

BOUNDS ON THE GLOBAL OFFENSIVE k -ALLIANCE NUMBER IN GRAPHS

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Abstract

Let $G = (V(G), E(G))$ be a graph, and let $k \geq 1$ be an integer. A set $S \subseteq V(G)$ is called a *global offensive k -alliance* if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $N(v)$ is the neighborhood of v . The global offensive k -alliance number $\gamma_o^k(G)$ is the minimum cardinality of a global offensive k -alliance in G . We present different bounds on $\gamma_o^k(G)$ in terms of order, maximum degree, independence number, chromatic number and minimum degree.

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1. TERMINOLOGY

Let $G = (V, E) = (V(G), E(G))$ be a finite and simple graph. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of v , denoted by $d_G(v)$, is $|N(v)|$. By $n(G) = n$, $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the *order*, the *maximum degree* and the *minimum degree* of the graph G , respectively. If $A \subseteq V(G)$, then $G[A]$ is the graph induced by the vertex set A . We denote by K_n the *complete graph* of order n , and by $K_{r,s}$ the *complete bipartite graph* with partite sets X and Y such that $|X| = r$ and $|Y| = s$. A set $D \subseteq V(G)$ is a *k-dominating set* of G if every vertex of $V(G) - D$ has at least $k \geq 1$ neighbors in D . The *k-domination number* $\gamma_k(G)$ is the cardinality of a minimum k -dominating set. The case $k = 1$ leads to the classical *domination number* $\gamma(G) = \gamma_1(G)$.

In [11], Kristiansen, Hedetniemi and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive k -alliances, given by Shafique and Dutton [14, 15]. A set S of vertices of a graph G is called a *global offensive k-alliance* if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $k \geq 1$ is an integer. The *global offensive k-alliance number* $\gamma_o^k(G)$ is the minimum cardinality of a global offensive k -alliance in G . If S is a global k -offensive alliance of G and $|S| = \gamma_o^k(G)$, then we say that S is a $\gamma_o^k(G)$ -set. A global offensive 1-alliance is a global offensive alliance and a global offensive 2-alliance is a global strong offensive alliance. In [7], Fernau, Rodríguez and Sigarreta show that the problem of finding optimal global offensive k -alliances is *NP*-complete.

If $k \geq 1$ is an integer, then let $L_k(G) = \{x \in V(G) : d_G(x) \leq k - 1\}$. Denote by $\alpha(G)$ the *independence number*, by $\chi(G)$ the *chromatic number*, and by $\omega(G)$ the *clique number* of G , respectively. The *corona graph* $G \circ K_1$ of a graph G is the graph constructed from a copy of G , where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. Next assume that G_1 and G_2 are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

2. UPPER BOUNDS

We begin this section by giving an upper bound on the global offensive k -alliance number for an r -partite graph G in terms of its order and $|L_k(G)|$.

Theorem 1. *Let $k \geq 1$ be an integer. If G is an r -partite graph, then*

$$\gamma_o^k(G) \leq \frac{(r-1)n(G) + |L_k(G)|}{r}.$$

Proof. Clearly, the set $L_k(G)$ is contained in every $\gamma_o^k(G)$ -set. In the case that $|L_k(G)| = |V(G)|$, we are finished. In the remaining case that $|L_k(G)| < |V(G)|$, let V_1, V_2, \dots, V_r be a partition of the r -partite graph $G - L_k(G)$ such that $|V_1| \geq |V_2| \geq \dots \geq |V_r|$, where $V_i = \emptyset$ is possible for $i \geq 2$. Then every vertex of V_1 has degree at least k in G , and all its neighbors are in $V(G) - V_1$. Thus $V(G) - V_1$ is a global offensive k -alliance of G . Since

$$|V_1| \geq \frac{|V_1| + |V_2| + \dots + |V_r|}{r} = \frac{n(G) - |L_k(G)|}{r},$$

we obtain

$$\gamma_o^k(G) \leq n(G) - |V_1| \leq n(G) - \frac{n(G) - |L_k(G)|}{r} = \frac{(r-1)n(G) + |L_k(G)|}{r},$$

and the proof is complete. ■

The case $k = r = 2$ in Theorem 1 leads to the next result.

Corollary 2 (Chellali [4]). *If G is a bipartite graph, then*

$$\gamma_o^2(G) \leq \frac{n(G) + |L_2(G)|}{2}.$$

Observation 3. *If $k \geq 1$ is an integer, then $\gamma_o^k(G) \geq \gamma_k(G)$ for any graph G .*

Proof. If S is any $\gamma_o^k(G)$ -set, then every vertex of $V(G) - S$ has at least k neighbors in S . Thus S is a k -dominating set of G and so $\gamma_k(G) \leq |S| = \gamma_o^k(G)$. ■

Using Theorem 1 for $r = 2$ and Observation 3, we obtain the known theorem by Blidia, Chellali and Volkmann [2].

Corollary 4 (Blidia, Chellali, Volkmann [2] 2006). *Let k be a positive integer. If G is a bipartite graph, then*

$$\gamma_k(G) \leq \frac{n(G) + |L_k(G)|}{2}.$$

Since every graph G is $\chi(G)$ -partite and $n(G) \leq \chi(G)\alpha(G)$, we obtain also the following corollaries from Theorem 1.

