $\chi = \chi(G)$  colour classes  $U_1, U_2, \ldots, U_{\chi}$  such that  $|U_1| = |U_2| = \cdots = |U_{\chi}| = \alpha(G)$ . Let v be a vertex of minimum degree  $\delta(G) = k$ , and assume, without loss of generality, that  $v \in U_1$ . As  $d_G(v) = k$ and  $\chi(G) = k + 2$ , there exists a colour class  $U_j$  with  $2 \leq j \leq \chi$  such that v is not adjacent to any vertex in  $U_j$ . Therefore  $U_j \cup \{v\}$  is an independent set. This is a contradiction to the fact that  $|U_j| = \alpha(G)$ , and the desired inequality  $\Delta(G) - \delta(G) \leq 1$  is proved.

Next assume that  $k \ge 2$ , and suppose on the contrary that  $\delta(G) \ge k+2$ . Then  $k \le \Delta(G) - 2$  and (5) and Theorem 17 lead to the contradiction

$$\frac{\Delta(G)-1}{\Delta(G)}n(G) = \gamma_o^k(G) \le \frac{k}{k+1}n(G) \le \frac{\Delta(G)-2}{\Delta(G)-1}n(G).$$

Thus  $\delta(G) \leq k \leq \delta(G) + 1$  when  $k \geq 2$ , and the proof of Theorem 18 is complete.

**Example 19.** 1. Let  $H_1, H_2, \ldots, H_t$  be  $t \ge 2$  copies of the complete graph  $K_{k+1}$ , and let  $u_i, v_i \in E(H_i)$  for  $1 \le i \le t$ . Define the graph G as the disjoint union  $H_1 \cup H_2 \cup \cdots \cup H_t$  together with the edge set  $\{v_1u_2, v_2u_3, \ldots, v_{t-1}u_t\}$ . Then it is easy to verify that  $\Delta(G) = k + 1, \, \delta(G) = k, \, \alpha(G) = t, \, \gamma_o^k(G) = tk$  and thus  $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$ .

2. Let  $F_1$  and  $F_2$  be 2 copies of the complete graph  $K_{k+1}$  with the vertex sets  $V(F_1) = \{x_1, x_2, \ldots, x_{k+1}\}$  and  $V(F_2) = \{y_1, y_2, \ldots, y_{k+1}\}$ . Define the graph H as the disjoint union  $F_1 \cup F_2$  together wit the edge set  $\{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ . If  $H_1, H_2, \ldots, H_t$  are  $t \geq 2$  copies of H, then let

 $u_{2i-1}$  and  $u_{2i}$  be the vertices of degree k in  $H_i$  for all  $i \in \{1, 2, \ldots, t\}$ . Define the graph G as the disjoint union  $H_1 \cup H_2 \cup \cdots \cup H_t$  together with the edge set  $\{u_2u_3, u_4u_5, \ldots, u_{2t}u_1\}$ . Then G is a (k + 1)-regular graph with  $\alpha(G) = 2t, \gamma_o^k(G) = 2kt$  and thus  $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$ .

3. Let  $k \geq 2$ , and let  $F_1$  and  $F_2$  be 2 copies of the complete graph  $K_k$  such that  $V(F_1) = \{x_1, x_2, \ldots, x_k\}$  and  $V(F_2) = \{y_1, y_2, \ldots, y_k\}$ . Define the graph H as the disjoint union  $F_1 \cup F_2$  together wit the edge set  $\{x_1y_1, x_2y_2, \ldots, x_{k-1}y_{k-1}\}$ . If  $H_1, H_2, \ldots, H_t$  are  $t \geq 2$  copies of H, then let  $u_{2i-1}$  and  $u_{2i}$  be the vertices of degree k-1 in  $H_i$  for all  $i \in \{1, 2, \ldots, t\}$ . Define the graph G as the disjoint union  $H_1 \cup H_2 \cup \cdots \cup H_t$  together with the edge set  $\{u_2u_3, u_4u_5, \ldots, u_{2t}u_1\}$ . Then G is a k-regular graph with  $\alpha(G) = 2t$ ,  $\gamma_o^k(G) = 2(k-1)t$  and thus  $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$ .

