

## POTENTIALLY $H$ -BIGRAPHIC SEQUENCES

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### Abstract

We extend the notion of a potentially  $H$ -graphic sequence as follows. Let  $A$  and  $B$  be nonnegative integer sequences. The sequence pair  $S = (A, B)$  is said to be *bigraphic* if there is some bipartite graph  $G = (X \cup Y, E)$  such that  $A$  and  $B$  are the degrees of the vertices in  $X$  and  $Y$ , respectively. If  $S$  is a bigraphic pair, let  $\sigma(S)$  denote the sum of the terms in  $A$ .

Given a bigraphic pair  $S$ , and a fixed bipartite graph  $H$ , we say that  $S$  is *potentially  $H$ -bigraphic* if there is some realization of  $S$  containing  $H$  as a subgraph. We define  $\sigma(H, m, n)$  to be the minimum integer  $k$  such that every bigraphic pair  $S = (A, B)$  with  $|A| = m, |B| = n$  and  $\sigma(S) \geq k$  is potentially  $H$ -bigraphic. In this paper, we determine  $\sigma(K_{s,t}, m, n)$ ,  $\sigma(P_t, m, n)$  and  $\sigma(C_{2t}, m, n)$ .

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## 1. INTRODUCTION

Let  $S = (A, B) = (a_1, \dots, a_m; b_1, \dots, b_n)$  be a pair of positive integer sequences. We say that  $S$  is a *bigraphic pair* if there exists some simple bipartite graph  $G$  with partite sets  $X = \{x_1, \dots, x_m\}$  and  $Y = \{y_1, \dots, y_n\}$  such that the degree of  $x_i$  is  $a_i$  and the degree of  $y_j$  is  $b_j$ . In this case, we say that  $G$  is a bigraphic realization of  $S$ . In this paper, as the bipartite context is clear, we will simply call  $G$  a realization of  $S$ . One easy method to determine if a given sequence pair is bigraphic is the Gale-Ryser condition [3, 11]. Given a bipartite graph  $H$  and a bigraphic pair  $S$ , we say that  $S$  is *potentially  $H$ -bigraphic* if there is some realization of  $S$  that contains  $H$  as a subgraph. This is a weakening of the Zarankiewicz problem [12], which is the bipartite analogue to determining the extremal function for arbitrary subgraphs. This seemingly innocent variant to the classical Turán problem has proven to be much more challenging over time. A good discussion of the problem and its rich history can be found in [1].

Given a bigraphic sequence pair  $S = (A, B)$ , let  $\sigma(S)$  denote the sum of the terms in either  $A$  or  $B$  (which are necessarily equal). For a given bipartite graph  $H$ , let  $\sigma(H, m, n)$  denote the minimum integer  $k$  such that any bigraphic pair  $S = (A, B)$  with  $|A| = m, |B| = n$  and  $\sigma(S) \geq k$  is potentially  $H$ -bigraphic. This is a natural extension of the notion of a potentially  $H$ -graphic sequence, which has been widely studied.

In this paper, we will determine  $\sigma(H, m, n)$  for several graphs  $H$ . In Section 2, we determine  $\sigma(K_{s,t}, m, n)$ , where  $K_{s,t}$  is the complete bipartite graph with vertex sets of size  $s$  and  $t$ . In Section 3 we find  $\sigma(P_t, m, n)$ , where  $P_t$  is the path on  $t$  vertices. Finally, in Section 4, we use the two previous results to determine  $\sigma(C_{2t}, m, n)$  for even cycles  $C_{2t}$ .

The following useful lemma is an extension of a result found in [4].