Corollary 5. *If G is a graph and k a positive integer, then*

$$\gamma_o^k(G) \leq \frac{(\chi(G) - 1)n(G) + |L_k(G)|}{\chi(G)}.$$

Corollary 6. *Let $k \geq 1$ be an integer. If G is a graph with $\delta(G) \geq k$, then*

$$\gamma_o^k(G) \leq (\chi(G) - 1)\alpha(G).$$

Theorem 7 (Brooks [3] 1941). *If G is a connected graph different from the complete graph and from a cycle of odd length, then $\chi(G) \leq \Delta(G)$.*

Combining Brooks' Theorem and Corollary 6, we can prove the following result.

Theorem 8. *Let $k \geq 1$ be an integer, and let G be a connected graph with $\delta(G) \geq k$. Then*

$$(1) \quad \gamma_o^k(G) \leq (\Delta(G) - 1)\alpha(G)$$

if and only if G is neither isomorphic to the complete graphs K_{k+1} or K_{k+2} nor to a cycle of odd length when $1 \leq k \leq 2$.

Proof. If G is the complete graph K_n , then $\Delta(G) = \delta(G) = n - 1 \geq k \geq 1$ and $\alpha(G) = 1$. Since $\gamma_o^k(K_{k+1}) = k$ and $\gamma_o^k(K_{k+2}) = k + 1$, inequality (1) is not true for these two complete graphs. However, in the remaining case that $n \geq k + 3$, we observe that $\gamma_o^k(G) \leq n - 2$, and we arrive at the desired bound

$$\gamma_o^k(G) \leq n - 2 = \Delta(G) - 1 = (\Delta(G) - 1)\alpha(G).$$

Assume next that $1 \leq k \leq 2$. If G is a cycle of odd length, then $\Delta(G) = 2$, $\gamma_o^1(G) = \gamma_o^2(G) = \lceil n(G)/2 \rceil$ and $\alpha(G) = \lfloor n(G)/2 \rfloor$ and thus (1) is not valid in these cases.

For all other graphs inequality (1) follows directly from Brooks' Theorem and Corollary 6. ■

Lemma 9 (Hansberg, Meierling, Volkmann [10]). *Let $k \geq 1$ be an integer. If G is a connected graph with $\delta(G) \leq k - 1$ and $\Delta(G) \leq k$, then*

$$k\alpha(G) \geq n(G).$$

Theorem 10. *Let $k \geq 1$ be an integer. If G is a connected r -partite graph with $\Delta(G) \geq k$, then*

$$\gamma_o^k(G) \leq \frac{\alpha(G)}{r}((r-1)r + k - 1).$$

Proof. Assume that $k = 1$. Since G is connected and $\Delta(G) \geq 1$, we note that $|L_1(G)| = 0$. Applying Theorem 1, and using the fact that $r\alpha(G) \geq n(G)$, we receive the desired inequality immediately.

Assume next that $k \geq 2$. Since G is connected and $G - L_k(G)$ is not empty, every component Q of $G[L_k(G)]$ fulfills $\delta(Q) \leq k-2$ and $\Delta(Q) \leq k-1$. Hence Lemma 9 implies $(k-1)\alpha(Q) \geq n(Q)$. If Q_1, Q_2, \dots, Q_t are the components of $G[L_k(G)]$, we therefore deduce that

$$\alpha(G) \geq \alpha(G[L_k(G)]) = \sum_{i=1}^t \alpha(Q_i) \geq \frac{|L_k(G)|}{k-1}.$$

Combining $n(G) \leq r\alpha(G)$ with Theorem 1, we receive the desired inequality as follows:

$$\begin{aligned} \gamma_o^k(G) &\leq \frac{(r-1)n(G) + |L_k(G)|}{r} \\ &\leq \frac{(r-1)r\alpha(G) + (k-1)\alpha(G)}{r} \\ &= \frac{\alpha(G)}{r}((r-1)r + k - 1). \end{aligned} \quad \text{■}$$

The case $r = 2$ in Theorem 10 leads to the next result.

Corollary 11. *Let $k \geq 1$ be an integer. If G is a connected bipartite graph with $\Delta(G) \geq k$, then*

$$\gamma_o^k(G) \leq \frac{(k+1)\alpha(G)}{2}.$$

Using Observation 3, we obtain the following known bounds on the 2-domination number.

Corollary 12 (Fujisawa, Hansberg, Kubo, Saito, Sugita, Volkmann [9] 2008). *If G is a connected bipartite graph of order at least 3, then*

$$\gamma_2(G) \leq \frac{3\alpha(G)}{2}.$$

Corollary 13 (Blidia, Chellali, Favaron [1] 2005). *If T is a tree of order at least 3, then*

$$\gamma_2(T) \leq \frac{3\alpha(T)}{2}.$$

Theorem 14 (Favaron, Hansberg, Volkmann [6] 2008). *Let G be a graph. If $r \geq 1$ is an integer, then there is a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ of $V(G)$ such that*

$$(2) \quad |N_G(u) \cap V_i| \leq \frac{d_G(u)}{r}$$

for each $i \in \{1, 2, \dots, r\}$ and each $u \in V_i$.

Theorem 15. *Let $k \geq 1$ be an integer. If G is a graph of order n with minimum degree $\delta \geq k$, then*

$$(3) \quad \gamma_o^k(G) \leq \frac{k+1}{k+2}n,$$

and the bound given in (3) is best possible.