4. Let  $H_1$  and  $H_2$  be 2 copies of the complete graph  $K_{k+2}$ , and let  $x \in E(H_1)$  and  $y \in E(H_2)$ . Define the graph G' as the disjoint union  $H_1 \cup H_2$  together with the edge xy. Then  $\Delta(G') = k + 2$ ,  $\delta(G') = k + 1$ ,  $\alpha(G') = 2$ ,  $\gamma_o^k(G') = 2(k+1)$  and thus  $\gamma_o^k(G') = (\Delta(G') - 1)\alpha(G')$ .

These four examples show that  $\Delta = k + 1$  and  $\delta = k$ ,  $\Delta = \delta = k + 1$ ,  $\Delta = \delta = k$  as well as  $\Delta = k + 2$  and  $\delta = k + 1$  in Theorem 18 are possible.

**Theorem 20.** If G is a graph and k an integer such that  $1 \le k \le \delta(G) - 1$ , then

$$\gamma_o^{k+1}(G) \le \frac{\gamma_o^k(G) + n(G)}{2}.$$

**Proof.** Let S be a  $\gamma_o^k(G)$ -set, and let A be the set of isolated vertices in the subgraph induced by the vertex set V(G) - S. Then the subgraph induced by  $V(G) - (S \cup A)$  contains no isolated vertices. If D is a minimum dominating set of  $G[V(G) - (S \cup A)]$ , then the well-known inequality of Ore [12] implies

$$|D| \le \frac{|V(G) - (S \cup A)|}{2} \le \frac{|V(G) - S|}{2} = \frac{n(G) - \gamma_o^k(G)}{2}.$$

Since  $\delta(G) \ge k+1$ , every vertex of A has at least k+1 neighbors in S, and therefore  $D \cup S$  is a global offensive (k+1)-alliance of G. Thus we obtain the desired bound as follows:

$$\gamma_o^{k+1}(G) \le |S \cup D| \le \gamma_o^k(G) + \frac{n(G) - \gamma_o^k(G)}{2} = \frac{\gamma_o^k(G) + n(G)}{2}.$$

The graphs G of even order and without isolated vertices with  $\gamma(G) = n/2$  have been characterized independently by Payan and Xuong [13] and Fink, Jacobson, Kinch and Roberts [8].

**Theorem 21** (Payan, Xuong [13] 1982 and Fink, Jacobson, Kinch, Roberts [8] 1985). Let G be a graph of even order n without isolated vertices. Then  $\gamma(G) = n/2$  if and only if each component of G is either a cycle  $C_4$  or the corona of a connected graph.

A graph is  $P_4$ -free if and only if it contains no induced subgraph isomorphic to the path  $P_4$  of order four. A graph is  $(K_4 - e)$ -free if and only if it contains no induced subgraph isomorphic to the graph  $K_4 - e$ , where e is an arbitrary edge of the complete graph  $K_4$ . The graph  $\overline{G}$  denotes the complement of the graph G. Next we give a characterization of some special graphs attaining equality in Theorem 20.

**Theorem 22.** Let G be a connected  $P_4$ -free graph such that  $\overline{G}$  is  $(K_4 - e)$ -free. If k is an integer with  $1 \le k \le \delta(G) - 1$ , then  $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$  if and only if

- 1.  $G = K_{k+3}$  or
- 2.  $\overline{G} = H \cup 2K_{1,1}$  such that n(H) = k + 2 and all components of H are isomorphic to  $K_{1,1}$ , to  $K_{3,3}$ , to  $K_{3,4}$  or to  $K_{4,4}$  or
- G = (Q<sub>1</sub>∪Q<sub>2</sub>)+F, where Q<sub>1</sub>, Q<sub>2</sub> and F are three pairwise disjoint graphs such that 1 ≤ |V(F)| ≤ k + 1, α(F) ≤ 2, and Q<sub>1</sub> and Q<sub>2</sub> are cliques with |V(Q<sub>1</sub>)| = |V(Q<sub>2</sub>)| = k + 3 |V(F)| such that |V(F)| ≤ 2 or α(F) = 1 and |V(F)| = k + 1 or α(F) = 2 and F = K<sub>k+1</sub> M, where M is a matching of F or α(F) = 2 and F = K<sub>k</sub> M, where M is a perfect matching of F or α(F) = 2 and |V(F)| = k + 1 t for 0 ≤ t ≤ k 2 with k ≥ 3t + 3 and all components of F are isomorphic to K<sub>t+2,t+2</sub>, to K<sub>t+2,t+3</sub> or to K<sub>t+3,t+3</sub>.

