# BOUNDS ON THE GLOBAL OFFENSIVE $k$-ALLIANCE NUMBER IN GRAPHS 

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#### Abstract

Let $G=(V(G), E(G))$ be a graph, and let $k \geq 1$ be an integer. A set $S \subseteq V(G)$ is called a global offensive $k$-alliance if $|N(v) \cap S| \geq$ $|N(v)-S|+k$ for every $v \in V(G)-S$, where $N(v)$ is the neighborhood of $v$. The global offensive $k$-alliance number $\gamma_{o}^{k}(G)$ is the minimum cardinality of a global offensive $k$-alliance in $G$. We present different bounds on $\gamma_{o}^{k}(G)$ in terms of order, maximum degree, independence number, chromatic number and minimum degree.


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## 1. Terminolgy

Let $G=(V, E)=(V(G), E(G))$ be a finite and simple graph. The open neighborhood of a vertex $v \in V$ is $N_{G}(v)=N(v)=\{u \in V \mid u v \in E\}$ and the closed neighborhood is $N_{G}[v]=N[v]=N(v) \cup\{v\}$. The degree of $v$, denoted by $d_{G}(v)$, is $|N(v)|$. By $n(G)=n, \Delta(G)=\Delta$ and $\delta(G)=\delta$ we denote the order, the maximum degree and the minimum degree of the graph $G$, respectively. If $A \subseteq V(G)$, then $G[A]$ is the graph induced by the vertex set $A$. We denote by $K_{n}$ the complete graph of order $n$, and by $K_{r, s}$ the complete bipartite graph with partite sets $X$ and $Y$ such that $|X|=r$ and $|Y|=s$. A set $D \subseteq V(G)$ is a $k$-dominating set of $G$ if every vertex of $V(G)-D$ has at least $k \geq 1$ neighbors in $D$. The $k$-domination number $\gamma_{k}(G)$ is the cardinality of a minimum $k$-dominating set. The case $k=1$ leads to the classical domination number $\gamma(G)=\gamma_{1}(G)$.

In [11], Kristiansen, Hedetniemi and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive $k$-alliances, given by Shafique and Dutton [14, 15]. A set $S$ of vertices of a graph $G$ is called a global offensive $k$-alliance if $|N(v) \cap S| \geq$ $|N(v)-S|+k$ for every $v \in V(G)-S$, where $k \geq 1$ is an integer. The global offensive $k$-alliance number $\gamma_{o}^{k}(G)$ is the minimum cardinality of a global offensive $k$-alliance in $G$. If $S$ is a global $k$-offensive alliance of $G$ and $|S|=\gamma_{o}^{k}(G)$, then we say that $S$ is a $\gamma_{o}^{k}(G)$-set. A global offensive 1 -alliance is a global offensive alliance and a global offensive 2 -alliance is a global strong offensive alliance. In [7], Fernau, Rodríguez and Sigarreta show that the problem of finding optimal global offensive $k$-alliances is $N P$-complete.

If $k \geq 1$ is an integer, then let $L_{k}(G)=\left\{x \in V(G): d_{G}(x) \leq k-1\right\}$. Denote by $\alpha(G)$ the independence number, by $\chi(G)$ the chromatic number, and by $\omega(G)$ the clique number of $G$, respectively. The corona graph $G \circ K_{1}$ of a graph $G$ is the graph constructed from a copy of $G$, where for each vertex $v \in V(G)$, a new vertex $v^{\prime}$ and a pendant edge $v v^{\prime}$ are added. Next assume that $G_{1}$ and $G_{2}$ are two graphs with disjoint vertex sets. The union $G=G_{1} \cup G_{2}$ has $V(G)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and $E(G)=E\left(G_{1}\right) \cup E\left(G_{2}\right)$. The join $G=G_{1}+G_{2}$ has $V(G)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and

$$
E(G)=E\left(G_{1}\right) \cup E\left(G_{2}\right) \cup\left\{u v: u \in V\left(G_{1}\right) \text { and } v \in V\left(G_{2}\right)\right\} .
$$

## 2. Upper Bounds

We begin this section by giving an upper bound on the global offensive $k$ -alliance number for an $r$-partite graph $G$ in terms of its order and $\left|L_{k}(G)\right|$.

Theorem 1. Let $k \geq 1$ be an integer. If $G$ is an $r$-partite graph, then

$$
\gamma_{o}^{k}(G) \leq \frac{(r-1) n(G)+\left|L_{k}(G)\right|}{r}
$$

Proof. Clearly, the set $L_{k}(G)$ is contained in every $\gamma_{o}^{k}(G)$-set. In the case that $\left|L_{k}(G)\right|=|V(G)|$, we are finished. In the remaining case that $\left|L_{k}(G)\right|<|V(G)|$, let $V_{1}, V_{2}, \ldots, V_{r}$ be a partition of the $r$-partite graph $G-L_{k}(G)$ such that $\left|V_{1}\right| \geq\left|V_{2}\right| \geq \cdots \geq\left|V_{r}\right|$, where $V_{i}=\emptyset$ is possible for $i \geq 2$. Then every vertex of $V_{1}$ has degree at least $k$ in $G$, and all its neighbors are in $V(G)-V_{1}$. Thus $V(G)-V_{1}$ is a global offensive $k$-alliance of $G$. Since

$$
\left|V_{1}\right| \geq \frac{\left|V_{1}\right|+\left|V_{2}\right|+\cdots+\left|V_{r}\right|}{r}=\frac{n(G)-\left|L_{k}(G)\right|}{r},
$$

we obtain

$$
\gamma_{o}^{k}(G) \leq n(G)-\left|V_{1}\right| \leq n(G)-\frac{n(G)-\left|L_{k}(G)\right|}{r}=\frac{(r-1) n(G)+\left|L_{k}(G)\right|}{r}
$$

and the proof is complete.
The case $k=r=2$ in Theorem 1 leads to the next result.
Corollary 2 (Chellali [4]). If $G$ is a bipartite graph, then

$$
\gamma_{o}^{2}(G) \leq \frac{n(G)+\left|L_{2}(G)\right|}{2}
$$

Observation 3. If $k \geq 1$ is an integer, then $\gamma_{o}^{k}(G) \geq \gamma_{k}(G)$ for any graph $G$.

Proof. If $S$ is any $\gamma_{o}^{k}(G)$-set, then every vertex of $V(G)-S$ has at least $k$ neighbors in $S$. Thus $S$ is a $k$-dominating set of $G$ and so $\gamma_{k}(G) \leq|S|=$ $\gamma_{o}^{k}(G)$.

Using Theorem 1 for $r=2$ and Observation 3, we obtain the known theorem by Blidia, Chellali and Volkmann [2].