Proof. Choose $r = k+2$ in Theorem 14, and let V_1, V_2, \dots, V_r be a partition of $V(G)$ as in Theorem 14 such that $|V_1| \geq |V_2| \geq \cdots \geq |V_r|$. If $D = V_2 \cup V_3 \cup \cdots \cup V_r$, then it follows from (2) and the hypothesis that $\delta \geq k$ for each $v \in V_1 = V(G) - D$ that

$$\begin{aligned} |N_G(v) \cap D| &\geq \left\lceil \frac{k+1}{k+2} d_G(v) \right\rceil \geq \left\lfloor \frac{d_G(v)}{k+2} \right\rfloor + k \\ &\geq |N_G(v) \cap V_1| + k = |N_G(v) - D| + k. \end{aligned}$$

Thus D is a global offensive k -alliance of G such that $|D| \leq (k+1)n/(k+2)$, and (3) is proved.

Let H be a connected graph, and let $G_k = H \circ K_{k+1}$. Then it is easy to see that $\gamma_o^k(G_k) = (k+1)n(G_k)/(k+2)$, and therefore (3) is the best possible. ■

Corollary 16 (Favaron, Fricke, Goddard, Hedetniemi, Hedetniemi, Kristiansen, Laskar, Skaggs [5] 2004). *Let G be graph of order n and minimum degree δ .*

If $\delta \geq 1$, then $\gamma_o^1(G) \leq 2n/3$.

If $\delta \geq 2$, then $\gamma_o^2(G) \leq 3n/4$.

In the case that $\delta \geq k+2$, we obtain the following bound, improving the bound of Theorem 15.

Theorem 17. *Let $k \geq 2$ be an integer, and let G be a graph of order n with minimum degree $\delta \geq k+2$. Then*

$$(4) \quad \gamma_o^k(G) \leq \frac{k}{k+1}n.$$

Proof. Choose $r = k+1$ in Theorem 14, and let V_1, V_2, \dots, V_r be a partition of $V(G)$ as in Theorem 14 such that $|V_1| \geq |V_2| \geq \dots \geq |V_r|$. If $D = V_2 \cup V_3 \cup \dots \cup V_r$, then it follows from (2) and the hypothesis $\delta \geq k+2$ for each $v \in V_1 = V(G) - D$ that

$$\begin{aligned} |N_G(v) \cap D| &\geq \left\lceil \frac{k}{k+1}d_G(v) \right\rceil \geq \left\lfloor \frac{d_G(v)}{k+1} \right\rfloor + k \\ &\geq |N_G(v) \cap V_1| + k = |N_G(v) - D| + k. \end{aligned}$$

Thus D is a global offensive k -alliance of G such that $|D| \leq kn/(k+1)$, and (4) is proved. ■

Theorem 18. *Let $k \geq 1$ be an integer, and let G be a connected non-complete graph such that $\delta(G) \geq k$ and $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$. Then $\Delta(G) \leq k+2$, $\Delta(G) - \delta(G) \leq 1$ and if $k \geq 2$, then $\delta(G) \leq k+1$.*

Proof. Because of $\chi(G)\alpha(G) \geq n(G)$, Corollary 5 and the hypothesis imply that

$$(\Delta(G) - 1)\alpha(G) = \gamma_o^k(G) \leq \frac{(\chi(G) - 1)n(G)}{\chi(G)} \leq (\chi(G) - 1)\alpha(G).$$

Since G is neither a complete graph nor a cycle of odd length, it follows from Brooks' Theorem that $\Delta(G) = \chi(G)$, $\chi(G)\alpha(G) = n(G)$ and

$$(5) \quad \gamma_o^k(G) = \frac{(\chi(G) - 1)n(G)}{\chi(G)} = \frac{(\Delta(G) - 1)n(G)}{\Delta(G)}.$$

If we suppose on the contrary that $\Delta(G) \geq k + 3$, then it follows from (5) and Theorem 15 that

$$\frac{\Delta(G) - 1}{\Delta(G)}n(G) = \gamma_o^k(G) \leq \frac{k + 1}{k + 2}n(G) \leq \frac{\Delta(G) - 2}{\Delta(G) - 1}n(G).$$

This contradiction shows that $\Delta(G) \leq k + 2$.

If we suppose on the contrary that $\Delta(G) - \delta(G) \geq 2$, then we deduce that $\delta(G) = k$ and $\Delta(G) = k + 2 = \chi(G)$. Since $\chi(G)\alpha(G) = n(G)$, there exists a partition of $V(G)$ in $\chi = \chi(G)$ colour classes U_1, U_2, \dots, U_χ such that $|U_1| = |U_2| = \dots = |U_\chi| = \alpha(G)$. Let v be a vertex of minimum degree $\delta(G) = k$, and assume, without loss of generality, that $v \in U_1$. As $d_G(v) = k$ and $\chi(G) = k + 2$, there exists a colour class U_j with $2 \leq j \leq \chi$ such that v is not adjacent to any vertex in U_j . Therefore $U_j \cup \{v\}$ is an independent set. This is a contradiction to the fact that $|U_j| = \alpha(G)$, and the desired inequality $\Delta(G) - \delta(G) \leq 1$ is proved.