**Proof.** Assume that  $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$ . Following the same notation as used in the proof of Theorem 20, we obtain  $|D| = \frac{|V(G)-S|}{2}$ , and we observe that  $S \cup D$  is a  $\gamma_o^{k+1}(G)$ -set. It follows that G[V(G) - S] has no isolated vertices and so by Theorem 21, each component of G[V(G) - S] is either a cycle  $C_4$  or the corona of some connected graph. Using the

hypothesis that G is  $P_4$ -free, we deduce that each component of G[V(G) - S]is isomorphic to  $K_2$  or to  $C_4$ . Since  $\overline{G}$  is  $(K_4 - e)$ -free, there remain exactly the three cases that G[V(G) - S] is isomorphic to  $K_2$ , to  $C_4$  or to  $2K_2$ .

Case 1. First assume that  $G[V(G) - S] = K_2$ . Suppose that G has an independent set Q of size at least two. Then the hypothesis  $\delta(G) \ge k + 1$  implies that V(G) - Q is a global offensive (k + 1)-alliance of G of size  $n - |Q| < |S \cup D| = n - 1$ , a contradiction. Therefore  $\alpha(G) = 1$  and thus  $G = K_{k+3}$ .

Case 2. Second assume that G[V(G)-S] is a cycle  $C_4 = x_0x_1x_2x_3x_0$ . In the following the indices of the vertices  $x_i$  are taken modulo 4. Recall that  $S \cup D$  is a  $\gamma_o^{k+1}(G)$ -set, and D contains two vertices of the cycle  $C_4$ . Clearly, since S is a  $\gamma_o^k(G)$ -set, every vertex of the cycle  $C_4$  has degree at least k+4. Suppose that  $d_G(x_i) \ge k+5$  for an  $i \in \{0,1,2,3\}$ . Then  $\{x_{i+2}\} \cup S$  is a global offensive (k+1)-alliance of G of size  $|S|+1 < |S \cup D| = |S|+2$ , a contradiction. Thus  $d_G(x_i) = k+4$  for every  $i \in \{0,1,2,3\}$ . Now if Qis an  $\alpha(G)$ -set, then  $|Q| \le 2$ , for otherwise the hypothesis  $\delta(G) \ge k+1$ implies that V(G) - Q is a global offensive (k+1)-alliance of G of size  $|V(G) - Q| < |S \cup D| = n(G) - 2$ , a contradiction too. Since there are two non-adjacent vertices on the cycle  $C_4$  and G is  $P_4$ -free, it follows that every vertex of S has at least three neighbors on the cycle  $C_4$ .

Subcase 2.1. Assume that  $\alpha(G[S]) = 1$ . Then the subgraph induced by S is complete and  $|S| \ge k + 2$ . If |S| = k + 2, then we observe that every vertex of S has exactly four neighbours on the cycle  $C_4$ . Thus, in each case, we deduce that  $d_G(y) \ge k + 5$  for every  $y \in S$ . But then for any subset W of S of size three, the set V(G) - W is a global offensive (k + 1)-alliance of G of size less than  $|S \cup D|$ , a contradiction.

Subcase 2.2. Assume that  $\alpha(G[S]) = 2$ . Suppose that there exists a vertex  $w \in S$  with at least k+1 neighbors in S. Then, since  $|N(w) \cap V(C_4)| \geq 3$ , say  $\{x_0, x_1, x_2\} \subseteq N(w) \cap V(C_4)$ , we observe that  $(S - \{w\}) \cup \{x_0, x_2\}$  is a global offensive (k+1)-alliance of G of size  $|S|+1 < |S \cup D|$ , a contradiction. Thus every vertex of S has at most k neighbors in S.