Corollary 4 (Blidia, Chellali, Volkmann [2] 2006). Let $k$ be a positive integer. If $G$ is a bipartite graph, then

$$
\gamma_{k}(G) \leq \frac{n(G)+\left|L_{k}(G)\right|}{2}
$$

Since every graph $G$ is $\chi(G)$-partite and $n(G) \leq \chi(G) \alpha(G)$, we obtain also the following corollaries from Theorem 1.

Corollary 5. If $G$ is a graph and $k$ a positve integer, then

$$
\gamma_{o}^{k}(G) \leq \frac{(\chi(G)-1) n(G)+\left|L_{k}(G)\right|}{\chi(G)} .
$$

Corollary 6. Let $k \geq 1$ be an integer. If $G$ is a graph with $\delta(G) \geq k$, then

$$
\gamma_{o}^{k}(G) \leq(\chi(G)-1) \alpha(G)
$$

Theorem 7 (Brooks [3] 1941). If $G$ is a connected graph different from the complete graph and from a cycle of odd length, then $\chi(G) \leq \Delta(G)$.

Combining Brooks' Theorem and Corollary 6, we can prove the following result.

Theorem 8. Let $k \geq 1$ be an integer, and let $G$ be a connected graph with $\delta(G) \geq k$. Then

$$
\begin{equation*}
\gamma_{o}^{k}(G) \leq(\Delta(G)-1) \alpha(G) \tag{1}
\end{equation*}
$$

if and only if $G$ is neither isomorphic to the complete graphs $K_{k+1}$ or $K_{k+2}$ nor to a cycle of odd length when $1 \leq k \leq 2$.

Proof. If $G$ is the complete graph $K_{n}$, then $\Delta(G)=\delta(G)=n-1 \geq k \geq 1$ and $\alpha(G)=1$. Since $\gamma_{o}^{k}\left(K_{k+1}\right)=k$ and $\gamma_{o}^{k}\left(K_{k+2}\right)=k+1$, inequality (1) is not true for these two complete graphs. However, in the remaining case that $n \geq k+3$, we observe that $\gamma_{o}^{k}(G) \leq n-2$, and we arrive at the desired bound

$$
\gamma_{o}^{k}(G) \leq n-2=\Delta(G)-1=(\Delta(G)-1) \alpha(G) .
$$

Assume next that $1 \leq k \leq 2$. If $G$ is a cycle of odd length, then $\Delta(G)=2$, $\gamma_{o}^{1}(G)=\gamma_{o}^{2}(G)=\lceil n(G) / 2\rceil$ and $\alpha(G)=\lfloor n(G) / 2\rfloor$ and thus (1) is not valid in these cases.

For all other graphs inequality (1) follows directly from Brooks' Theorem and Corollary 6.

Lemma 9 (Hansberg, Meierling, Volkmann [10]). Let $k \geq 1$ be an integer. If $G$ is a connected graph with $\delta(G) \leq k-1$ and $\Delta(G) \leq k$, then

$$
k \alpha(G) \geq n(G)
$$

Theorem 10. Let $k \geq 1$ be an integer. If $G$ is a connected $r$-partite graph with $\Delta(G) \geq k$, then

$$
\gamma_{o}^{k}(G) \leq \frac{\alpha(G)}{r}((r-1) r+k-1)
$$

Proof. Assume that $k=1$. Since $G$ is connected and $\Delta(G) \geq 1$, we note that $\left|L_{1}(G)\right|=0$. Applying Theorem 1, and using the fact that $\operatorname{r\alpha }(G) \geq$ $n(G)$, we receive the desired inequality immediately.

Assume next that $k \geq 2$. Since $G$ is connected and $G-L_{k}(G)$ is not empty, every component $Q$ of $G\left[L_{k}(G)\right]$ fufills $\delta(Q) \leq k-2$ and $\Delta(Q) \leq k-1$. Hence Lemma 9 implies $(k-1) \alpha(Q) \geq n(Q)$. If $Q_{1}, Q_{2}, \ldots, Q_{t}$ are the components of $G\left[L_{k}(G)\right]$, we therefore deduce that

$$
\alpha(G) \geq \alpha\left(G\left[L_{k}(G)\right]\right)=\sum_{i=1}^{t} \alpha\left(Q_{i}\right) \geq \frac{\left|L_{k}(G)\right|}{k-1}
$$

Combining $n(G) \leq r \alpha(G)$ with Theorem 1 , we receive the desired inequality as follows:

$$
\begin{aligned}
\gamma_{o}^{k}(G) & \leq \frac{(r-1) n(G)+\left|L_{k}(G)\right|}{r} \\
& \leq \frac{(r-1) r \alpha(G)+(k-1) \alpha(G)}{r} \\
& =\frac{\alpha(G)}{r}((r-1) r+k-1)
\end{aligned}
$$

The case $r=2$ in Theorem 10 leads to the next result.

Corollary 11. Let $k \geq 1$ be an integer. If $G$ is a connected bipartite graph with $\Delta(G) \geq k$, then

$$
\gamma_{o}^{k}(G) \leq \frac{(k+1) \alpha(G)}{2}
$$

Using Observation 3, we obtain the following known bounds on the 2domination number.

Corollary 12 (Fujisawa, Hansberg, Kubo, Saito, Sugita, Volkmann [9] 2008). If $G$ is a connected bipartite graph of order at least 3 , then

$$
\gamma_{2}(G) \leq \frac{3 \alpha(G)}{2}
$$

Corollary 13 (Blidia, Chellali, Favaron [1] 2005). If $T$ is a tree of order at least 3, then

$$
\gamma_{2}(T) \leq \frac{3 \alpha(T)}{2}
$$

Theorem 14 (Favaron, Hansberg, Volkmann [6] 2008). Let $G$ be a graph. If $r \geq 1$ is an integer, then there is a partition $V(G)=V_{1} \cup V_{2} \cup \cdots \cup V_{r}$ of $V(G)$ such that

$$
\begin{equation*}
\left|N_{G}(u) \cap V_{i}\right| \leq \frac{d_{G}(u)}{r} \tag{2}
\end{equation*}
$$

for each $i \in\{1,2, \ldots, r\}$ and each $u \in V_{i}$.
Theorem 15. Let $k \geq 1$ be an integer. If $G$ is a graph of order $n$ with minimum degree $\delta \geq k$, then

$$
\begin{equation*}
\gamma_{o}^{k}(G) \leq \frac{k+1}{k+2} n \tag{3}
\end{equation*}
$$

and the bound given in (3) is best possible.
Proof. Choose $r=k+2$ in Theorem 14, and let $V_{1}, V_{2}, \ldots, V_{r}$ be a partition of $V(G)$ as in Theorem 14 such that $\left|V_{1}\right| \geq\left|V_{2}\right| \geq \cdots \geq\left|V_{r}\right|$. If $D=V_{2} \cup V_{3} \cup \cdots \cup V_{r}$, then it follows from (2) and the hypothesis that $\delta \geq k$ for each $v \in V_{1}=V(G)-D$ that

$$
\begin{aligned}
\left|N_{G}(v) \cap D\right| & \geq\left\lceil\frac{k+1}{k+2} d_{G}(v)\right\rceil \geq\left\lfloor\frac{d_{G}(v)}{k+2}\right\rfloor+k \\
& \geq\left|N_{G}(v) \cap V_{1}\right|+k=\left|N_{G}(v)-D\right|+k
\end{aligned}
$$

Thus $D$ is a global offensive $k$-alliance of $G$ such that $|D| \leq(k+1) n /(k+2)$, and (3) is proved.