Next assume that $k \geq 2$, and suppose on the contrary that $\delta(G) \geq k + 2$. Then $k \leq \Delta(G) - 2$ and (5) and Theorem 17 lead to the contradiction

$$\frac{\Delta(G) - 1}{\Delta(G)}n(G) = \gamma_o^k(G) \leq \frac{k}{k + 1}n(G) \leq \frac{\Delta(G) - 2}{\Delta(G) - 1}n(G).$$

Thus $\delta(G) \leq k \leq \delta(G) + 1$ when $k \geq 2$, and the proof of Theorem 18 is complete. ■

Example 19. 1. Let H_1, H_2, \dots, H_t be $t \geq 2$ copies of the complete graph K_{k+1} , and let $u_i, v_i \in E(H_i)$ for $1 \leq i \leq t$. Define the graph G as the disjoint union $H_1 \cup H_2 \cup \dots \cup H_t$ together with the edge set $\{v_1u_2, v_2u_3, \dots, v_{t-1}u_t\}$. Then it is easy to verify that $\Delta(G) = k + 1$, $\delta(G) = k$, $\alpha(G) = t$, $\gamma_o^k(G) = tk$ and thus $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$.

2. Let F_1 and F_2 be 2 copies of the complete graph K_{k+1} with the vertex sets $V(F_1) = \{x_1, x_2, \dots, x_{k+1}\}$ and $V(F_2) = \{y_1, y_2, \dots, y_{k+1}\}$. Define the graph H as the disjoint union $F_1 \cup F_2$ together with the edge set $\{x_1y_1, x_2y_2, \dots, x_ky_k\}$. If H_1, H_2, \dots, H_t are $t \geq 2$ copies of H , then let

u_{2i-1} and u_{2i} be the vertices of degree k in H_i for all $i \in \{1, 2, \dots, t\}$. Define the graph G as the disjoint union $H_1 \cup H_2 \cup \dots \cup H_t$ together with the edge set $\{u_2u_3, u_4u_5, \dots, u_{2t}u_1\}$. Then G is a $(k+1)$ -regular graph with $\alpha(G) = 2t$, $\gamma_o^k(G) = 2kt$ and thus $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$.

3. Let $k \geq 2$, and let F_1 and F_2 be 2 copies of the complete graph K_k such that $V(F_1) = \{x_1, x_2, \dots, x_k\}$ and $V(F_2) = \{y_1, y_2, \dots, y_k\}$. Define the graph H as the disjoint union $F_1 \cup F_2$ together with the edge set $\{x_1y_1, x_2y_2, \dots, x_{k-1}y_{k-1}\}$. If H_1, H_2, \dots, H_t are $t \geq 2$ copies of H , then let u_{2i-1} and u_{2i} be the vertices of degree $k-1$ in H_i for all $i \in \{1, 2, \dots, t\}$. Define the graph G as the disjoint union $H_1 \cup H_2 \cup \dots \cup H_t$ together with the edge set $\{u_2u_3, u_4u_5, \dots, u_{2t}u_1\}$. Then G is a k -regular graph with $\alpha(G) = 2t$, $\gamma_o^k(G) = 2(k-1)t$ and thus $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$.

4. Let H_1 and H_2 be 2 copies of the complete graph K_{k+2} , and let $x \in E(H_1)$ and $y \in E(H_2)$. Define the graph G' as the disjoint union $H_1 \cup H_2$ together with the edge xy . Then $\Delta(G') = k+2$, $\delta(G') = k+1$, $\alpha(G') = 2$, $\gamma_o^k(G') = 2(k+1)$ and thus $\gamma_o^k(G') = (\Delta(G') - 1)\alpha(G')$.

These four examples show that $\Delta = k+1$ and $\delta = k$, $\Delta = \delta = k+1$, $\Delta = \delta = k$ as well as $\Delta = k+2$ and $\delta = k+1$ in Theorem 18 are possible.

Theorem 20. *If G is a graph and k an integer such that $1 \leq k \leq \delta(G) - 1$, then*

$$\gamma_o^{k+1}(G) \leq \frac{\gamma_o^k(G) + n(G)}{2}.$$

Proof. Let S be a $\gamma_o^k(G)$ -set, and let A be the set of isolated vertices in the subgraph induced by the vertex set $V(G) - S$. Then the subgraph induced by $V(G) - (S \cup A)$ contains no isolated vertices. If D is a minimum dominating set of $G[V(G) - (S \cup A)]$, then the well-known inequality of Ore [12] implies

$$|D| \leq \frac{|V(G) - (S \cup A)|}{2} \leq \frac{|V(G) - S|}{2} = \frac{n(G) - \gamma_o^k(G)}{2}.$$

Since $\delta(G) \geq k+1$, every vertex of A has at least $k+1$ neighbors in S , and therefore $D \cup S$ is a global offensive $(k+1)$ -alliance of G . Thus we obtain the desired bound as follows:

$$\gamma_o^{k+1}(G) \leq |S \cup D| \leq \gamma_o^k(G) + \frac{n(G) - \gamma_o^k(G)}{2} = \frac{\gamma_o^k(G) + n(G)}{2}. \quad \blacksquare$$

The graphs G of even order and without isolated vertices with $\gamma(G) = n/2$ have been characterized independently by Payan and Xuong [13] and Fink, Jacobson, Kinch and Roberts [8].

Theorem 21 (Payan, Xuong [13] 1982 and Fink, Jacobson, Kinch, Roberts [8] 1985). *Let G be a graph of even order n without isolated vertices. Then $\gamma(G) = n/2$ if and only if each component of G is either a cycle C_4 or the corona of a connected graph.*

A graph is P_4 -free if and only if it contains no induced subgraph isomorphic to the path P_4 of order four. A graph is $(K_4 - e)$ -free if and only if it contains no induced subgraph isomorphic to the graph $K_4 - e$, where e is an arbitrary edge of the complete graph K_4 . The graph \overline{G} denotes the complement of the graph G . Next we give a characterization of some special graphs attaining equality in Theorem 20.