**Lemma 1.1.** *Let  $S$  be a bigraphic pair with realization  $G = (X \cup Y, E)$  having partite sets  $X$  and  $Y$ . Let  $H = (X' \cup Y', E')$  be a subgraph of  $G$  such that  $X'$  and  $Y'$  are contained in  $X$  and  $Y$ , respectively. Then there exists a realization  $G_1 = (X \cup Y, E_1)$  of  $S$  containing  $H$  as a subgraph such that  $X'$  and  $Y'$  lie on the vertices of highest degree in  $X$  and  $Y$ , respectively.*

**Proof.** Let  $G = G(X \cup Y, E)$  be a realization of bigraphic sequence  $S$  containing a graph  $H$  as a subgraph, such that  $\{u, v\} \subset X$ , (or  $\{u, v\} \subset Y$ ),  $u \notin V(H)$ ,  $v \in V(H)$ , and  $\deg_G(u) \geq \deg_G(v)$ . Let  $T = N_H(v) \setminus N_G(u, v)$  be the neighbors of  $v$  in  $H$  that are not neighbors of  $u$ . Since  $|N_G(u)| \geq |N_G(v)|$ ,

we have that

$$|N_G(u) \setminus N_G(u, v)| \geq |N_G(v) \setminus N_G(u, v)| \geq |T|,$$

thus there exists subset  $T'$  of  $N_G(u) \setminus N_G(u, v)$  of size  $|T|$ . Let  $G' = G'(X \cup Y, E')$  where

$$E' = E \setminus (E(u + T') \cup E(v + T)) \cup (E(u + T) \cup E(v + T')).$$

Then  $G'$  is a realization of  $S$  containing a copy of  $H$  with the vertex  $u$  in place of the vertex  $v$ . The lemma follows. ■

Throughout this paper, we will assume each sequence in a given sequence pair is nonincreasing. We will also often use exponential notation for a degree sequence. That is, we will write  $(a_1^{\alpha_1}, \dots, a_r^{\alpha_r}; b_1^{\beta_1}, \dots, b_s^{\beta_s})$  to denote the sequence pair

$$(a_1, \dots, a_1, a_2, \dots, a_2, \dots, a_r, \dots, a_r; b_1, \dots, b_1, b_2, \dots, b_2, \dots, b_s, \dots, b_s)$$

in which  $a_i$  and  $b_j$  occur  $\alpha_i$  and  $\beta_j$  times respectively.

## 2. COMPLETE BIPARTITE GRAPHS

In this section, we determine  $\sigma(K_{s,t}, m, n)$ . The problem of determining when a graphic sequence contains a copy of  $K_{s,t}$  has been studied, and the interested reader may wish to compare the corresponding results, found in [8] and [9]. In the bipartite setting, determining  $\sigma(K_{s,t}, m, n)$  might be considered analagous to determining when a graphic sequence has a realization containing a copy of  $K_t$ , as in [2, 4, 6], and [7].

**Theorem 2.1.** *For all  $1 \leq s \leq t$ , there exists  $m_0$  such that for  $n \geq m \geq m_0$  the following holds.*

$$\sigma(K_{s,t}, m, n) = n(s - 1) + m(t - 1) - (t - 1)(s - 1) + 1.$$

**Proof.** We begin by exhibiting a bigraphic pair  $S$  with  $\sigma(S) = n(s - 1) + m(t - 1) - (t - 1)(s - 1)$  which is not potentially  $K_{s,t}$ -bigraphic. Consider the sequence pair

$$S = (n^{s-1}, (t - 1)^{m-s+1}; m^{s-1}, (t - 1)^{m-s+1}, (s - 1)^{n-m}).$$

This sequence is bigraphic, and neither partite set in any realization of  $S$  has  $s$  vertices of degree  $t$ . Hence  $S$  is not potentially  $K_{s,t}$ -graphic.

Moving forward, let  $S$  be a bigraphic pair with  $\sigma(S)$  at least  $n(s-1) + m(t-1) - (t-1)(s-1) + 1$ . Let  $G$  be a realization of  $S$  with partite sets  $X$  and  $Y$ , with  $|X| = n$  and  $|Y| = m$ . Let  $X_t$  be the set of  $t$  highest degree vertices of  $X$ , and  $Y_s$  be the set of  $s$  highest degree vertices of  $Y$ . Assume that  $G$  is a realization of  $S$  that maximizes the number of edges between  $X_t$  and  $Y_s$ . If the graph on  $X_t \cup Y_s$  is  $K_{s,t}$  we are done, so assume otherwise. Let  $x$  and  $y$  be nonadjacent members of  $X_t$  and  $Y_s$ , and let  $H_X = X_t \setminus \{x\}$  and  $H_Y = Y_s \setminus \{y\}$ .

