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(H,k) STABLE BIPARTITE GRAPHS WITH MINIMUM SIZE

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Abstract

Let us call a graph G $(H;k)$ vertex stable if it contains a subgraph H after removing any of its k vertices. In this paper we are interested in finding the $(K_{n,n+1};1)$ (respectively $(K_{n,n};1)$) vertex stable graphs with minimum size.

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1. Introduction

We deal with simple graphs without loops and multiple edges. As usual $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. $|G|$, $e(G)$ are the order and the size of G, respectively, whereas $deg_G(v)$ is the degree of $v \in V(G)$. Let $(B,W;E)$ be a complete bipartite graph with vertex bipartition sets defined as follows: $B = \{x_1, x_2, \ldots, x_m\}$, $W = \{y_1, y_2, \ldots, y_n\}$ and the edge set $E = \{x_i y_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$. For simplicity of notation we will write $K_{m,n}$.

By C_n we denote the cycle of order n, by P_n a path of order n. By K_r we denote the complete graph on r vertices and by $K_{p,1}$ the star on $1+p$ vertices. By $G-e$ we shall denote the graph without the edge e and by $G-v$ the graph G without the vertex $v \in V(G)$ and with all edges incident to v deleted as well.

In [2] G.Y. Katona and P. Frankl were interested in finding minimum size of an r-uniform hypergraph such that after removing any k hyperedges there is still a hamiltonian chain in the hypergraph. To find a lower bound of minimum size of the above mentioned r-uniform hypergraphs the authors of [2] define the (P_4, k) edge stable graph as the graph in which after removing any k edges there is still P_4 and they ask about minimum size of the (P_4, k) edge stable graph. It was natural to try to relate minimum size of the (P_4, k) edge stable graph. In [3] G.Y. Katona and I. Horváth address the problem of minimum size of (P_n, k) edge stable graphs. In [1] a similar problem is considered but in a vertex version based on the following definition:

Definition 1. Let us call a graph $(H; k)$ *vertex stable* if it contains a connected subgraph H ever after removing any of its k vertices. By $Q(H; k)$ we will denote minimum size of the $(H; k)$ vertex stable graph.

For convenience of the reader, we repeat the relevant material from [1] without proofs, thus making our exposition self-contained.

Theorem 1. $Q(C_3; k) = 3k + 3$.

Theorem 2. $Q(C_4; k) = 4k + 4$.

Theorem 3. $Q(C_n; k) \leq kn + n, (n \geq 3)$.

Theorem 4.

$$Q(K_4; k) = \begin{cases} 6 & \textit{for } k = 0; \\ 5k + 5 & \textit{for } k \geq 1. \end{cases}$$

For a sufficiently large k, there exists an upper bound:

Theorem 5. *There is an integer $k(s)$ such that $Q(K_s, k) \leq (2s - 3)(k + 1)$ for $k > k(s)$.*

Theorem 6. *For every $k \in N$ there exists $s(k)$ such that $Q(K_s, k) = \binom{s+k}{2}$ for every $s \geq s(k)$.*

It is worth pointing out that [1] was the first paper concerning vertex stable graphs with minimum size. Moreover, the main aim of [1] was to give only minimum size of (H, k) vertex stable graphs, however, it was also shown that K_{s+k} is the only (K_s, k) stable graph with minimum size for $s \geq 2k^2+5k+2$.

It is natural to ask about the characterization of all $(H;k)$ vertex stable graphs with minimum size for a fixed graph H. This is the purpose of the paper for $H = K_{n,n+1}$, $H = K_{n,n+1}$ and $k = 1$.

For simplicity we will write $(H;k)$ stable instead of $(H;k)$ vertex stable. Observe that if we find a graph G which is an $(H;k)$ stable graph with minimum size then adding isolated vertices we still have an $(H;k)$ stable graph with the same size. In this paper we will concentrate on $(H;k)$ stable graph with minimum size and without isolated vertices. (Observe that for $H \neq K_1$ there always exists an $(H;k)$ stable graph without isolated vertices.) From now on we make this assumption and we will not repeat it in any theorem or its proof.

The proofs in this paper are based on the facts given below (see [1]).