Theorem 22. *Let G be a connected P_4 -free graph such that \overline{G} is $(K_4 - e)$ -free. If k is an integer with $1 \leq k \leq \delta(G) - 1$, then $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$ if and only if*

1. $G = K_{k+3}$ or
2. $\overline{G} = H \cup 2K_{1,1}$ such that $n(H) = k + 2$ and all components of H are isomorphic to $K_{1,1}$, to $K_{3,3}$, to $K_{3,4}$ or to $K_{4,4}$ or
3. $G = (Q_1 \cup Q_2) + F$, where Q_1, Q_2 and F are three pairwise disjoint graphs such that $1 \leq |V(F)| \leq k + 1$, $\alpha(F) \leq 2$, and Q_1 and Q_2 are cliques with $|V(Q_1)| = |V(Q_2)| = k + 3 - |V(F)|$ such that
 - $|V(F)| \leq 2$ or
 - $\alpha(F) = 1$ and $|V(F)| = k + 1$ or
 - $\alpha(F) = 2$ and $F = K_{k+1} - M$, where M is a matching of F or
 - $\alpha(F) = 2$ and $F = K_k - M$, where M is a perfect matching of F or
 - $\alpha(F) = 2$ and $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ with $k \geq 3t + 3$ and all components of \overline{F} are isomorphic to $K_{t+2,t+2}$, to $K_{t+2,t+3}$ or to $K_{t+3,t+3}$.

Proof. Assume that $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$. Following the same notation as used in the proof of Theorem 20, we obtain $|D| = \frac{|V(G) - S|}{2}$, and we observe that $S \cup D$ is a $\gamma_o^{k+1}(G)$ -set. It follows that $G[V(G) - S]$ has no isolated vertices and so by Theorem 21, each component of $G[V(G) - S]$ is either a cycle C_4 or the corona of some connected graph. Using the

hypothesis that G is P_4 -free, we deduce that each component of $G[V(G) - S]$ is isomorphic to K_2 or to C_4 . Since \overline{G} is $(K_4 - e)$ -free, there remain exactly the three cases that $G[V(G) - S]$ is isomorphic to K_2 , to C_4 or to $2K_2$.

Case 1. First assume that $G[V(G) - S] = K_2$. Suppose that G has an independent set Q of size at least two. Then the hypothesis $\delta(G) \geq k + 1$ implies that $V(G) - Q$ is a global offensive $(k + 1)$ -alliance of G of size $n - |Q| < |S \cup D| = n - 1$, a contradiction. Therefore $\alpha(G) = 1$ and thus $G = K_{k+3}$.

Case 2. Second assume that $G[V(G) - S]$ is a cycle $C_4 = x_0x_1x_2x_3x_0$. In the following the indices of the vertices x_i are taken modulo 4. Recall that $S \cup D$ is a $\gamma_o^{k+1}(G)$ -set, and D contains two vertices of the cycle C_4 . Clearly, since S is a $\gamma_o^k(G)$ -set, every vertex of the cycle C_4 has degree at least $k + 4$. Suppose that $d_G(x_i) \geq k + 5$ for an $i \in \{0, 1, 2, 3\}$. Then $\{x_{i+2}\} \cup S$ is a global offensive $(k + 1)$ -alliance of G of size $|S| + 1 < |S \cup D| = |S| + 2$, a contradiction. Thus $d_G(x_i) = k + 4$ for every $i \in \{0, 1, 2, 3\}$. Now if Q is an $\alpha(G)$ -set, then $|Q| \leq 2$, for otherwise the hypothesis $\delta(G) \geq k + 1$ implies that $V(G) - Q$ is a global offensive $(k + 1)$ -alliance of G of size $|V(G) - Q| < |S \cup D| = n(G) - 2$, a contradiction too. Since there are two non-adjacent vertices on the cycle C_4 and G is P_4 -free, it follows that every vertex of S has at least three neighbors on the cycle C_4 .

Subcase 2.1. Assume that $\alpha(G[S]) = 1$. Then the subgraph induced by S is complete and $|S| \geq k + 2$. If $|S| = k + 2$, then we observe that every vertex of S has exactly four neighbours on the cycle C_4 . Thus, in each case, we deduce that $d_G(y) \geq k + 5$ for every $y \in S$. But then for any subset W of S of size three, the set $V(G) - W$ is a global offensive $(k + 1)$ -alliance of G of size less than $|S \cup D|$, a contradiction.

Subcase 2.2. Assume that $\alpha(G[S]) = 2$. Suppose that there exists a vertex $w \in S$ with at least $k + 1$ neighbors in S . Then, since $|N(w) \cap V(C_4)| \geq 3$, say $\{x_0, x_1, x_2\} \subseteq N(w) \cap V(C_4)$, we observe that $(S - \{w\}) \cup \{x_0, x_2\}$ is a global offensive $(k + 1)$ -alliance of G of size $|S| + 1 < |S \cup D|$, a contradiction. Thus every vertex of S has at most k neighbors in S .