Let  $A$  denote  $N(y) \setminus H_X$  and let  $B$  denote  $N(x) \setminus H_Y$ . Note that neither  $A$  nor  $B$  is empty, as it is straightforward to show that  $x$  and  $y$  have degrees at least  $s$  and  $t$ , respectively.

**Claim 2.2.** Let  $a$  and  $b$  lie in  $A$  and  $B$  respectively. Then  $ab$  is an edge of  $G$ .

*Proof.* Assume otherwise, and exchange the edges  $ya$  and  $xb$  for the nonedges  $ab$  and  $xy$ . This preserves the degree sequence of  $G$ , but contradicts our assumption that  $G$  had the maximum number of edges between  $X_t$  and  $Y_s$  among all realizations of  $S$ . ■

Claim 2.2 implies that the subgraph of  $G$  induced by  $A$  and  $B$  is a complete bipartite graph.

**Claim 2.3.** For each  $b$  in  $B$  there exists a vertex  $h_x$  in  $H_X$  such that  $b$  is not adjacent to  $h_x$ . Similarly, for each  $a$  in  $A$  there exists a vertex  $h_y$  in  $H_Y$  such that  $a$  is not adjacent to  $h_y$ .

*Proof.* We prove the first statement. The proof of the second is similar. Assume the first statement is false. Then, as  $b$  is adjacent to  $x$ ,

$$d(b) \geq |H_X| + |A| + 1 > d(y).$$

This contradicts the fact that  $y$  is one of the  $s$  highest degree vertices in  $Y$ . ■

Claim 2.3 immediately implies the following two claims.

**Claim 2.4.** Let  $b$  and  $h_x$  be nonadjacent vertices in  $B$  and  $H_X$  respectively. Then for all  $a$  in  $A$  and all  $v$  in  $N(h_x) \setminus (Y_s \cup B)$ ,  $av$  is an edge of  $G$ .

The analogous statement about nonadjacent  $a$  and  $h_y$  in  $A$  and  $H_Y$  respectively, is also true.

**Proof.** Again, we prove just the first statement. Assume it is false, i.e., that there is some  $a \in A$  and some  $v \in N(h_x) \setminus (Y_t \cup B)$  that are not adjacent. Then we could exchange the edges  $ay, bx$  and  $h_x v$  for the nonedges  $h_x b, av$  and  $xy$ . This is, again, a contradiction to our choice of  $G$ . ■

This allows us to bound the number of vertices in  $A$  and  $B$  as follows.

**Claim 2.5.** Let  $A$  and  $B$  be as defined above. Then both  $A$  and  $B$  contain at most

$$(s - 1)(t - 1)$$

vertices.

**Proof.** We prove  $|B| \leq (s - 1)(t - 1)$ . The proof for  $|A|$  is similar. Assume that  $|B| > (s - 1)(t - 1)$ . By Claim 2.3 and the pigeonhole principle there must be some  $h_x$  in  $H_X$  that is nonadjacent to at least  $s$  vertices in  $B$ . Its neighborhood is

$$N(h_x) = [N(h_x) \setminus (Y_s \cup B)] \cup [N(h_x) \cap Y_s] \cup [N(h_x) \cap B],$$

so we have that

$$d(h_x) \leq |N(h_x) \setminus (Y_s \cup B)| + s + (|B| - s) = |N(h_x) \setminus (Y_s \cup B)| + |B|.$$

On the other hand, for any vertex  $a$  in  $A$ , Claim 2.4 and the comment following Claim 2.2 implies that the neighborhood of  $a$  contains

$$[N(h_x) \setminus (Y_s \cup B)] \cup B \cup \{y\},$$

so we have

$$(1) \quad d(a) \geq |N(h_x) \setminus (Y_s \cup B)| + |B| + 1 > d(h_x).$$

This contradicts the fact that  $h_x$  is in  $X_t$ . ■

**Claim 2.6.** Let  $h_x$  and  $h_y$  be as given above. Then

$$d(h_x) < 2s + |B| \quad \text{and} \quad d(h_y) < 2t + |A|.$$

**Proof.** Assume  $d(h_x) \geq |B| + 2s$ . Then by equation (1), we have for any  $a$  in  $A$  that

$$d(a) \geq [d(h_x) - |(Y_s \cup B)|] + |B| + 1 > |B| + s > d(x).$$

This contradicts the assumption that  $d(x) \geq d(a)$ . The proof for  $h_y$  is similar. ■