Proposition 1. *If G is an $(H;k)$ stable graph with minimum size, then every vertex as well as every edge of G belongs to some subgraph of G isomorphic to H.*

2. $Q(K_{m,n};1)$

Without loss of generality we may assume in $K_{m,n}$ that $m \geq n$ in $K_{m,n}$, and from now on we always make this assumption. Let us define $K_{m,n} + w$ as a graph $G = (V_1, E_1)$ where $V_1 := B \cup W \cup \{w\}$ and $E_1 := E \cup \{x_i w, y_j w\}$ for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ and $K_{m,n} + (w_1, w_2)$ as a graph $H = (V_2, E_2)$ where $V_2 := B \cup W \cup \{w_1, w_2\}$ and $E_2 := E \cup \{x_i w_1, y_j w_2\}$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Proposition 2. $Q(K_{m,n};1) \leq nm + n + m$.

Proof. It is enough to consider the graph $K_{m,n} + (w_1, w_2)$ or $K_{m,n} + w$. ■

The above mentioned graphs will play an important role in this paper. Observe that Proposition 2 follows from the evident inequality $Q(H;1) \leq e(H) + |H|$.

Proposition 3. $Q(K_{m,n};1) \geq nm + m$.

Proof. Let G be a $(K_{m,n};1)$ stable graph with $e(G) = Q(K_{m,n};1)$. It is evident that there exists v such that $deg_G(v) \geq m$.

From the inequalities $mn \leq e(G-v) \leq e(G)-m$ we conclude $Q(K_{m,n};1) \geq nm + m$. ■

For $n = 1$ the bipartite graph $K_{m,n}$ is a star which was considered in [1] as a case of the following Theorem:

Theorem 7. *For* $m \geq 3$ $Q(K_{m,1}; k) = mk + m$.

($Q(K_{1,1}; 1)$ will be given in Theorem 8, $Q(K_{2,1}; 1)$ will be given in Theorem 9.)

Now we will present the main results of this paper.

Theorem 8. $Q(K_{2,1}; 1) = 4$ *and* $K_{2,2}$ *as well as* $2K_{2,1}$ *are the only* $(K_{2,1}; 1)$ *stable graphs with minimum size. For* $n \geq 2$, $Q(K_{n+1,n}; 1) = (n+1)^2$ *and* $K_{n+1,n+1}$ *is the unique* $(K_{n+1,n}; 1)$ *stable graph with minimum size.*

Proof. It is understood that $K_{n+1,n+1}$ is a $(K_{n+1,n}; 1)$ stable graph so $Q(K_{n+1,n}; 1) \leq (n+1)^2$. From Proposition 3 it follows that
$Q(K_{n+1,n}; 1) \geq (n+1)^2$.

Hence $e(G) = (n+1)^2$.

Now we will show that $K_{2,2}$ and $2K_{2,1}$ are the only $(K_{2,1}; 1)$ stable graphs with minimum size and that for $n \geq 2$, $K_{n+1,n+1}$ is the unique $(K_{n+1,n}; 1)$ stable graph with minimum size.

Let G be a $(K_{n+1,n}; 1)$ stable graph with $e(G) = Q(K_{n+1,n;}; 1)$. It is clear that $|G| \geq 2(n+1)$. The proof falls naturally into 2 cases.

Case 1. $|G| = 2(n+1)$.

We first prove that $deg_G(v) \geq n+1$ for every $v \in V(G)$. From Proposition 1 we have $deg_G(v) \geq n$. Suppose indirectly that for some $v_0 \in V(G)$ we have $deg_G(v_0) = n$. Let $v_1 \in N_G(v_0)$. Deleting v_1 we get $deg_{(G-v_1)}(v_0) \leq n-1$ which together with $|G - v_1| = 2n+1$ contradicts the fact that $K_{n+1,n}$ is isomorphic to some subgraph of $G - v_1$.

We have just proved that $G = K_{n+1,n+1}$ is the unique $(K_{n+1,n}; 1)$ stable graph of order $2(n+1)$ with minimum size.

Case 2. $|G| \geq 2n+3$.

Note that there is always a subgraph isomorphic to $K_{n+1,n}$ in G and there are two vertices not belonging to this $K_{n+1,n}$.