Let $H$ be a connected graph, and let $G_{k}=H \circ K_{k+1}$. Then it is easy to see that $\gamma_{o}^{k}\left(G_{k}\right)=(k+1) n\left(G_{k}\right) /(k+2)$, and therefore (3) is the best possible.

Corollary 16 (Favaron, Fricke, Goddard, Hedetniemi, Hedetniemi, Kristiansen, Laskar, Skaggs [5] 2004). Let $G$ be graph of order $n$ and minimum degree $\delta$.
If $\delta \geq 1$, then $\gamma_{o}^{1}(G) \leq 2 n / 3$.
If $\delta \geq 2$, then $\gamma_{o}^{2}(G) \leq 3 n / 4$.
In the case that $\delta \geq k+2$, we obtain the following bound, improving the bound of Theorem 15 .

Theorem 17. Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n$ with minimum degree $\delta \geq k+2$. Then

$$
\begin{equation*}
\gamma_{o}^{k}(G) \leq \frac{k}{k+1} n \tag{4}
\end{equation*}
$$

Proof. Choose $r=k+1$ in Theorem 14, and let $V_{1}, V_{2}, \ldots, V_{r}$ be a partition of $V(G)$ as in Theorem 14 such that $\left|V_{1}\right| \geq\left|V_{2}\right| \geq \cdots \geq\left|V_{r}\right|$. If $D=V_{2} \cup V_{3} \cup \cdots \cup V_{r}$, then it follows from (2) and the hypothesis $\delta \geq k+2$ for each $v \in V_{1}=V(G)-D$ that

$$
\begin{aligned}
\left|N_{G}(v) \cap D\right| & \geq\left\lceil\frac { k } { k + 1 } d _ { G } ( v ) \left|\geq\left|\frac{d_{G}(v)}{k+1}\right|+k\right.\right. \\
& \geq\left|N_{G}(v) \cap V_{1}\right|+k=\left|N_{G}(v)-D\right|+k .
\end{aligned}
$$

Thus $D$ is a global offensive $k$-alliance of $G$ such that $|D| \leq k n /(k+1)$, and (4) is proved.

Theorem 18. Let $k \geq 1$ be an integer, and let $G$ be a connected noncomplete graph such that $\delta(G) \geq k$ and $\gamma_{o}^{k}(G)=(\Delta(G)-1) \alpha(G)$. Then $\Delta(G) \leq k+2, \Delta(G)-\delta(G) \leq 1$ and if $k \geq 2$, then $\delta(G) \leq k+1$.

Proof. Because of $\chi(G) \alpha(G) \geq n(G)$, Corollary 5 and the hypothesis imply that

$$
(\Delta(G)-1) \alpha(G)=\gamma_{o}^{k}(G) \leq \frac{(\chi(G)-1) n(G)}{\chi(G)} \leq(\chi(G)-1) \alpha(G)
$$

Since $G$ is neither a complete graph nor a cycle of odd length, it follows from Brooks' Theorem that $\Delta(G)=\chi(G), \chi(G) \alpha(G)=n(G)$ and

$$
\begin{equation*}
\gamma_{o}^{k}(G)=\frac{(\chi(G)-1) n(G)}{\chi(G)}=\frac{(\Delta(G)-1) n(G)}{\Delta(G)} . \tag{5}
\end{equation*}
$$

If we suppose on the contrary that $\Delta(G) \geq k+3$, then it follows from (5) and Theorem 15 that

$$
\frac{\Delta(G)-1}{\Delta(G)} n(G)=\gamma_{o}^{k}(G) \leq \frac{k+1}{k+2} n(G) \leq \frac{\Delta(G)-2}{\Delta(G)-1} n(G) .
$$

This contradiction shows that $\Delta(G) \leq k+2$.
If we suppose on the contrary that $\Delta(G)-\delta(G) \geq 2$, then we deduce that $\delta(G)=k$ and $\Delta(G)=k+2=\chi(G)$. Since $\chi(G) \alpha(G)=n(G)$, there exists a partition of $V(G)$ in $\chi=\chi(G)$ colour classes $U_{1}, U_{2}, \ldots, U_{\chi}$ such that $\left|U_{1}\right|=\left|U_{2}\right|=\cdots=\left|U_{\chi}\right|=\alpha(G)$. Let $v$ be a vertex of minimum degree $\delta(G)=k$, and assume, without loss of generality, that $v \in U_{1}$. As $d_{G}(v)=k$ and $\chi(G)=k+2$, there exists a colour class $U_{j}$ with $2 \leq j \leq \chi$ such that $v$ is not adjacent to any vertex in $U_{j}$. Therefore $U_{j} \cup\{v\}$ is an independent set. This is a contradiction to the fact that $\left|U_{j}\right|=\alpha(G)$, and the desired inequality $\Delta(G)-\delta(G) \leq 1$ is proved.

Next assume that $k \geq 2$, and suppose on the contrary that $\delta(G) \geq k+2$. Then $k \leq \Delta(G)-2$ and (5) and Theorem 17 lead to the contradiction

$$
\frac{\Delta(G)-1}{\Delta(G)} n(G)=\gamma_{o}^{k}(G) \leq \frac{k}{k+1} n(G) \leq \frac{\Delta(G)-2}{\Delta(G)-1} n(G) .
$$

Thus $\delta(G) \leq k \leq \delta(G)+1$ when $k \geq 2$, and the proof of Theorem 18 is complete.