Now since both  $d(x)$  and  $d(h_x)$  are bounded by  $2s + |B|$ , and they are both in  $X_t$  the number of edges in  $G$  that are incident to vertices of either  $X_t$  or  $A$  is at most

$$(t - 2)m + (2s + |B|)(|A| + 2).$$

Similarly the number of edges incident to  $Y_s$  or  $B$  is at most

$$(s - 2)n + (2t + |A|)(|B| + 2).$$

By Claim 2.5, this accounts for at most

$$(t - 2)m + (s - 2)n + (2s + (s - 1)(t - 1))((s - 1)(t - 1) + 2) \\ + (2t + (s - 1)(t - 1))((s - 1)(t - 1) + 2),$$

which is less than  $(t - 2)m + (s - 2)n + 6s^2t^2$ , edges from  $G$ . Taking  $m$  and  $n$  larger than  $4s^2t^2$  this is strictly less than  $\sigma(S)$ .

Furthermore,  $A$  has at most  $|A|d(h_x)$  neighbors, which by Claims 2.5 and 2.6, is at most

$$(2s + (s - 1)(t - 1))(s - 1)(t - 1) < 3s^2t^2.$$

Each of these vertices which is outside of  $Y_s$  has at most  $d(h_y) < 2t + (s - 1)(t - 1)$  neighbors. Thus at most  $9s^4t^3$  vertices in  $X$  have neighbors outside of  $Y_s$  which are adjacent to vertices of  $A$ . Assuming that  $m_0 = 9s^4t^4$ , together  $n > m_0 > 9s^4t^3$  and  $m > m_0 > 9t^4s^3$ , ensure that there exists some edge  $e = x'y'$ , with  $x' \in X - X_t - A$  and  $y' \in Y - Y_t - B$ , and vertices  $a$  and  $b$  in  $A$  and  $B$  respectively, such that  $x'b$  and  $y'a$  are not edges in  $G$ . We can then exchange the edges  $ab$  and  $e$  for the non-edges  $x'a$  and  $y'b$ , contradicting Claim 2.2, and completing the proof. ■

We note in the proof that the sets  $A$  and  $B$  induce a complete bipartite graph. Hence at least one of  $A$  and  $B$  contains at most  $t - 1$  vertices, and if either contains more than  $t - 1$  vertices, the other set contains at most  $s - 1$  vertices. This would be useful if one were interested in finding smaller bounds on the  $n$  and  $m$  necessary to assure Theorem 2.1.

3. PATHS

Recall that  $P_t$  denotes the path on  $t$  vertices. In this section we determine the quantity  $\sigma(P_t, m, n)$ . In particular, we prove the following.

**Theorem 3.1.** *For  $t \geq 2$  and integers  $n \geq m \geq t + 1$ ,*

$$\sigma(P_{2t+1}, m, n) = \sigma(P_{2t+2}, m, n) = n(t - 1) + m - (t - 1) + 1.$$

To see that both  $\sigma(P_{2t+1}, m, n)$  and  $\sigma(P_{2t+2}, m, n)$  are greater than  $n(t - 1) + m - (t - 1)$ , consider the sequence pair  $S = (m^1, (t - 1)^{n-1}; n^{t-1}, 1^{m-t+1})$ . This pair has  $\sigma(S) = n(t - 1) + m - (t - 1)$  and has a unique realization, which contains no  $P_{2t+1}$ .

The remainder of the section is dedicated to showing that a bigraphic sequence with the above sum has a realization containing a  $P_{2t+1}$  and a realization containing a  $P_{2t+2}$ . The proof will be by induction on  $t$ . The following lemma is sufficient to act as a basis for this induction, and is also of interest for the sake of completeness.