Let $S = X \cup Y$ such that every vertex of X has exactly three and every vertex of Y exactly 4 neighbors on C_4 . We shall show that $X = \emptyset$. If $X \neq \emptyset$, then let $S_{x_i} \subseteq X$ be the set of vertices such that each vertex of S_{x_i} is not adjacent to x_{i+2} for $i \in \{0, 1, 2, 3\}$. Because of $\alpha(G) = 2$, we observe that

the set $S_{x_i} \cup \{x_i\}$ induces a complete graph for each $i \in \{0, 1, 2, 3\}$. In addition, since G is P_4 -free it is straightforward to verify that all vertices of $X \cup C_4$ are adjacent to all vertices of Y and that $S_{x_i} \cup S_{x_{i+1}} \cup \{x_i, x_{i+1}\}$ induces a complete graph for each $i \in \{0, 1, 2, 3\}$. Now assume, without loss of generality, that $S_{x_0} \neq \emptyset$, and let $w \in S_{x_0}$. On the one hand we have seen above that $d_G(w) \leq k + 3$. On the other hand, we observe that $d_G(w) = d_G(x_0)$. But since $d_G(x_0) = k + 4$, we have a contradiction.

Hence we have shown that $X = \emptyset$, and this leads to $|S| = k + 2$. If we define $H = \overline{G[S]}$, then $\omega(H) = 2$, $\delta(H) \geq 1$ and $\Delta(H) \leq 4$. Since H is also P_4 -free, H does not contain an induced cycle of odd length. Using $\omega(H) = 2$, we deduce that H is a bipartite graph. Now let H_i be a component of H . If H_i is not a complete bipartite graph, then H_i contains a P_4 , a contradiction. Thus the components of H consists of $K_{1,1}$, $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, $K_{2,2}$, $K_{2,3}$, $K_{2,4}$, $K_{3,3}$, $K_{3,4}$ or $K_{4,4}$.

If $K_{1,2}$ is a component of H , then $V(G) - V(K_{1,2})$ is a global offensive $(k + 1)$ -alliance of G of size $n - 3$, a contradiction.

If $K_{1,3}$ is a component of H with a leaf u , then $(V(G) - V(K_{1,3})) \cup \{u\}$ is a global offensive $(k + 1)$ -alliance of G of size $n - 3$, a contradiction.

If $K_{1,4}$ is a component of H and u, v are two leaves of this star, then $(V(G) - V(K_{1,3})) \cup \{u, v\}$ is a global offensive $(k + 1)$ -alliance of G of size $n - 3$, a contradiction.

If $K_{2,2}$ is a component of H , then $V(G) - V(K_{2,2})$ is a global offensive $(k + 1)$ -alliance of G of size $n - 4$, a contradiction.

Next let $K_{2,3}$ be a component of H with the bipartition $\{v_1, v_2, v_3\}$ and $\{u_1, u_2\}$. Then $V(G) - \{u_1, v_1, v_2\}$ is a global offensive $(k + 1)$ -alliance of G of size $n - 3$, a contradiction.

Finally, let $K_{2,4}$ be a component of H with the bipartition $\{v_1, v_2, v_3, v_4\}$ and $\{u_1, u_2\}$. Then $V(G) - \{u_1, v_1, v_2\}$ is a global offensive $(k + 1)$ -alliance of G of size $n - 3$, a contradiction.

Case 3. Third assume that $G[V(G) - S] = 2K_2$. Let $2K_2 = J_1 \cup J_2 = J$ such that $V(J_1) = \{u_1, u_2\}$ and $V(J_2) = \{u_3, u_4\}$. If $\alpha(G) \geq 3$, then we obtain the contradiction $\gamma_o^{k+1}(G) \leq n - 3$. Thus $\alpha(G) = 2$. Since S is a $\gamma_o^k(G)$ -set, every vertex of J has degree at least $k + 2$. Suppose that $d_G(u_1) \geq k + 3$ and $d_G(u_2) \geq k + 3$. Then $\{u_3\} \cup S$ is a global offensive $(k + 1)$ -alliance of G of size $|S| + 1 < |S \cup D| = |S| + 2$, a contradiction. Thus J_1 contains at least one vertex of degree $k + 2$, and for reason of symmetry, also J_2 contains a vertex of degree $k + 2$. Since $\alpha(G) = 2$, every vertex of

S has at least two neighbors in J_1 or in J_2 . Now let $x \in S$. If x has two neighbors in J_i and one neighbor in J_{3-i} for $i = 1, 2$, then the hypothesis that G is P_4 -free implies that x is adjacent to each vertex of J . Consequently, S can be partitioned in three subsets S_1, S_2 and A such that all vertices of S_1 are adjacent to all vertices of J_1 and there is no edge between S_1 and J_2 , all vertices of S_2 are adjacent to all vertices of J_2 and there is no edge between S_2 and J_1 , all vertices of A are adjacent to all vertices of J . Since G is P_4 -free, it follows that there is no edge between S_1 and S_2 , and that all vertices of S_i are adjacent to all vertices of A for $i = 1, 2$. Furthermore, $\alpha(G) = 2$ shows that $G[S_1]$ and $G[S_2]$ are cliques. Altogether we see that $d_G(u_i) = k + 2$ for each $i \in \{1, 2, 3, 4\}$ and therefore $|S_1| + |A| = |S_2| + |A| = k + 1$. It follows that $|S_1| = |S_2|$ and $|S| + |A| = 2k + 2$. Since G is connected, we deduce that $|A| \geq 1$ and so $1 \leq |A| \leq k + 1$. If we define $F = G[A]$ and $Q_i = G[S_i \cup V(J_i)]$ for $i = 1, 2$, then we derive the desired structure, since $\alpha(G[A]) \leq 2$.