Example 19.1. Let $H_{1}, H_{2}, \ldots, H_{t}$ be $t \geq 2$ copies of the complete graph $K_{k+1}$, and let $u_{i}, v_{i} \in E\left(H_{i}\right)$ for $1 \leq i \leq t$. Define the graph $G$ as the disjoint union $H_{1} \cup H_{2} \cup \cdots \cup H_{t}$ together with the edge set $\left\{v_{1} u_{2}, v_{2} u_{3}, \ldots, v_{t-1} u_{t}\right\}$. Then it is easy to verify that $\Delta(G)=k+1, \delta(G)=k, \alpha(G)=t, \gamma_{o}^{k}(G)=t k$ and thus $\gamma_{o}^{k}(G)=(\Delta(G)-1) \alpha(G)$.
2. Let $F_{1}$ and $F_{2}$ be 2 copies of the complete graph $K_{k+1}$ with the vertex sets $V\left(F_{1}\right)=\left\{x_{1}, x_{2}, \ldots, x_{k+1}\right\}$ and $V\left(F_{2}\right)=\left\{y_{1}, y_{2}, \ldots, y_{k+1}\right\}$. Define the graph $H$ as the disjoint union $F_{1} \cup F_{2}$ together wit the edge set $\left\{x_{1} y_{1}, x_{2} y_{2}, \ldots, x_{k} y_{k}\right\}$. If $H_{1}, H_{2}, \ldots, H_{t}$ are $t \geq 2$ copies of $H$, then let
$u_{2 i-1}$ and $u_{2 i}$ be the vertices of degree $k$ in $H_{i}$ for all $i \in\{1,2, \ldots, t\}$. Define the graph $G$ as the disjoint union $H_{1} \cup H_{2} \cup \cdots \cup H_{t}$ together with the edge set $\left\{u_{2} u_{3}, u_{4} u_{5}, \ldots, u_{2 t} u_{1}\right\}$. Then $G$ is a $(k+1)$-regular graph with $\alpha(G)=2 t, \gamma_{o}^{k}(G)=2 k t$ and thus $\gamma_{o}^{k}(G)=(\Delta(G)-1) \alpha(G)$.
3. Let $k \geq 2$, and let $F_{1}$ and $F_{2}$ be 2 copies of the complete graph $K_{k}$ such that $V\left(F_{1}\right)=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ and $V\left(F_{2}\right)=\left\{y_{1}, y_{2}, \ldots, y_{k}\right\}$. Define the graph $H$ as the disjoint union $F_{1} \cup F_{2}$ together wit the edge set $\left\{x_{1} y_{1}, x_{2} y_{2}, \ldots, x_{k-1} y_{k-1}\right\}$. If $H_{1}, H_{2}, \ldots, H_{t}$ are $t \geq 2$ copies of $H$, then let $u_{2 i-1}$ and $u_{2 i}$ be the vertices of degree $k-1$ in $H_{i}$ for all $i \in\{1,2, \ldots, t\}$. Define the graph $G$ as the disjoint union $H_{1} \cup H_{2} \cup \cdots \cup H_{t}$ together with the edge set $\left\{u_{2} u_{3}, u_{4} u_{5}, \ldots, u_{2 t} u_{1}\right\}$. Then $G$ is a $k$-regular graph with $\alpha(G)=2 t$, $\gamma_{o}^{k}(G)=2(k-1) t$ and thus $\gamma_{o}^{k}(G)=(\Delta(G)-1) \alpha(G)$.
4. Let $H_{1}$ and $H_{2}$ be 2 copies of the complete graph $K_{k+2}$, and let $x \in E\left(H_{1}\right)$ and $y \in E\left(H_{2}\right)$. Define the graph $G^{\prime}$ as the disjoint union $H_{1} \cup H_{2}$ together with the edge $x y$. Then $\Delta\left(G^{\prime}\right)=k+2, \delta\left(G^{\prime}\right)=k+1$, $\alpha\left(G^{\prime}\right)=2, \gamma_{o}^{k}\left(G^{\prime}\right)=2(k+1)$ and thus $\gamma_{o}^{k}\left(G^{\prime}\right)=\left(\Delta\left(G^{\prime}\right)-1\right) \alpha\left(G^{\prime}\right)$.

These four examples show that $\Delta=k+1$ and $\delta=k, \Delta=\delta=k+1$, $\Delta=\delta=k$ as well as $\Delta=k+2$ and $\delta=k+1$ in Theorem 18 are possible.

Theorem 20. If $G$ is a graph and $k$ an integer such that $1 \leq k \leq \delta(G)-1$, then

$$
\gamma_{o}^{k+1}(G) \leq \frac{\gamma_{o}^{k}(G)+n(G)}{2}
$$

Proof. Let $S$ be a $\gamma_{o}^{k}(G)$-set, and let $A$ be the set of isolated vertices in the subgraph induced by the vertex set $V(G)-S$. Then the subgraph induced by $V(G)-(S \cup A)$ contains no isolated vertices. If $D$ is a minimum dominating set of $G[V(G)-(S \cup A)]$, then the well-known inequality of Ore [12] implies

$$
|D| \leq \frac{|V(G)-(S \cup A)|}{2} \leq \frac{|V(G)-S|}{2}=\frac{n(G)-\gamma_{o}^{k}(G)}{2}
$$

Since $\delta(G) \geq k+1$, every vertex of $A$ has at least $k+1$ neighbors in $S$, and therefore $D \cup S$ is a global offensive ( $k+1$ )-alliance of $G$. Thus we obtain the desired bound as follows:

$$
\gamma_{o}^{k+1}(G) \leq|S \cup D| \leq \gamma_{o}^{k}(G)+\frac{n(G)-\gamma_{o}^{k}(G)}{2}=\frac{\gamma_{o}^{k}(G)+n(G)}{2}
$$

The graphs $G$ of even order and without isolated vertices with $\gamma(G)=n / 2$ have been characterized independently by Payan and Xuong [13] and Fink, Jacobson, Kinch and Roberts [8].

Theorem 21 (Payan, Xuong [13] 1982 and Fink, Jacobson, Kinch, Roberts [8] 1985). Let $G$ be a graph of even order $n$ without isolated vertices. Then $\gamma(G)=n / 2$ if and only if each component of $G$ is either a cycle $C_{4}$ or the corona of a connected graph.

A graph is $P_{4}$-free if and only if it contains no induced subgraph isomorphic to the path $P_{4}$ of order four. A graph is $\left(K_{4}-e\right)$-free if and only if it contains no induced subgraph isomorphic to the graph $K_{4}-e$, where $e$ is an arbitrary edge of the complete graph $K_{4}$. The graph $\bar{G}$ denotes the complement of the graph $G$. Next we give a characterization of some special graphs attaining equality in Theorem 20.