**Lemma 3.2.** *Let  $n \geq m$  be integers. Then,*

- (i)  $\sigma(P_3, m, n) = m + 1$ , and
- (ii)  $\sigma(P_4, m, n) = n + 1$ .

**Proof.** That  $\sigma(P_3, m, n) \geq m + 1$  and  $\sigma(P_4, m, n) \geq n + 1$  is obvious. Equality for statement (i) follows from the fact that with degree sum  $m + 1$  some vertex in any realization must have degree 2, and hence be the center vertex of a  $P_3$ . For statement (ii) observe that with degree sum  $n + 1$ , at least one vertex in each partite set has degree 2 or more. Applying Lemma 1.1 with  $H = K_{1,1}$ , there exists a realization in which these vertices are adjacent, and hence lie in a  $P_4$ . ■

The induction follows immediately from the following two lemmas.

**Lemma 3.3.** *For  $t \geq 2$  and integers  $n \geq m \geq t + 1$ , if  $S$  is a bigraphic pair with*

$$\sigma(S) \geq n(t - 1) + m - (t - 1) + 1,$$

*and  $S$  is potentially  $P_{2t}$ -bigraphic, then  $S$  is potentially  $P_{2t+1}$ -bigraphic.*

**Lemma 3.4.** *For  $t \geq 2$  and integers  $n \geq m \geq t + 1$ , if  $S$  is a bigraphic pair with*

$$\sigma(S) \geq n(t - 1) + m - (t - 1) + 1,$$

*and  $S$  is potentially  $P_{2t+1}$ -bigraphic, then  $S$  is potentially  $P_{2t+2}$ -bigraphic.*

To finish the proof of Theorem 3.1 we thus prove Lemmas 3.3 and 3.4.

**Proof of Lemma 3.3.** Let  $S = (A, B) = (a_1 \dots, a_n; b_1, \dots, b_m)$  be a bigraphic pair with  $\sigma(S) = n(t - 1) + m - (t - 1) + 1$ , and let  $G = G(X \cup Y, E)$  be a realization of  $S$ , with  $|X| = m$  and  $|Y| = n$ , that contains a  $P_{2t}$ . By Lemma 1.1 we may assume that the copy of  $P_{2t}$  occurs on vertex sets  $X_t := \{x_1, \dots, x_t\}$  and  $Y_t := \{y_1, \dots, y_t\}$ . We must now show that some realization  $G'$  of  $S$  contains a  $P_{2t+1}$ . We proceed by contradiction and assume that no realization of  $S$ , including  $G$ , contains a  $P_{2t+1}$ .

The following claim allows us to further assume that there is no  $C_{2t}$  on  $X_t \cup Y_t$ .

**Claim 3.5.** *If the subgraph induced by  $X_t \cup Y_t$  contains a cycle  $C_{2t}$ , then  $S$  is potentially  $P_{2t+1}$ -bigraphic.*

**Proof.** Assume that the subgraph induced by  $X_t \cup Y_t$  contains a copy of  $C_{2t}$ . If there exists an edge with one endpoint in  $X_t \cup Y_t$  and one endpoint outside of this set then we are done. Thus we may assume there exists no such edge. Since  $m, n > t$ , there exists a pair of vertices  $x, y$  in  $V(G) - (X_t \cup Y_t)$ , and by assumption, each has degree at least 1. We may assume that  $xy$  is an edge for  $x \in X - X_t$  and  $y \in Y - Y_t$  and so for any edge  $x'y'$  in the  $C_{2t}$ ,  $x' \approx y \sim x \approx y' \sim x'$  is an alternating cycle in  $G$ . Removing the edges of this alternating cycle from  $G$  and putting the nonedges into  $G$ , we arrive at another realization of the same degree sequence, which contains a  $P_{2t+1}$ . ■