Assume that $|V(F)| \geq 3$ and $\alpha(F) = 1$. If x_1, x_2, x_3 are three arbitrary vertices in F , then let $S_0 = V(G) - \{x_1, x_2, x_3\}$. If $d_G(x_i) \geq k + 5$ for each $i = 1, 2, 3$, then S_0 is a global offensive $(k + 1)$ -alliance of G , a contradiction. Otherwise, we have $n - 1 = d_G(x_i) \leq k + 4$ for at least one $i \in \{1, 2, 3\}$ and so $n \leq k + 5$ and thus $|V(F)| = k + 1$.

Assume next that $|V(F)| \geq 3$ and $\alpha(F) = 2$. As we have seen in Case 2, all components of \overline{F} are complete bipartite graphs.

Subcase 3.1. Assume that $K_{1,1}$ is the greatest component of \overline{F} . Let u and v be the two vertices of the complete bipartite graph $K_{1,1}$. If $n \geq k + 7$, then let w be a further vertex in F , and it is easy to verify that $V(G) - \{u, v, w\}$ is a global offensive $(k + 1)$ -alliance of G of size $n - 3$, a contradiction. If $n = k + 6$ and there exists a vertex w in F of degree $k + 5$, then $V(G) - \{u, v, w\}$ is a global offensive $(k + 1)$ -alliance of G of size $n - 3$, a contradiction.

Subcase 3.2. Assume that $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ and \overline{F} contains a component $K_{p,q}$ with $1 \leq p \leq q$ and $p + q \geq 3$. Let $\{v_1, v_2, \dots, v_q\}$ and $\{u_1, u_2, \dots, u_p\}$ be a partition of $K_{p,q}$.

If $K_{1,s} \subseteq \overline{F}$ with $s \geq t + 4$, then $\delta(G) \leq k$, a contradiction to $\delta(G) \geq k + 1$. Thus $q \leq t + 3$.

If $q \leq t + 1$ or $q = t + 2$ and $p \leq t + 1$, then it is easy to see that $V(G) - \{u_1, v_1, v_2\}$ is a global offensive $(k + 1)$ -alliance of G of size $n - 3$, a contradiction.

Conversely, if $G = K_{k+3}$, then obviously $\gamma_o^k(G) = k + 1$, $\gamma_o^{k+1}(G) = k + 2$ and so $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$.

Now let $\overline{G} = H \cup 2K_{1,1}$ such that $n(H) = k + 2$ and the components of H are complete bipartite graphs $K_{1,1}$, $K_{3,3}$, $K_{3,4}$ or $K_{4,4}$. Thus $k + 1 \leq d_G(z) \leq k + 4$ for every $z \in V(G)$, and G contains a cycle C on four vertices, where each vertex of C has degree $k + 4$. Clearly, $V(H)$ is a global offensive k -alliance of G and so $\gamma_o^k(G) \leq n(G) - 4$. If D is a $\gamma_o^k(G)$ -set of size $|D| \leq n(G) - 5$, then, since $\alpha(G) = 2$, the induced subgraph $G[V(G) - D]$ contains a vertex x of degree at least two. This leads to the contradiction $|N_G(x) \cap D| \leq k + 1 < |N_G(x) - D| + k$. Hence we have shown that $\gamma_o^k(G) = n(G) - 4$.

Now let us prove that $\gamma_o^{k+1}(G) = n(G) - 2$. Clearly, $\gamma_o^{k+1}(G) \geq \gamma_o^k(G) \geq n(G) - 4$. Let D be a $\gamma_o^{k+1}(G)$ -set. First, assume that $\gamma_o^{k+1}(G) = n(G) - 4$. Then, since $n(G) = k + 6$ and $\alpha(G) = 2$, the induced subgraph $G[V(G) - D]$ is isomorphic to $2K_{1,1}$, say ab and cd , and every vertex of $V(G) - D$ is adjacent to all vertices of D . Since $d_G(x) = k + 3$ for every $x \in \{a, b, c, d\}$ it follows that a, b, c, d lie in one component C_4 of H , a contradiction. Second, assume that $\gamma_o^{k+1}(G) = n(G) - 3$. Since every vertex has degree at most $k + 4$, no vertex of $V(G) - D$ has two neighbors in $V(G) - D$. Moreover, since $\alpha(G) = 2$, $G[V(G) - D]$ is formed by two adjacent vertices x, y plus an isolated vertex w . Since w has degree at least two in \overline{G} , the vertices w, x, y lie in one component in H and so belong to $K_{3,3}$, $K_{3,4}$ or $K_{4,4}$. Thus each of x and y has at least two non-neighbors in D and hence $|N(x) \cap D| \leq k + 1$, a contradiction to the fact D is a $\gamma_o^{k+1}(G)$ -set. Thus $|D| \geq n(G) - 2$ and the equality follows from the fact that $V(G)$ minus any two non-adjacent vertices of C is a global offensive $(k + 1)$ -alliance of G . Therefore $\gamma_o^{k+1}(G) = n(G) - 2 = (\gamma_o^k(G) + n(G))/2$.