Subcase 2a. There are two nonadjacent vertices not belonging to the same subgraph $K_{n+1,n}$ of G.

One may estimate for $n \geq 2$ that $e(G) \geq n(n+1)+n+n \geq n(n+1)+n+2 > (n+1)^2$ to get a contradiction. It is easily seen that for $n = 1$, $2K_{2,1}$ is the unique $(K_{2,1}; 1)$ stable graph with minimum size.

Subcase 2b. Any two vertices not belonging to the same subgraph $K_{n,n+1}$ of G are adjacent.

For $n \geq 3$ one may estimate:
$e(G) \geq n(n+1) + 2(n-1) + 1 \geq n(n+1) + n + 2 > (n+1)^2$ to obtain a contradiction. We leave it to the reader to verify that trying to construct for $n = 1, 2$ of $(K_{n+1,n}; 1)$ stable graph of order at least $2n + 3$ with minimum size we obtain a contradiction with the definition of (H, k) stable graph.

Therefore, there is no $(K_{n+1,n}; 1)$ stable graph of order at least $2n + 3$ with minimum size. ■

Theorem 9. $Q(K_{n,n}; 1) = n^2 + 2n$ *for* $n \geq 2$*. Moreover,* $K_{2,2} + w$*,* $K_{2,2} + (w_1, w_2)$ *and* $2K_{2,2}$ *are the only* $(K_{2,2}; 1)$ *stable graphs with minimum size. For* $n \geq 3$*,* $K_{n,n} + w$ *and* $K_{n,n} + (w_1, w_2)$ *are the only* $(K_{n,n}; 1)$ *stable graphs with minimum size.*

Proof. It is evident that $Q(K_{1,1}; 1) = 2$ and $2K_2$ is the unique $(K_{1,1}; 1)$ stable graph with minimum size. We may suppose that $n \geq 2$.

By Proposition 2, $Q(K_{n,n}; 1) \leq n^2+2n$. We will prove that $Q(K_{n,n}; 1) \geq n^2 + 2n$. The proof will be divided into 2 cases. In both of the cases we will first prove that the inequality $Q(K_{n,n}, 1) \geq n^2+2n$ and as a conclusion from this part of the proof we will consider the existence of the unique $(K_{n,n}; 1)$ stable graph with minimum size.

Let G be a $(K_{n,n}; 1)$ stable graph with minimum size.
From Proposition 1 $deg_G(u) \geq n$ holds for every $u \in V(G)$.

Case 1. $|G| = 2n + 1$.
The same reasoning which was used to prove Case 1 of Theorem 8 gives us $deg_G(v) \geq n + 1$ for every $v \in V(G)$.

Subcase 1a. There is $v_1 \in V(G)$ such that $deg_G(v_1) = n + 1$.
Let $N_G(v_1) := \{w_1, w_2, \ldots, w_{n+1}\}$. We will show that the degree of some vertex belonging to $N_G(v_1)$ is at least $2n$. Let us delete from G a vertex $w_j \in N_G(v_1)$, $j \in \{1, 2, \ldots, n + 1\}$. $K_{n,n}$ must be isomorphic to some subgraph of $G - w_j$ and moreover, it must have bipartition sets as follows:

$\{w_1, \ldots, w_{n+1}\} \backslash \{w_j\}$ and $V(G) \backslash \{w_1, \ldots, w_{n+1}\} := \{v_1, \ldots, v_n\}$. Observe that every vertex from $\{w_1, \ldots, w_{n+1}\}$ must be adjacent to every vertex from $\{v_1, \ldots, v_n\}$. Let us now delete from G a vertex v_2. Of course, $K_{n,n}$ is a subgraph of $G - v_2$ which means that for some vertex from $\{w_1, \ldots, w_{n+1}\}$, say w_i, we have $deg_{G-v_2}(w_i) \geq n + n = 2n$.

Hence $e(G) = \frac{\sum_{v \in V(G)} deg_G(v)}{2} \geq \frac{2n(n+1)+2n}{2} = n^2 + 2n$.

Subcase 1b. For every $v \in V(G)$ $deg_G(v) \geq n + 2$.
We have: $e(G) = \frac{\sum_{v \in V(G)} deg_G(v)}{2} \geq \frac{(2n+1)(n+2)}{2} > n^2 + 2n$, a contradiction.