Theorem 22. Let $G$ be a connected $P_{4}$-free graph such that $\bar{G}$ is $\left(K_{4}-e\right)$ free. If $k$ is an integer with $1 \leq k \leq \delta(G)-1$, then $\gamma_{o}^{k+1}(G)=\left(\gamma_{o}^{k}(G)+\right.$ $n(G)) / 2$ if and only if

1. $G=K_{k+3}$ or
2. $\bar{G}=H \cup 2 K_{1,1}$ such that $n(H)=k+2$ and all components of $H$ are isomorphic to $K_{1,1}$, to $K_{3,3}$, to $K_{3,4}$ or to $K_{4,4}$ or
3. $G=\left(Q_{1} \cup Q_{2}\right)+F$, where $Q_{1}, Q_{2}$ and $F$ are three pairwise disjoint graphs such that $1 \leq|V(F)| \leq k+1, \alpha(F) \leq 2$, and $Q_{1}$ and $Q_{2}$ are cliques with $\left|V\left(Q_{1}\right)\right|=\left|V\left(Q_{2}\right)\right|=k+3-|V(F)|$ such that $|V(F)| \leq 2$ or $\alpha(F)=1$ and $|V(F)|=k+1$ or $\alpha(F)=2$ and $F=K_{k+1}-M$, where $M$ is a matching of $F$ or $\alpha(F)=2$ and $F=K_{k}-M$, where $M$ is a perfect matching of $F$ or $\alpha(F)=2$ and $|V(F)|=k+1-t$ for $0 \leq t \leq k-2$ with $k \geq 3 t+3$ and all components of $\bar{F}$ are isomorphic to $K_{t+2, t+2}$, to $K_{t+2, t+3}$ or to $K_{t+3, t+3}$.

Proof. Assume that $\gamma_{o}^{k+1}(G)=\left(\gamma_{o}^{k}(G)+n(G)\right) / 2$. Following the same notation as used in the proof of Theorem 20, we obtain $|D|=\frac{|V(G)-S|}{2}$, and we observe that $S \cup D$ is a $\gamma_{o}^{k+1}(G)$-set. It follows that $G[V(G)-S]$ has no isolated vertices and so by Theorem 21, each component of $G[V(G)-S]$ is either a cycle $C_{4}$ or the corona of some connected graph. Using the
hypothesis that $G$ is $P_{4}$-free, we deduce that each component of $G[V(G)-S]$ is isomorphic to $K_{2}$ or to $C_{4}$. Since $\bar{G}$ is $\left(K_{4}-e\right)$-free, there remain exactly the three cases that $G[V(G)-S]$ is isomorphic to $K_{2}$, to $C_{4}$ or to $2 K_{2}$.

Case 1. First assume that $G[V(G)-S]=K_{2}$. Suppose that $G$ has an independent set $Q$ of size at least two. Then the hypothesis $\delta(G) \geq k+1$ implies that $V(G)-Q$ is a global offensive $(k+1)$-alliance of $G$ of size $n-|Q|<|S \cup D|=n-1$, a contradiction. Therefore $\alpha(G)=1$ and thus $G=K_{k+3}$.

Case 2. Second assume that $G[V(G)-S]$ is a cycle $C_{4}=x_{0} x_{1} x_{2} x_{3} x_{0}$. In the following the indices of the vertices $x_{i}$ are taken modulo 4. Recall that $S \cup D$ is a $\gamma_{o}^{k+1}(G)$-set, and $D$ contains two vertices of the cycle $C_{4}$. Clearly, since $S$ is a $\gamma_{o}^{k}(G)$-set, every vertex of the cycle $C_{4}$ has degree at least $k+4$. Suppose that $d_{G}\left(x_{i}\right) \geq k+5$ for an $i \in\{0,1,2,3\}$. Then $\left\{x_{i+2}\right\} \cup S$ is a global offensive $(k+1)$-alliance of $G$ of size $|S|+1<|S \cup D|=|S|+2$, a contradiction. Thus $d_{G}\left(x_{i}\right)=k+4$ for every $i \in\{0,1,2,3\}$. Now if $Q$ is an $\alpha(G)$-set, then $|Q| \leq 2$, for otherwise the hypothesis $\delta(G) \geq k+1$ implies that $V(G)-Q$ is a global offensive $(k+1)$-alliance of $G$ of size $|V(G)-Q|<|S \cup D|=n(G)-2$, a contradiction too. Since there are two non-adjacent vertices on the cycle $C_{4}$ and $G$ is $P_{4}$-free, it follows that every vertex of $S$ has at least three neighbors on the cycle $C_{4}$.

Subcase 2.1. Assume that $\alpha(G[S])=1$. Then the subgraph induced by $S$ is complete and $|S| \geq k+2$. If $|S|=k+2$, then we observe that every vertex of $S$ has exactly four neighbours on the cycle $C_{4}$. Thus, in each case, we deduce that $d_{G}(y) \geq k+5$ for every $y \in S$. But then for any subset $W$ of $S$ of size three, the set $V(G)-W$ is a global offensive $(k+1)$-alliance of $G$ of size less than $|S \cup D|$, a contradiction.

Subcase 2.2. Assume that $\alpha(G[S])=2$. Suppose that there exists a vertex $w \in S$ with at least $k+1$ neighbors in $S$. Then, since $\left|N(w) \cap V\left(C_{4}\right)\right| \geq$ 3, say $\left\{x_{0}, x_{1}, x_{2}\right\} \subseteq N(w) \cap V\left(C_{4}\right)$, we observe that $(S-\{w\}) \cup\left\{x_{0}, x_{2}\right\}$ is a global offensive ( $k+1$ )-alliance of $G$ of size $|S|+1<|S \cup D|$, a contradiction. Thus every vertex of $S$ has at most $k$ neighbors in $S$.

Let $S=X \cup Y$ such that every vertex of $X$ has exactly three and every vertex of $Y$ exactly 4 neighbors on $C_{4}$. We shall show that $X=\emptyset$. If $X \neq \emptyset$, then let $S_{x_{i}} \subseteq X$ be the set of vertices such that each vertex of $S_{x_{i}}$ is not adjacent to $x_{i+2}$ for $i \in\{0,1,2,3\}$. Because of $\alpha(G)=2$, we observe that
the set $S_{x_{i}} \cup\left\{x_{i}\right\}$ induces a complete graph for each $i \in\{0,1,2,3\}$. In additon, since $G$ is $P_{4}$-free it is straightforward to verify that all vertices of $X \cup C_{4}$ are adjacent to all vertices of $Y$ and that $S_{x_{i}} \cup S_{x_{i+1}} \cup\left\{x_{i}, x_{i+1}\right\}$ induces a complete graph for each $i \in\{0,1,2,3\}$. Now assume, without loss of generality, that $S_{x_{0}} \neq \emptyset$, and let $w \in S_{x_{0}}$. On the one hand we have seen above that $d_{G}(w) \leq k+3$. On the other hand, we observe that $d_{G}(w)=d_{G}\left(x_{0}\right)$. But since $d_{G}\left(x_{0}\right)=k+4$, we have a contradiction.