Let the  $P_{2t}$  on  $X_t \cup Y_t$  be

$$v_1, v_2, \dots, v_{2t-1}, v_{2t}$$

where vertices with odd index are in  $X_t$  and those with even index are in  $Y_t$ . Clearly, neither of  $v_1$  and  $v_{2t}$  can have neighbors outside of  $X_t \cup Y_t$ . Moreover, where  $d_X = \deg(v_1)$  and  $d_Y = \deg(v_{2t})$ , the following is also true.

$$(2) \quad d_X + d_Y \leq t.$$



Indeed if  $G$  contained both of the edges  $v_1v_{2i}$  and  $v_{2t}v_{2i-1}$ , for any  $i = 1, \dots, t$ , then it would contain the  $2t$ -cycle

$$v_1, v_2, \dots, v_{2i-1}, v_{2t}, v_{2t-1}, \dots, v_{2i}, v_1.$$

This would contradict Claim 3.5.

Now the number of edges in  $G$  is the number of edges incident to  $(Y - Y_t) \cup \{v_{2t}\}$  or  $(X - X_t) \cup \{v_1\}$ , plus the number between  $X_t - \{v_1\}$  and  $Y_t - \{v_{2t}\}$ . This is at most

$$(m - (t - 1))d_X + (n - (t - 1))d_Y + (t - 1)^2.$$

Since  $n \geq m$ , and  $d_X \geq 1$ , this is at most

$$n(d_X + d_Y - 1) + m - (t - 1)(d_X + d_Y) + (t - 1)^2.$$

Since  $n > t$  and  $d_X + d_Y \leq t$ , this is maximized when  $d_X + d_Y = t$ , so is at most

$$n(t - 1) + m - (t - 1).$$

This, however, is one less than  $\sigma(S)$ , which is a contradiction. ■

**Proof of Lemma 3.4.** Let  $S$  be a bigraphic pair with  $\sigma(S) \geq n(t - 1) + m - (t - 1) + 1$ , and let  $G$  be a realization of  $S$  that contains a  $P_{2t+1}$ .

We first consider the case in which the endpoints of the  $P_{2t+1}$  occur in  $X$ . By Lemma 1.1 we may assume that the copy of  $P_{2t+1}$  occurs on vertex sets  $X_{t+1} := \{x_1, \dots, x_{t+1}\}$  and  $Y_t := \{y_1, \dots, y_t\}$ . We show that  $G$  contains a  $P_{2t+2}$ . The proof is, again, by contradiction.

Let  $e_X$  denote the number of vertices of  $X_{t+1}$  that are the endpoint of some  $P_{2t+1}$  on  $X_{t+1} \cup Y_t$ . Let  $x$  be any such endpoint, and observe that

$$(3) \quad e_X \geq \deg(x) + 1.$$

Indeed, let  $x = v_1, \dots, v_{2t+1}$  be a  $P_{2t+1}$  with  $x = v_1$  as an endpoint. For every edge  $v_1v_i$ , the following is a  $P_{2t+1}$  having  $v_{2t+1}$  as an endpoint:

$$v_{i-1}, v_{i-2}, \dots, v_1, v_i, v_{i+1}, v_{i+2}, \dots, v_{2t+1}.$$

As  $v_{2t+1}$  is also counted by  $e_X$ , the inequality holds.

Since each vertex of  $X_{t+1}$  which is counted by  $e_X$  has degree at most  $e_X - 1$ , we can bound the number of edges in  $G$  by

$$\begin{aligned} & n(t + 1 - e_X) + (e_X - 1)[m - (t + 1 - e_X)] \\ &= n(t - (e_X - 1)) + m(e_X - 1) - (e_X - 1)(t - (e_X - 1)). \end{aligned}$$

Because  $n \geq m$  and  $1 \leq e_X - 1 \leq t$ , this is maximized when  $e_X - 1 = 1$ , so is at most

$$n(t - 1) + m - (t - 1).$$

This is one less than  $\sigma(S)$ , so completes the proof in the case that the endpoints of the  $P_{2t+1}$  are in  $X$ .

When the endpoints of the  $P_{2t+1}$  are in  $Y$ , then analogous arguments allow us to bound the number of edges in  $G$  by

$$(4) \quad m(t - (e_Y - 1)) + n(e_Y - 1) - (e_Y - 1)(t - (e_Y - 1)),$$

where  $e_Y$  denotes the number of vertices in  $Y_{t+1}$  that are endpoints of a  $P_{2t+1}$  on  $X_t \cup Y_{t+1}$ .