Let  $S = X \cup Y$  such that every vertex of X has exactly three and every vertex of Y exactly 4 neighbors on  $C_4$ . We shall show that  $X = \emptyset$ . If  $X \neq \emptyset$ , then let  $S_{x_i} \subseteq X$  be the set of vertices such that each vertex of  $S_{x_i}$  is not adjacent to  $x_{i+2}$  for  $i \in \{0, 1, 2, 3\}$ . Because of  $\alpha(G) = 2$ , we observe that the set  $S_{x_i} \cup \{x_i\}$  induces a complete graph for each  $i \in \{0, 1, 2, 3\}$ . In additon, since G is  $P_4$ -free it is straightforward to verify that all vertices of  $X \cup C_4$  are adjacent to all vertices of Y and that  $S_{x_i} \cup S_{x_{i+1}} \cup \{x_i, x_{i+1}\}$ induces a complete graph for each  $i \in \{0, 1, 2, 3\}$ . Now assume, without loss of generality, that  $S_{x_0} \neq \emptyset$ , and let  $w \in S_{x_0}$ . On the one hand we have seen above that  $d_G(w) \leq k+3$ . On the other hand, we observe that  $d_G(w) = d_G(x_0)$ . But since  $d_G(x_0) = k + 4$ , we have a contradiction.

Hence we have shown that  $X = \emptyset$ , and this leads to |S| = k + 2. If we define  $H = \overline{G[S]}$ , then  $\omega(H) = 2$ ,  $\delta(H) \ge 1$  and  $\Delta(H) \le 4$ . Since H is also  $P_4$ -free, H does not contain an induced cycle of odd length. Using  $\omega(H) = 2$ , we deduce that H is a bipartite graph. Now let  $H_i$  be a component of H. If  $H_i$  is not a complete bipartite graph, then  $H_i$  contains a  $P_4$ , a contradiction. Thus the components of H consists of  $K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}, K_{2,2}, K_{2,3}, K_{2,4}, K_{3,3}, K_{3,4}$  or  $K_{4,4}$ .

If  $K_{1,2}$  is a component of H, then  $V(G) - V(K_{1,2})$  is a global offensive (k+1)-alliance of G of size n-3, a contradiction.

If  $K_{1,3}$  is a component of H with a leaf u, then  $(V(G) - V(K_{1,3})) \cup \{u\}$  is a global offensive (k + 1)-alliance of G of size n - 3, a contradiction.

If  $K_{1,4}$  is a component of H and u, v are two leaves of this star, then  $(V(G) - V(K_{1,3})) \cup \{u, v\}$  is a global offensive (k + 1)-alliance of G of size n-3, a contradiction.

If  $K_{2,2}$  is a component of H, then  $V(G) - V(K_{2,2})$  is a global offensive (k+1)-alliance of G of size n-4, a contradiction.

Next let  $K_{2,3}$  be a component of H with the bipartition  $\{v_1, v_2, v_3\}$  and  $\{u_1, u_2\}$ . Then  $V(G) - \{u_1, v_1, v_2\}$  is a global offensive (k + 1)-alliance of G of size n - 3, a contradiction.

Finally, let  $K_{2,4}$  be a component of H with the bipartition  $\{v_1, v_2, v_3, v_4\}$ and  $\{u_1, u_2\}$ . Then  $V(G) - \{u_1, v_1, v_2\}$  is a global offensive (k + 1)-alliance of G of size n - 3, a contradiction.