Therefore: $e(G) = n^2 + 2n$.
Note that we have actually proved more: if G is a $(K_{n,n}; 1)$ stable graph of order $2n+1$ with minimum size, then all vertices of G, except one vertex of degree $2n$, have their degree equal to $n+1$. Otherwise we have $e(G) > n^2+2n$ in Subcase 1a, a contradiction.

It follows immediately that $K_{n,n}+w$ is the unique $(K_{n,n}; 1)$ stable graph of order $2n + 1$ with minimum size which finishes the proof in this case.

Case 2. $|G| \geq 2n + 2$.
To avoid repetition let us denote $K_{n,n} = (B, W, E)$ where $B = \{x_1, x_2, \ldots, x_k, \ldots, x_n\}$ and $W = \{y_1, y_2, \ldots, y_s, \ldots, y_n\}$.

Subcase 2a. $|G| \geq 2n + 3$.
Observe that G must contain at least three vertices v_1, v_2, v_3 not belonging to the same subgraph $K_{n,n}$.

Hence: $e(G) \geq n^2 + 3n - 3 \geq n^2 + 2n$ for $n > 2$.
One may easily prove that for $n = 2$ $e(G) \geq 4$ holds and the graph $2K_{2,2}$ is the unique $(K_{2,2}; 1)$ stable graph with minimum size.

One may also verify that for $n = 3$ there is no $(K_{3,3}; 1)$ stable graph in this subcase. Observe that for $n > 3$ we have $e(G) > n^2 + 2n$ so there is no $(K_{n,n}; 1)$ stable graph of order at least $2n + 3$ with minimum size.

Subcase 2b. $|G| = 2n + 2$.
Note that there is always a subgraph isomorphic to $K_{n,n}$ in G and there are two vertices not belonging to this $K_{n,n}$. We will consider two subcases.

Subcase 2b.1. There are two nonadjacent vertices not belonging to the same subgraph $K_{n,n}$ of G, say u_1, u_2.

Then $e(G) \geq n^2 + 2n$. Hence we have $e(G) = n^2 + 2n$.

From Proposition 1 and the fact that in $E(G)$ there are n^2 edges of $K_{n,n}$ and only $2n$ additional edges we obtain $deg_G(u_1) = deg_G(u_2) = n$. We will show that every vertex of $K_{n,n}$ is adjacent either to u_1 or to u_2. Suppose the contrary, then there is $x_i \in B$, say x_1,(we could consider a vertex from W by symmetry) such that $x_1u_1, x_1u_2 \notin E(G)$. It is clear that there is $y_i \in N_G(x_1)$, say y_1, such that $y_1u_1 \in E(G)$. Deleting y_1 from G we get $d_{G-y_1}(x_1) < n$ and $d_{G-y_1}(u_1) < n$. Therefore $K_{n,n}$ cannot be isomorphic to any subgraph of $G - y_1$, a contradiction.

Now we will show that every vertex of $K_{n,n}$ is adjacent to exactly one vertex u_1 or u_2. If there is $x_i, i = 1, 2, \ldots, n$, say x_2, such that $x_2u_1 \in E(G)$ and $x_2u_2 \in E(G)$, then there must be at least one vertex from $(B\backslash\{x_2\}) \cup W$ which is not adjacent to u_1 and u_2, a contradiction.

We may suppose that $u_1x_1, u_1x_2, \ldots, u_1x_k, u_2x_{k+1}, u_2x_n, u_2y_1, u_2y_2, \ldots, u_2y_s, u_1y_{s+1}, \ldots, u_1y_n \in E(G)$. Observe that u_2 has its neighbours in both sets B and W. Deleting x_1 we have $|N_{G-x_1}(u_2)| = n$ where $N_{G-x_1}(u_2) = \{x_{k+1}, x_n, y_1, \ldots, y_s\}$. Therefore $N_{G-x_1}(u_2)$ and $\{x_2, \ldots, x_k, y_{s+1}, \ldots, y_n, u_2\}$ should create a bipartition set of some subgraph isomorphic to $K_{n,n}$ in $G - x_1$ which is impossible, a contradiction.