**Claim 3.6.** We have the following inequality,

$$1 \leq e_Y - 1 < t.$$

**Proof.** It is trivial from the definition that  $1 \leq e_Y - 1 \leq t$ . Assume now that  $e_Y - 1 = t$ , so all vertices in  $Y_{t+1}$  are endpoints of a  $P_{2t+1}$ . Then there are no edges from  $Y_{t+1}$  to  $X - X_t$ , or else we have a  $P_{2t+2}$ . So every vertex of  $Y_{t+1}$  has degree at most  $t$ . Since by the degree sum, some vertex of  $Y_{t+1}$  must have degree at least  $t$ , there is some vertex  $y$  of  $Y_{t+1}$  that is adjacent to every vertex in  $X_t$ . In particular, when

$$y = v_1, v_2, \dots, v_{2t+1}$$

is the  $P_{2t+1}$  with  $y$  as endpoint,  $y$  is adjacent to  $v_{2t}$ .

Now since  $m > t + 1$ , there is a vertex  $x_0 \in X - X_t$  which must have some edge. Let  $y_0$  in  $Y - Y_{t+1}$  be the other endpoint of this edge. If  $y_0 \sim v_{2t}$  then we have the  $P_{2t+2}$

$$v_1, \dots, v_{2t}, y_0, x_0.$$

If  $y_0 \approx v_{2t}$ , then we have the alternating path

$$v_1 \sim v_{2t} \approx y_0 \sim x_0 \approx v_1.$$

Removing the edges of this path from  $G$  and replacing them with the nonedges, we get a new realization of the same degree sequence which has the  $P_{2t+2}$

$$x_0, v_1, v_2, \dots, v_{2t}, y_0.$$

This completes the proof of the claim. ■

With this claim we have that the  $(e_Y - 1)(t - (e_Y - 1))$  of equation (4) is minimized when  $e_Y - 1 = 1$  and because  $m \leq n$ , the positive terms are maximized when  $e_Y - 1 = t - 1$ , thus it is bounded above by

$$m(t - (t - 1)) + n(t - 1) - (1)(t - (1)) = n(t - 1) + m - (t - 1).$$

Again, this is one less than  $\sigma(S)$ , so is a contradiction.

This completes the proof of the Lemma 3.4 and therefore completes the proof of Theorem 3.1. ■

#### 4. EVEN CYCLES

The minimum degree sum necessary to assure a graphic sequence has a realization containing a copy of  $C_t$  was determined in [5], and [10]. Here, we look at the similar problem of determining the minimum sum needed to assure that a bigraphic pair has a realization containing a copy of  $C_{2t}$ .

**Theorem 4.1.** *Given  $t \geq 2$ , and  $n \geq m \geq 2(t + 1)$ ,*

$$\sigma(C_{2t}, m, n) = n(t - 1) + m - (t - 1) + 1.$$

**Proof.** The case  $t = 2$  follows from Theorem 2.1, so we assume that  $t > 2$ . The fact that the given value is a lower bound for  $\sigma(C_{2t}, m, n)$  is established by the same bigraphic sequence given in Theorem 3.1. We now show that it is also an upper bound.

Let  $S$  be a bigraphic pair with  $\sigma(S) \geq n(t - 1) + m - (t - 1) + 1$ . By Theorem 3.1 we get a realization  $G$  of  $S$  with a copy of  $P_{2t+2}$ . By Lemma 1.1 we may assume that this  $P_{2t+2}$  occurs on  $X_{t+1} = \{x_1, \dots, x_{t+1}\}$  and  $Y_{t+1} = \{y_1, \dots, y_{t+1}\}$ . The following claim allows us to assume that  $G$  contains a  $C_{2t+2}$ .

**Claim 4.2.** Let  $P$  be a copy of  $P_{2t+2}$  in  $G$ . If the endpoints of  $P$  are not adjacent, then  $S$  is potentially  $C_{2t}$ -bigraphic.