Case 3. Third assume that  $G[V(G) - S] = 2K_2$ . Let  $2K_2 = J_1 \cup J_2 = J$ such that  $V(J_1) = \{u_1, u_2\}$  and  $V(J_2) = \{u_3, u_4\}$ . If  $\alpha(G) \ge 3$ , then we obtain the contradiction  $\gamma_o^{k+1}(G) \le n-3$ . Thus  $\alpha(G) = 2$ . Since S is a  $\gamma_o^k(G)$ -set, every vertex of J has degree at least k + 2. Suppose that  $d_G(u_1) \ge k + 3$  and  $d_G(u_2) \ge k + 3$ . Then  $\{u_3\} \cup S$  is a global offensive (k+1)-alliance of G of size  $|S|+1 < |S \cup D| = |S|+2$ , a contradiction. Thus  $J_1$  contains at least one vertex of degree k + 2. Since  $\alpha(G) = 2$ , every vertex of S has at least two neighbors in  $J_1$  or in  $J_2$ . Now let  $x \in S$ . If x has two neighbors in  $J_i$  and one neighbor in  $J_{3-i}$  for i = 1, 2, then the hypothesis that G is  $P_4$ -free implies that x is adjacent to each vertex of J. Consequently, S can be partioned in three subsets  $S_1, S_2$  and A such that all vertices of  $S_1$ are adjacent to all vertices of  $J_1$  and there is no edge between  $S_1$  and  $J_2$ , all vertices of  $S_2$  are adjacent to all vertices of  $J_2$  and there is no edge between  $S_2$  and  $J_1$ , all vertices of A are adjacent to all vertices of J. Since G is  $P_4$ -free, it follows that there is no edge between  $S_1$  and  $S_2$ , and that all vertices of  $S_i$ are adjacent to all vertices of A for i = 1, 2. Furthermore,  $\alpha(G) = 2$  shows that  $G[S_1]$  and  $G[S_2]$  are cliques. Altogether we see that  $d_G(u_i) = k + 2$ for each  $i \in \{1, 2, 3, 4\}$  and therefore  $|S_1| + |A| = |S_2| + |A| = k + 1$ . It follows that  $|S_1| = |S_2|$  and |S| + |A| = 2k + 2. Since G is connected, we deduce that  $|A| \ge 1$  and so  $1 \le |A| \le k + 1$ . If we define F = G[A] and  $Q_i = G[S_i \cup V(J_i)]$  for i = 1, 2, then we derive the desired structure, since  $\alpha(G[A]) \le 2$ .

Assume that  $|V(F)| \ge 3$  and  $\alpha(F) = 1$ . If  $x_1, x_2, x_3$  are three arbitrary vertices in F, then let  $S_0 = V(G) - \{x_1, x_2, x_3\}$ . If  $d_G(x_i) \ge k + 5$  for each i = 1, 2, 3, then  $S_0$  is a global offensive (k+1)-alliance of G, a contradiction. Otherwise, we have  $n - 1 = d_G(x_i) \le k + 4$  for at least one  $i \in \{1, 2, 3\}$  and so  $n \le k + 5$  and thus |V(F)| = k + 1.

Assume next that  $|V(F)| \ge 3$  and  $\alpha(F) = 2$ . As we have seen in Case 2, all components of  $\overline{F}$  are complete bipartite graphs.

Subcase 3.1. Assume that  $K_{1,1}$  is the greatest component of  $\overline{F}$ . Let u and v be the two vertices of the complete bipartite graph  $K_{1,1}$ . If  $n \ge k+7$ , then let w be a further vertex in F, and it is easy to verify that  $V(G) - \{u, v, w\}$  is a global offensive (k+1)-alliance of G of size n-3, a contradiction. If n = k+6 and there exists a vertex w in F of degree k+5, then  $V(G) - \{u, v, w\}$  is a global offensive (k+1)-alliance of G of size n-3, a contradiction.

Subcase 3.2. Assume that |V(F)| = k + 1 - t for  $0 \le t \le k - 2$  and  $\overline{F}$  contains a component  $K_{p,q}$  with  $1 \le p \le q$  and  $p+q \ge 3$ . Let  $\{v_1, v_2, \ldots, v_q\}$  and  $\{u_1, u_2, \ldots, u_p\}$  be a partition of  $K_{p,q}$ .

If  $K_{1,s} \subseteq \overline{F}$  with  $s \ge t + 4$ , then  $\delta(G) \le k$ , a contradiction to  $\delta(G) \ge k + 1$ . Thus  $q \le t + 3$ .

If  $q \leq t + 1$  or q = t + 2 and  $p \leq t + 1$ , then it is easy to see that  $V(G) - \{u_1, v_1, v_2\}$  is a global offensive (k + 1)-alliance of G of size n - 3, a contradiction.

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Conversely, if  $G = K_{k+3}$ , then obviously  $\gamma_o^k(G) = k+1$ ,  $\gamma_o^{k+1}(G) = k+2$ and so  $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$ .