The proof above gives that $K_{n,n} + (u_1, u_2)$ is the unique $(K_{n,n}; 1)$ stable graph of order $2n + 2$ with minimum size in this subcase.

Subcase 2b.2. Any two vertices not belonging to the same subgraph $(K_{n,n})$ of G are adjacent.

Let $K_{n,n} = (B, W, E)$ where $B = \{x_1, x_2, \ldots, x_n\}$ and $W = \{y_1, y_2, \ldots, y_n\}$. Let u_1, u_2 be vertices of G not belonging to the defined $K_{n,n}$. Suppose on the contrary that $e(G) \leq n^2 + 2n - 1$. From Proposition 1 and the fact that in $E(G)$ there are n^2 edges of $K_{n,n}$ and only $2n - 1$ additional edges we obtain $deg_G(u_1) = deg_G(u_2) = n$. One may deduce that there is at least one vertex from B, say x_1, (by symmetry we may also assume the existence a vertex from W) such that $x_1u_1 \notin E(G)$ and $x_1u_2 \notin E(G)$. Observe that there exist a vertex from $N_G(x_1)$, say y_1, such that $y_1u_1 \in E(G)$ or $y_1u_2 \in E(G)$, say $y_1u_1 \in E(G)$. After deleting y_1 we have $deg_{G-y_1}(u_1) < n$ and $deg_{G-y_1}(x_1) < n$. It means that $K_{n,n}$ is not isomorphic to any subgraph of $G - y_1$, a contradiction.

Hence we have $e(G) \geq n^2 + 2n$ and in consequence $e(G) = n^2 + 2n$.

Now we will take under consideration the existence of the unique $(K_{n,n}; 1)$ stable graph with minimum size. Suppose that G is a $(K_{n,n}; 1)$ stable graph of order $2n + 2$ with minimum size equal to $n^2 + 2n$.

Observe that there are only two possibilities, namely, both vertices u_1, u_2 have their degree equal to n or one of the vertices has its degree equal to n and the other to $n+1$.

Suppose first that $deg_G(u_1) = deg_G(u_2) = n$. It is obvious that there exists a vertex from W, say $y_1 \in W$, (it could be by symmetry $x_i \in B$) such that: $y_1u_1 \in E(G)$ or $y_1u_2 \in E(G)$, say $y_1u_1 \in E(G)$. After deleting y_1 we have $deg_{G-y_1}(u_1) < n$ so $u_1 \notin V(K_{n,n})$ in $G - y_1$. It is evident that also $u_2 \notin V(K_{n,n})$ in $G - y_1$. So $K_{n,n}$ is not isomorphic to any subgraph of $G - y_1$, a contradiction.

By symmetry we may assume now that $deg_G(u_1) = n+1$ and $deg_G(u_2) = n$. It is obvious that there exists a vertex of $K_{n,n}$ which is not adjacent neither to u_1 nor to u_2 (by symmetry we may also assume a vertex from W or B). Hence it is enough to consider two cases:

There is no $x_i \in B$ such that $x_iu_2 \in E(G)$.
Observe that there is exactly one vertex in $W \cup B$, say z, such that $zu_1 \notin E(G)$ and $zu_2 \notin E(G)$. Therefore $deg_G(z) = deg_G(u_2) = n$ and all other vertices have their degree equal to $n+1$. Hence we have a situation from Subcase 2b.1, a contradiction.

There is $x_i \in B$ such that $x_iu_2 \in E(G)$, say x_1.
Deleting x_1 we get: $deg_{G-x_1}(y_1) < n$ and $deg_{G-x_1}(u_2) < n$ so $K_{n,n}$ is no isomorphic to any subgraph of $G - x_1$, a contradiction.

Note that in Subcase 2b.2 there is no $(K_{n,n}; 1)$ stable graph of order $2n+2$ with minimum size. Therefore the unique $(K_{n,n}; 1)$ stable graph with minimum size of order $2n+2$ in Subcase 2b is $K_{n,n} + (u_1, u_2)$. ∎

Conjecture 1. For $m \neq 1$ and $|m-n| \neq 1$ we have $Q(K_{m,n}, 1) = mn+m+n$.

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