Hence we have shown that $X=\emptyset$, and this leads to $|S|=k+2$. If we define $H=\overline{G[S]}$, then $\omega(H)=2, \delta(H) \geq 1$ and $\Delta(H) \leq 4$. Since $H$ is also $P_{4}$-free, $H$ does not contain an induced cycle of odd length. Using $\omega(H)=2$, we deduce that $H$ is a bipartite graph. Now let $H_{i}$ be a component of $H$. If $H_{i}$ is not a complete bipartite graph, then $H_{i}$ contains a $P_{4}$, a contradiction. Thus the components of $H$ consists of $K_{1,1}, K_{1,2}, K_{1,3}, K_{1,4}, K_{2,2}, K_{2,3}, K_{2,4}$, $K_{3,3}, K_{3,4}$ or $K_{4,4}$.

If $K_{1,2}$ is a component of $H$, then $V(G)-V\left(K_{1,2}\right)$ is a global offensive ( $k+1$ )-alliance of $G$ of size $n-3$, a contradiction.

If $K_{1,3}$ is a component of $H$ with a leaf $u$, then $\left(V(G)-V\left(K_{1,3}\right)\right) \cup\{u\}$ is a global offensive $(k+1)$-alliance of $G$ of size $n-3$, a contradiction.

If $K_{1,4}$ is a component of $H$ and $u, v$ are two leaves of this star, then $\left(V(G)-V\left(K_{1,3}\right)\right) \cup\{u, v\}$ is a global offensive $(k+1)$-alliance of $G$ of size $n-3$, a contradiction.

If $K_{2,2}$ is a component of $H$, then $V(G)-V\left(K_{2,2}\right)$ is a global offensive ( $k+1$ )-alliance of $G$ of size $n-4$, a contradiction.

Next let $K_{2,3}$ be a component of $H$ with the bipartition $\left\{v_{1}, v_{2}, v_{3}\right\}$ and $\left\{u_{1}, u_{2}\right\}$. Then $V(G)-\left\{u_{1}, v_{1}, v_{2}\right\}$ is a global offensive $(k+1)$-alliance of $G$ of size $n-3$, a contradiction.

Finally, let $K_{2,4}$ be a component of $H$ with the bipartition $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and $\left\{u_{1}, u_{2}\right\}$. Then $V(G)-\left\{u_{1}, v_{1}, v_{2}\right\}$ is a global offensive $(k+1)$-alliance of $G$ of size $n-3$, a contradiction.

Case 3. Third assume that $G[V(G)-S]=2 K_{2}$. Let $2 K_{2}=J_{1} \cup J_{2}=J$ such that $V\left(J_{1}\right)=\left\{u_{1}, u_{2}\right\}$ and $V\left(J_{2}\right)=\left\{u_{3}, u_{4}\right\}$. If $\alpha(G) \geq 3$, then we obtain the contradiction $\gamma_{o}^{k+1}(G) \leq n-3$. Thus $\alpha(G)=2$. Since $S$ is a $\gamma_{o}^{k}(G)$-set, every vertex of $J$ has degree at least $k+2$. Suppose that $d_{G}\left(u_{1}\right) \geq k+3$ and $d_{G}\left(u_{2}\right) \geq k+3$. Then $\left\{u_{3}\right\} \cup S$ is a global offensive $(k+1)$-alliance of $G$ of size $|S|+1<|S \cup D|=|S|+2$, a contradiction. Thus $J_{1}$ contains at least one vertex of degree $k+2$, and for reason of symmetry, also $J_{2}$ contains a vertex of degree $k+2$. Since $\alpha(G)=2$, every vertex of
$S$ has at least two neighbors in $J_{1}$ or in $J_{2}$. Now let $x \in S$. If $x$ has two neighbors in $J_{i}$ and one neighbor in $J_{3-i}$ for $i=1,2$, then the hypothesis that $G$ is $P_{4}$-free implies that $x$ is adjacent to each vertex of $J$. Consequently, $S$ can be partioned in three subsets $S_{1}, S_{2}$ and $A$ such that all vertices of $S_{1}$ are adjacent to all vertices of $J_{1}$ and there is no edge between $S_{1}$ and $J_{2}$, all vertices of $S_{2}$ are adjacent to all vertices of $J_{2}$ and there is no edge between $S_{2}$ and $J_{1}$, all vertices of $A$ are adjacent to all vertices of $J$. Since $G$ is $P_{4}$-free, it follows that there is no edge between $S_{1}$ and $S_{2}$, and that all vertices of $S_{i}$ are adjacent to all vertices of $A$ for $i=1,2$. Furthermore, $\alpha(G)=2$ shows that $G\left[S_{1}\right]$ and $G\left[S_{2}\right]$ are cliques. Altogether we see that $d_{G}\left(u_{i}\right)=k+2$ for each $i \in\{1,2,3,4\}$ and therefore $\left|S_{1}\right|+|A|=\left|S_{2}\right|+|A|=k+1$. It follows that $\left|S_{1}\right|=\left|S_{2}\right|$ and $|S|+|A|=2 k+2$. Since $G$ is connected, we deduce that $|A| \geq 1$ and so $1 \leq|A| \leq k+1$. If we define $F=G[A]$ and $Q_{i}=G\left[S_{i} \cup V\left(J_{i}\right)\right]$ for $i=1,2$, then we derive the desired structure, since $\alpha(G[A]) \leq 2$.

Assume that $|V(F)| \geq 3$ and $\alpha(F)=1$. If $x_{1}, x_{2}, x_{3}$ are three arbitrary vertices in $F$, then let $S_{0}=V(G)-\left\{x_{1}, x_{2}, x_{3}\right\}$. If $d_{G}\left(x_{i}\right) \geq k+5$ for each $i=1,2,3$, then $S_{0}$ is a global offensive ( $k+1$ )-alliance of $G$, a contradiction. Otherwise, we have $n-1=d_{G}\left(x_{i}\right) \leq k+4$ for at least one $i \in\{1,2,3\}$ and so $n \leq k+5$ and thus $|V(F)|=k+1$.

Assume next that $|V(F)| \geq 3$ and $\alpha(F)=2$. As we have seen in Case 2, all components of $\bar{F}$ are complete bipartite graphs.

Subcase 3.1. Assume that $K_{1,1}$ is the greatest component of $\bar{F}$. Let $u$ and $v$ be the two vertices of the complete bipartite graph $K_{1,1}$. If $n \geq k+7$, then let $w$ be a further vertex in $F$, and it is easy to verify that $V(G)-\{u, v, w\}$ is a global offensive ( $k+1$ )-alliance of $G$ of size $n-3$, a contradiction. If $n=k+6$ and there exists a vertex $w$ in $F$ of degree $k+5$, then $V(G)-\{u, v, w\}$ is a global offensive $(k+1)$-alliance of $G$ of size $n-3$, a contradiction.