Now let  $\overline{G} = H \cup 2K_{1,1}$  such that n(H) = k + 2 and the components of H are complete bipartite graphs  $K_{1,1}$ ,  $K_{3,3}$ ,  $K_{3,4}$  or  $K_{4,4}$ . Thus  $k + 1 \leq d_G(z) \leq k + 4$  for every  $z \in V(G)$ , and G contains a cycle C on four vertices, where each vertex of C has degree k + 4. Clearly, V(H) is a global offensive k-alliance of G and so  $\gamma_o^k(G) \leq n(G) - 4$ . If D is a  $\gamma_o^k(G)$ -set of size  $|D| \leq n(G) - 5$ , then, since  $\alpha(G) = 2$ , the induced subgraph G[V(G) - D]contains a vertex x of degree at least two. This leads to the contradiction  $|N_G(x) \cap D| \leq k + 1 < |N_G(x) - D| + k$ . Hence we have shown that  $\gamma_o^k(G) = n(G) - 4$ .

Now let us prove that  $\gamma_o^{k+1}(G) = n(G) - 2$ . Clearly,  $\gamma_o^{k+1}(G) \ge \gamma_o^k(G) \ge n(G) - 4$ . Let D be a  $\gamma_o^{k+1}(G)$ -set. First, assume that  $\gamma_o^{k+1}(G) = n(G) - 4$ . Then, since n(G) = k + 6 and  $\alpha(G) = 2$ , the induced subgraph G[V(G) - D] is isomorphic to  $2K_{1,1}$ , say ab and cd, and every vertex of V(G) - D is adjacent to all vertices of D. Since  $d_G(x) = k + 3$  for every  $x \in \{a, b, c, d\}$  it follows that a, b, c, d lie in one component  $C_4$  of H, a contradiction. Second, assume that  $\gamma_o^{k+1}(G) = n(G) - 3$ . Since every vertex has degree at most k + 4, no vertex of V(G) - D has two neighbors in V(G) - D. Moreover, since  $\alpha(G) = 2$ , G[V(G) - D] is formed by two adjacent vertices x, y plus an isolated vertex w. Since w has degree at least two in  $\overline{G}$ , the vertices w, x, y lie in one component in H and so belong to  $K_{3,3}, K_{3,4}$  or  $K_{4,4}$ . Thus each of x and y has at least two non-neighbors in D and hence  $|N(x) \cap D| \le k + 1$ , a contradiction to the fact D is a  $\gamma_o^{k+1}(G)$ -set. Thus  $|D| \ge n(G) - 2$  and the equality follows from the fact that V(G) minus any two non-adjacent vertices of C is a global offensive (k+1)-alliance of G. Therefore  $\gamma_o^{k+1}(G) = n(G) - 2 = (\gamma_o^k(G) + n(G))/2$ .

Finally, let  $G = (Q_1 \cup Q_2) + F$ , where  $Q_1, Q_2$  and F are three pairwise disjoint graphs such that  $1 \leq |V(F)| \leq k+1$ ,  $\alpha(F) \leq 2$ , and  $Q_1$  and  $Q_2$  are cliques with  $|V(Q_1)| = |V(Q_2)| = k+3 - |V(F)|$  such that  $|V(F)| \leq 2$  or  $\alpha(F) = 1$  and |V(F)| = k+1 or

 $\alpha(F) = 2$  and  $F = K_{k+1} - M$ , where M is matching of F or  $\alpha(F) = 2$  and  $F = K_k - M$ , where M is a perfect matching of F or  $\alpha(F) = 2$  and |V(F)| = k + 1 - t for  $0 \le t \le k - 2$  with  $k \ge 3t + 3$  and all components of  $\overline{F}$  are isomorphic to  $K_{t+2,t+2}$ , to  $K_{t+2,t+3}$  or to  $K_{t+3,t+3}$ .

Let D be a global offensive (k + 1)-alliance of G. Since each vertex of  $Q_i$  has degree k + 2, the set V(G) - D contains at most one vertex of  $Q_i$  for every i = 1, 2. Moreover, if  $(V(G) - D) \cap V(Q_i) \neq \emptyset$ , then  $V(F) \subseteq D$ .

Now suppose that  $\gamma_o^{k+1}(G) \leq n-3$ , and assume, without loss of generality, that  $V(G) - D = \{u, v, w\}$ . Then as noted above  $V(Q_1) \cup V(Q_2) \subseteq D$ , and hence the vertices u, v, w belong to V(F). It follows that  $|V(F)| \geq 3$ .

Obviously, we obtain a contradiction when  $\alpha(F) = 1$  and |V(F)| = k+1. Assume next that  $\alpha(F) = 2$ . This implies that at least two vertices of V(G) - D are adjacent in G.