**Proof.** Let  $x$  and  $y$  be the endpoints of  $P$  and  $y'$  and  $x'$  be their respective neighbors in  $P$ . If  $x'$  is adjacent to  $y'$  then we have a  $C_{2t}$  and are done. Thus we assume  $x' \approx y'$ . Now if  $x \approx y$  then  $x \approx y \sim x' \approx y' \sim x$  is an alternating cycle whose reversal yields a  $C_{2t}$  in  $G$ . Thus we may assume that  $x$  and  $y$  are adjacent. ■

We therefore make the assumption that  $G$  contains a  $C_{2t+2}$  on the vertices  $v_1, v_2, \dots, v_{2t+2}$ , where the vertices with even index are in  $X$  and those with odd index are in  $Y$ .

The following claim allows us to assume that the  $C_{2t+2}$  is induced.

**Claim 4.3.** If  $G$  contains a cycle  $C$  of length  $2t + 2$  that is not induced, then  $S$  is potentially  $C_{2t}$ -bigraphic.

**Proof.** Assume that  $C$  contains a chord, wlog  $v_1 \sim v_{2j}$  for some  $j$ ,  $2 \leq j \leq t$ . Then we have the  $P_{2t+2}$

$$v_2, v_3, \dots, v_{2j}, v_1, v_{2t+2}, v_{2t+1}, \dots, v_{2j+1}$$

with endpoints  $v_2$  and  $v_{2j+1}$ . By Claim 4.2, we may assume that these endpoints are adjacent. The same argument applied to the chord  $v_2v_{2j+1}$  shows that  $v_3$  and  $v_{2j+2}$  are adjacent as well. However, this implies that  $G$  contains the  $C_{2t}$   $v_3, v_4, \dots, v_{2j}, v_1, v_{2t+2}, v_{2t+1}, \dots, v_{2j+2}, v_3$ . This proves the claim. ■

Now let  $C$  refer to the induced copy of  $C_{2t+2}$  in  $G$  and consider its vertices  $v_1$  and  $v_6$ .

**Claim 4.4.** We may assume that  $v_1$  and  $v_6$  each have degree at least 3.

**Proof.** We show that there is some pair  $\{v_i, v_{i+5}\}$  with each vertex having degree at least 3. This will suffice. By the degree sum of  $S$ , there are at least  $t - 1$  vertices in each of  $X$  and  $Y$  with degree at least 3. Thus by Lemma 1.1 we may infer that at least  $t - 1$  vertices of each of  $V(C) \cap X$  and  $V(C) \cap Y$  have degree at least 3. We consider now two cases.

When  $t = 3$  there are at least two vertices in  $V(C) \cap X$  that have degree at least 3. Every vertex in  $V(C) \cap Y$  is distance 5 from one of these vertices, so since at least one of the vertices on  $V(C) \cap Y$  has degree at least 3, we are done.

When  $t \geq 4$  there are at least three vertices in  $V(C) \cap X$  that have degree at least three. So there are at least three vertices in  $V(C) \cap Y$  that distance 5 from one of these vertices. Since at most 2 vertices of  $V(C) \cap Y$  have degree less than 3 at least one of these 3 vertices has degree at least 3. ■

Now  $v_1$  has neighbor  $y$  in  $Y - V(C)$  and  $v_6$  has neighbor  $x$  in  $X - V(C)$ . If  $x \sim y$ , then we have the  $C_{2t}$

$$v_6, v_7, \dots, v_{2t+2}, v_1, y, x, v_6.$$

On the other hand, if  $x \not\sim y$  then we have the alternating path  $v_1 \sim y \not\sim x \sim v_6 \not\sim v_1$ . Reversing this in  $G$  we arrive at a realization of the same degree sequence that contains a non-induced  $C_{2t+2}$ . By Claim 4.3, this suffices to complete the proof. ■

## 5. CONCLUSION

This paper serves only as an initial investigation into the subject of potentially  $H$ -bigraphic sequence pairs. Looking forward, it may be interesting to consider other broad classes of bipartite graphs, particularly those graphs  $H$  for which the standard potential number is known.

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