Subcase 3.2. Assume that $|V(F)|=k+1-t$ for $0 \leq t \leq k-2$ and $\bar{F}$ contains a component $K_{p, q}$ with $1 \leq p \leq q$ and $p+q \geq 3$. Let $\left\{v_{1}, v_{2}, \ldots, v_{q}\right\}$ and $\left\{u_{1}, u_{2}, \ldots, u_{p}\right\}$ be a partition of $K_{p, q}$.

If $K_{1, s} \subseteq \bar{F}$ with $s \geq t+4$, then $\delta(G) \leq k$, a contradiction to $\delta(G) \geq$ $k+1$. Thus $q \leq t+3$.

If $q \leq t+1$ or $q=t+2$ and $p \leq t+1$, then it is easy to see that $V(G)-\left\{u_{1}, v_{1}, v_{2}\right\}$ is a global offensive $(k+1)$-alliance of $G$ of size $n-3$, a contradiction.

Conversely, if $G=K_{k+3}$, then obviously $\gamma_{o}^{k}(G)=k+1, \gamma_{o}^{k+1}(G)=k+2$ and so $\gamma_{o}^{k+1}(G)=\left(\gamma_{o}^{k}(G)+n(G)\right) / 2$.

Now let $\bar{G}=H \cup 2 K_{1,1}$ such that $n(H)=k+2$ and the components of $H$ are complete bipartite graphs $K_{1,1}, K_{3,3}, K_{3,4}$ or $K_{4,4}$. Thus $k+1 \leq$ $d_{G}(z) \leq k+4$ for every $z \in V(G)$, and $G$ contains a cycle $C$ on four vertices, where each vertex of $C$ has degree $k+4$. Clearly, $V(H)$ is a global offensive $k$-alliance of $G$ and so $\gamma_{o}^{k}(G) \leq n(G)-4$. If $D$ is a $\gamma_{o}^{k}(G)$-set of size $|D| \leq n(G)-5$, then, since $\alpha(G)=2$, the induced subgraph $G[V(G)-D]$ contains a vertex $x$ of degree at least two. This leads to the contradiction $\left|N_{G}(x) \cap D\right| \leq k+1<\left|N_{G}(x)-D\right|+k$. Hence we have shown that $\gamma_{o}^{k}(G)=n(G)-4$.

Now let us prove that $\gamma_{o}^{k+1}(G)=n(G)-2$. Clearly, $\gamma_{o}^{k+1}(G) \geq \gamma_{o}^{k}(G) \geq$ $n(G)-4$. Let $D$ be a $\gamma_{o}^{k+1}(G)$-set. First, assume that $\gamma_{o}^{k+1}(G)=n(G)-4$. Then, since $n(G)=k+6$ and $\alpha(G)=2$, the induced subgraph $G[V(G)-D]$ is isomorphic to $2 K_{1,1}$, say $a b$ and $c d$, and every vertex of $V(G)-D$ is adjacent to all vertices of $D$. Since $d_{G}(x)=k+3$ for every $x \in\{a, b, c, d\}$ it follows that $a, b, c, d$ lie in one component $C_{4}$ of $H$, a contradiction. Second, assume that $\gamma_{o}^{k+1}(G)=n(G)-3$. Since every vertex has degree at most $k+4$, no vertex of $V(G)-D$ has two neighbors in $V(G)-D$. Moreover, since $\alpha(G)=2, G[V(G)-D]$ is formed by two adjacent vertices $x, y$ plus an isolated vertex $w$. Since $w$ has degree at least two in $\bar{G}$, the vertices $w, x, y$ lie in one component in $H$ and so belong to $K_{3,3}, K_{3,4}$ or $K_{4,4}$. Thus each of $x$ and $y$ has at least two non-neighbors in $D$ and hence $|N(x) \cap D| \leq k+1$, a contradiction to the fact $D$ is a $\gamma_{o}^{k+1}(G)$-set. Thus $|D| \geq n(G)-2$ and the equality follows from the fact that $V(G)$ minus any two non-adjacent vertices of $C$ is a global offensive $(k+1)$-alliance of $G$. Therefore $\gamma_{o}^{k+1}(G)=$ $n(G)-2=\left(\gamma_{o}^{k}(G)+n(G)\right) / 2$.

Finally, let $G=\left(Q_{1} \cup Q_{2}\right)+F$, where $Q_{1}, Q_{2}$ and $F$ are three pairwise disjoint graphs such that $1 \leq|V(F)| \leq k+1, \alpha(F) \leq 2$, and $Q_{1}$ and $Q_{2}$ are cliques with $\left|V\left(Q_{1}\right)\right|=\left|V\left(Q_{2}\right)\right|=k+3-|V(F)|$ such that $|V(F)| \leq 2$ or $\alpha(F)=1$ and $|V(F)|=k+1$ or $\alpha(F)=2$ and $F=K_{k+1}-M$, where $M$ is matching of $F$ or $\alpha(F)=2$ and $F=K_{k}-M$, where $M$ is a perfect matching of $F$ or $\alpha(F)=2$ and $|\underline{V}(F)|=k+1-t$ for $0 \leq t \leq k-2$ with $k \geq 3 t+3$ and all components of $\bar{F}$ are isomorphic to $K_{t+2, t+2}$, to $K_{t+2, t+3}$ or to $K_{t+3, t+3}$.

Let $D$ be a global offensive $(k+1)$-alliance of $G$. Since each vertex of $Q_{i}$ has degree $k+2$, the set $V(G)-D$ contains at most one vertex of $Q_{i}$ for every $i=1,2$. Moreover, if $(V(G)-D) \cap V\left(Q_{i}\right) \neq \emptyset$, then $V(F) \subseteq D$.

Now suppose that $\gamma_{o}^{k+1}(G) \leq n-3$, and assume, without loss of generality, that $V(G)-D=\{u, v, w\}$. Then as noted above $V\left(Q_{1}\right) \cup V\left(Q_{2}\right) \subseteq D$, and hence the vertices $u, v, w$ belong to $V(F)$. It follows that $|V(F)| \geq 3$.

Obviously, we obtain a contradiction when $\alpha(F)=1$ and $|V(F)|=k+1$.
Assume next that $\alpha(F)=2$. This implies that at least two vertices of $V(G)-D$ are adjacent in $G$.

First assume that $F=K_{k}-M$, where $M$ is a perfect matching of $F$. Note that every vertex of $V(F)$ has degree $k+4$. Since $M$ is perfect, $\{u, v, w\}$ induces either a path $P_{3}$ or a clique $K_{3}$ with center vertex, say $v$, in $G$. But then $v$ has a non-neighbor in $D$ for which it is matched in $M$, and so $v$ has exaclty $k+2$ neighbors in $D$ against two in $V(G)-D$, a contradiction.