First assume that  $F = K_k - M$ , where M is a perfect matching of F. Note that every vertex of V(F) has degree k+4. Since M is perfect,  $\{u, v, w\}$  induces either a path  $P_3$  or a clique  $K_3$  with center vertex, say v, in G. But then v has a non-neighbor in D for which it is matched in M, and so v has exactly k+2 neighbors in D against two in V(G) - D, a contradiction.

Second assume that  $F = K_{k+1} - M$ , where M is a matching of F. Note that n = k + 5 and |D| = k + 2. As above,  $\{u, v, w\}$  induces either a path  $P_3$  or a clique  $K_3$  with center vertex, say v, in G. But then v has at most k + 2 neighbors in D against two in V(G) - D, a contradiction.

Assume now that  $\alpha(F) = 2$  and |V(F)| = k+1-t for  $0 \le t \le k-2$  with  $k \ge 3t+3$  and all components of  $\overline{F}$  are isomorphic to  $K_{t+2,t+2}$ , to  $K_{t+2,t+3}$  or to  $K_{t+3,t+3}$ . Note that in this case n = k+5+t and so |D| = n-3 = k+2+t. Assume, without loss of generality, that u and v are adjacent in G. This leads to  $|N_G(u) \cap D| \le (k+5+t) - (t+2+2) = k+1$ , a contradiction to the assumption that D is a global offensive (k+1)-alliance of G.

Altogether, we have shown that  $\gamma_o^{k+1}(G) = n-2$ . Finally, it is a simple matter to obtain  $\gamma_o^k(G) = n-4$ , and the proof of Theorem 22 is complete.

#### 3. Lower Bounds

Our aim in this section is to give lower bounds on the global offensive kalliance number of a graph in terms of its order n, minimum degree  $\delta$  and maximum degree  $\Delta$ .

**Theorem 23.** Let k be a positive integer. If G is a graph of order n, minimum degree  $\delta$  and maximum degree  $\Delta$ , then

(6) 
$$\gamma_o^k(G) \ge \frac{n(\delta+k)}{2\Delta+\delta+k}.$$

**Proof.** If S is any  $\gamma_o^k(G)$ -set, then

$$\Delta \gamma_o^k(G) = \Delta |S| \ge \sum_{v \in S} d_G(v) \ge \sum_{v \in V(G) - S} \frac{d_G(v) + k}{2}$$

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$$\geq |V(G) - S| \frac{\delta + k}{2} = (n - \gamma_o^k(G)) \frac{\delta + k}{2}.$$

This leads to

$$\gamma_o^k(G)(2\Delta + \delta + k) \ge n(\delta + k),$$

and (6) is proved.

**Theorem 24.** Let  $k \geq 1$  be an integer, and let G be a graph of order n, minimum degree  $\delta$  and maximum degree  $\Delta$ . If  $\delta$  is even and k odd or  $\delta$  odd and k even, then

(7) 
$$\gamma_o^k(G) \ge \frac{n(\delta+k+1)}{2\Delta+\delta+k+1}.$$

**Proof.** If S is any  $\gamma_o^k(G)$ -set, then

$$\begin{split} \Delta \gamma_o^k(G) &= \Delta |S| \ge \sum_{v \in S} d_G(v) \\ &\ge \sum_{v \in V(G) - S, \ d_G(v) = \delta} \frac{d_G(v) + k + 1}{2} + \sum_{v \in V(G) - S, \ d_G(v) > \delta} \frac{d_G(v) + k}{2} \\ &\ge |V(G) - S| \frac{\delta + k + 1}{2} = (n - \gamma_o^k(G)) \frac{\delta + k + 1}{2}. \end{split}$$

This leads to

$$\gamma_o^k(G)(2\Delta + \delta + k + 1) \ge n(\delta + k + 1),$$

and (7) is proved.

**Example 25.** Let G be a k-regular bipartite graph of order n with the partite sets X and Y. Then

$$\gamma_0^k(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta + k)}{2\Delta + \delta + k}$$

and

$$\gamma_0^{k-1}(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta + (k-1) + 1)}{2\Delta + \delta + (k-1) + 1}$$

for  $k \geq 2$ . This family of graphs demonstrate that the bounds in Theorems 23 and 24 are best possible.

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