Finally, let $G = (Q_1 \cup Q_2) + F$, where Q_1, Q_2 and F are three pairwise disjoint graphs such that $1 \leq |V(F)| \leq k + 1$, $\alpha(F) \leq 2$, and Q_1 and Q_2 are cliques with $|V(Q_1)| = |V(Q_2)| = k + 3 - |V(F)|$ such that $|V(F)| \leq 2$ or $\alpha(F) = 1$ and $|V(F)| = k + 1$ or

$\alpha(F) = 2$ and $F = K_{k+1} - M$, where M is matching of F or

$\alpha(F) = 2$ and $F = K_k - M$, where M is a perfect matching of F or

$\alpha(F) = 2$ and $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ with $k \geq 3t + 3$ and all components of \overline{F} are isomorphic to $K_{t+2,t+2}$, to $K_{t+2,t+3}$ or to $K_{t+3,t+3}$.

Let D be a global offensive $(k + 1)$ -alliance of G . Since each vertex of Q_i has degree $k + 2$, the set $V(G) - D$ contains at most one vertex of Q_i for every $i = 1, 2$. Moreover, if $(V(G) - D) \cap V(Q_i) \neq \emptyset$, then $V(F) \subseteq D$.

Now suppose that $\gamma_o^{k+1}(G) \leq n - 3$, and assume, without loss of generality, that $V(G) - D = \{u, v, w\}$. Then as noted above $V(Q_1) \cup V(Q_2) \subseteq D$, and hence the vertices u, v, w belong to $V(F)$. It follows that $|V(F)| \geq 3$.

Obviously, we obtain a contradiction when $\alpha(F) = 1$ and $|V(F)| = k + 1$.

Assume next that $\alpha(F) = 2$. This implies that at least two vertices of $V(G) - D$ are adjacent in G .

First assume that $F = K_k - M$, where M is a perfect matching of F . Note that every vertex of $V(F)$ has degree $k + 4$. Since M is perfect, $\{u, v, w\}$ induces either a path P_3 or a clique K_3 with center vertex, say v , in G . But then v has a non-neighbor in D for which it is matched in M , and so v has exactly $k + 2$ neighbors in D against two in $V(G) - D$, a contradiction.

Second assume that $F = K_{k+1} - M$, where M is a matching of F . Note that $n = k + 5$ and $|D| = k + 2$. As above, $\{u, v, w\}$ induces either a path P_3 or a clique K_3 with center vertex, say v , in G . But then v has at most $k + 2$ neighbors in D against two in $V(G) - D$, a contradiction.

Assume now that $\alpha(F) = 2$ and $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ with $k \geq 3t + 3$ and all components of \bar{F} are isomorphic to $K_{t+2, t+2}$, to $K_{t+2, t+3}$ or to $K_{t+3, t+3}$. Note that in this case $n = k + 5 + t$ and so $|D| = n - 3 = k + 2 + t$. Assume, without loss of generality, that u and v are adjacent in G . This leads to $|N_G(u) \cap D| \leq (k + 5 + t) - (t + 2 + 2) = k + 1$, a contradiction to the assumption that D is a global offensive $(k + 1)$ -alliance of G .

Altogether, we have shown that $\gamma_o^{k+1}(G) = n - 2$. Finally, it is a simple matter to obtain $\gamma_o^k(G) = n - 4$, and the proof of Theorem 22 is complete. ■

3. LOWER BOUNDS

Our aim in this section is to give lower bounds on the global offensive k -alliance number of a graph in terms of its order n , minimum degree δ and maximum degree Δ .

Theorem 23. *Let k be a positive integer. If G is a graph of order n , minimum degree δ and maximum degree Δ , then*

$$(6) \quad \gamma_o^k(G) \geq \frac{n(\delta + k)}{2\Delta + \delta + k}.$$

Proof. If S is any $\gamma_o^k(G)$ -set, then

$$\Delta \gamma_o^k(G) = \Delta |S| \geq \sum_{v \in S} d_G(v) \geq \sum_{v \in V(G) - S} \frac{d_G(v) + k}{2}$$

$$\geq |V(G) - S| \frac{\delta + k}{2} = (n - \gamma_o^k(G)) \frac{\delta + k}{2}.$$

This leads to

$$\gamma_o^k(G)(2\Delta + \delta + k) \geq n(\delta + k),$$

and (6) is proved. ■

Theorem 24. *Let $k \geq 1$ be an integer, and let G be a graph of order n , minimum degree δ and maximum degree Δ . If δ is even and k odd or δ odd and k even, then*

$$(7) \quad \gamma_o^k(G) \geq \frac{n(\delta + k + 1)}{2\Delta + \delta + k + 1}.$$

Proof. If S is any $\gamma_o^k(G)$ -set, then

$$\begin{aligned} \Delta \gamma_o^k(G) &= \Delta |S| \geq \sum_{v \in S} d_G(v) \\ &\geq \sum_{v \in V(G) - S, d_G(v) = \delta} \frac{d_G(v) + k + 1}{2} + \sum_{v \in V(G) - S, d_G(v) > \delta} \frac{d_G(v) + k}{2} \\ &\geq |V(G) - S| \frac{\delta + k + 1}{2} = (n - \gamma_o^k(G)) \frac{\delta + k + 1}{2}. \end{aligned}$$

This leads to

$$\gamma_o^k(G)(2\Delta + \delta + k + 1) \geq n(\delta + k + 1),$$

and (7) is proved. ■

Example 25. Let G be a k -regular bipartite graph of order n with the partite sets X and Y . Then

$$\gamma_0^k(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta + k)}{2\Delta + \delta + k}$$

and

$$\gamma_0^{k-1}(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta + (k-1) + 1)}{2\Delta + \delta + (k-1) + 1}$$

for $k \geq 2$. This family of graphs demonstrate that the bounds in Theorems 23 and 24 are best possible.

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