Second assume that $F=K_{k+1}-M$, where $M$ is a matching of $F$. Note that $n=k+5$ and $|D|=k+2$. As above, $\{u, v, w\}$ induces either a path $P_{3}$ or a clique $K_{3}$ with center vertex, say $v$, in $G$. But then $v$ has at most $k+2$ neighbors in $D$ against two in $V(G)-D$, a contradiction.

Assume now that $\alpha(F)=2$ and $|V(F)|=k+1-t$ for $0 \leq t \leq k-2$ with $k \geq 3 t+3$ and all components of $\bar{F}$ are isomorphic to $K_{t+2, t+2}$, to $K_{t+2, t+3}$ or to $K_{t+3, t+3}$. Note that in this case $n=k+5+t$ and so $|D|=n-3=k+2+t$. Assume, without loss of generality, that $u$ and $v$ are adjacent in $G$. This leads to $\left|N_{G}(u) \cap D\right| \leq(k+5+t)-(t+2+2)=k+1$, a contradiction to the assumption that $D$ is a global offensive $(k+1)$-alliance of $G$.

Altogether, we have shown that $\gamma_{o}^{k+1}(G)=n-2$. Finally, it is a simple matter to obtain $\gamma_{o}^{k}(G)=n-4$, and the proof of Theorem 22 is complete.

## 3. Lower Bounds

Our aim in this section is to give lower bounds on the global offensive $k$ alliance number of a graph in terms of its order $n$, minimum degree $\delta$ and maximum degree $\Delta$.

Theorem 23. Let $k$ be a positive integer. If $G$ is a graph of order $n$, minimum degree $\delta$ and maximum degree $\Delta$, then

$$
\begin{equation*}
\gamma_{o}^{k}(G) \geq \frac{n(\delta+k)}{2 \Delta+\delta+k} \tag{6}
\end{equation*}
$$

Proof. If $S$ is any $\gamma_{o}^{k}(G)$-set, then

$$
\Delta \gamma_{o}^{k}(G)=\Delta|S| \geq \sum_{v \in S} d_{G}(v) \geq \sum_{v \in V(G)-S} \frac{d_{G}(v)+k}{2}
$$

$$
\geq|V(G)-S| \frac{\delta+k}{2}=\left(n-\gamma_{o}^{k}(G)\right) \frac{\delta+k}{2} .
$$

This leads to

$$
\gamma_{o}^{k}(G)(2 \Delta+\delta+k) \geq n(\delta+k)
$$

and (6) is proved.
Theorem 24. Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n$, minimum degree $\delta$ and maximum degree $\Delta$. If $\delta$ is even and $k$ odd or $\delta$ odd and $k$ even, then

$$
\begin{equation*}
\gamma_{o}^{k}(G) \geq \frac{n(\delta+k+1)}{2 \Delta+\delta+k+1} \tag{7}
\end{equation*}
$$

Proof. If $S$ is any $\gamma_{o}^{k}(G)$-set, then

$$
\begin{aligned}
\Delta \gamma_{o}^{k}(G) & =\Delta|S| \geq \sum_{v \in S} d_{G}(v) \\
& \geq \sum_{v \in V(G)-S, d_{G}(v)=\delta} \frac{d_{G}(v)+k+1}{2}+\sum_{v \in V(G)-S, d_{G}(v)>\delta} \frac{d_{G}(v)+k}{2} \\
& \geq|V(G)-S| \frac{\delta+k+1}{2}=\left(n-\gamma_{o}^{k}(G)\right) \frac{\delta+k+1}{2} .
\end{aligned}
$$

This leads to

$$
\gamma_{o}^{k}(G)(2 \Delta+\delta+k+1) \geq n(\delta+k+1),
$$

and (7) is proved.
Example 25. Let $G$ be a $k$-regular bipartite graph of order $n$ with the partite sets $X$ and $Y$. Then

$$
\gamma_{0}^{k}(G)=|X|=|Y|=\frac{n}{2}=\frac{n(\delta+k)}{2 \Delta+\delta+k}
$$

and

$$
\gamma_{0}^{k-1}(G)=|X|=|Y|=\frac{n}{2}=\frac{n(\delta+(k-1)+1)}{2 \Delta+\delta+(k-1)+1}
$$

for $k \geq 2$. This family of graphs demonstrate that the bounds in Theorems 23 and 24 are best possible.

## References

[1] M. Blidia, M. Chellali and O. Favaron, Independence and 2-domination in trees, Australas. J. Combin. 33 (2005) 317-327.
[2] M. Blidia, M. Chellali and L. Volkmann, Some bounds on the p-domintion number in trees, Discrete Math. 306 (2006) 2031-2037.
[3] R.L. Brooks, On colouring the nodes of a network, Proc. Cambridge Philos. Soc. 37 (1941) 194-197.
[4] M. Chellali, Offensive alliances in bipartite graphs, J. Combin. Math. Combin. Comput., to appear.
[5] O. Favaron, G. Fricke, W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, R.C. Laskar and D.R. Skaggs, Offensive alliances in graphs, Dicuss. Math. Graph Theory 24 (2004) 263-275.
[6] O. Favaron, A. Hansberg and L. Volkmann, On $k$-domination and minimum degree in graphs, J. Graph Theory 57 (2008) 33-40.
[7] H. Fernau, J.A. Rodríguez and J.M. Sigarreta, Offensive r-alliance in graphs, Discrete Appl. Math. 157 (2009) 177-182.
[8] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, On graphs having domination number half their order, Period. Math. Hungar. 16 (1985) 287-293.
[9] J. Fujisawa, A. Hansberg, T. Kubo, A. Saito, M. Sugita and L. Volkmann, Independence and 2-domination in bipartite graphs, Australas. J. Combin. 40 (2008) 265-268.
[10] A. Hansberg, D. Meierling and L. Volkmann, Independence and p-domination in graphs, submitted.
[11] P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi, Alliances in graphs, J. Combin. Math. Combin. Comput. 48 (2004) 157-177.
[12] O. Ore, Theory of Graphs (Amer. Math. Soc. Colloq. Publ. 38, 1962).
[13] C. Payan and N.H. Xuong, Domination-balanced graphs, J. Graph Theory 6 (1982) 23-32.
[14] K. H. Shafique and R.D. Dutton, Maximum alliance-free and minimum alliance-cover sets, Congr. Numer. 162 (2003) 139-146.
[15] K. H. Shafique and R.D. Dutton, A tight bound on the cardinalities of maximum alliance-free and minimum alliance-cover sets, J. Combin. Math. Combin. Comput. 56 (2006) 139-145.

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