

DECOMPOSITIONS OF NEARLY COMPLETE DIGRAPHS INTO t ISOMORPHIC PARTS

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Abstract

An arc decomposition of the complete digraph \mathcal{DK}_n into t isomorphic subdigraphs is generalized to the case where the numerical divisibility condition is not satisfied. Two sets of nearly t th parts are constructively proved to be nonempty. These are the floor t th class $(\mathcal{DK}_n - R)/t$ and the ceiling t th class $(\mathcal{DK}_n + S)/t$, where R and S comprise (possibly copies of) arcs whose number is the smallest possible. The existence of cyclically 1-generated decompositions of \mathcal{DK}_n into cycles \vec{C}_{n-1} and into paths \vec{P}_n is characterized.

Keywords: decomposition, cyclically 1-generated, remainder, surplus, universal part.

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1. INTRODUCTION

Let t be an integer, $t \geq 2$. We focus our considerations on decompositions into t isomorphic parts. One of the most significant results in the graph decomposition theory is that a t th part of (or one t th of) the complete digraph exists whenever the size of the digraph is divisible by t . In case $t = 2$ parts are *halves* and they are called *self-complementary digraphs*; their existence is proved by Read [15]. The relevant result for any t is proved in Harary *et al.* [8]. However, the related problem of characterizing t th parts remains open if the order of the complete digraph is large enough.

Given the complete digraph \mathcal{DK}_n on n vertices (with $n(n-1)$ arcs), the numerical divisibility condition $t|n(n-1)$ is also known [8] to ensure

that self-converse t th parts of \mathcal{DK}_n exist. Our first aim is to extend this classical result to the case where parts are to be self-converse oriented graphs. Secondly, if the numerical divisibility condition is not satisfied, we consider t th parts of a corresponding nearly complete digraph obtained from \mathcal{DK}_n either by adding a *surplus* S or by deleting a *remainder* R . Then S comprises copies of arcs and R is a subset of arcs, both S and R are to have cardinalities as small as possible, $|R| = n(n-1) \bmod t$ and $|S| = (t - |R|) \bmod t$. Thus R is a set and S is possibly a multiset. Following Skupień [16], the classes of such t th parts are denoted by $[\mathcal{DK}_n/t]_S := (\mathcal{DK}_n + S)/t$ and $[\mathcal{DK}_n/t]_R := (\mathcal{DK}_n - R)/t$, and are called the *ceiling t th class* and the *floor t th class*, respectively. Call elements of those classes (also if $S = \emptyset = R$) to be *near- t th parts* of \mathcal{DK}_n ; more precisely, these are *ceiling- S t th parts* and *floor- R t th parts*, respectively.

The proof of theorem on divisibility of \mathcal{DK}_n by t in Harary *et al.* [8] gives the following result.

Proposition 1. *$\mathcal{DK}_n/2$ contains a self-converse oriented graph, e.g. the transitive tournament T_n .*

For $t = 3 \leq n$, $R = \emptyset = S$ unless $n \equiv 2 \pmod{3}$ and then $|S| = 1$, $|R| = 2$ and there are five configurations of R , which we call *admissible*, three of them being self-converse. In [11, 12] we have proved the following three theorems on third parts.

Theorem 2 [11]. *For each $n \geq 3$ and any admissible and self-converse R , the floor third class $[\mathcal{DK}_n/3]_R$ contains a self-converse oriented graph unless either $n = 3$ or possibly $n = 8$ and R induces a path \vec{P}_3 .*

A computer has not found such a member in case where $n = 8$ and R induces \vec{P}_3 .

Theorem 3 [11]. *If $k \in \mathbb{N}$, $n = 3k - 1 \neq 5$, and $t = 3$, then $|S| = 1$ and the ceiling third class $[\mathcal{DK}_n/3]_S$ contains a self-converse oriented graph.*

Theorem 4 [12]. *For $n = 5$, there is no ceiling third part of the complete digraph \mathcal{DK}_5 which could be a self-converse oriented graph.*

The main result of the paper provides a complete solution to the related existence problem for $t \geq 4$. It turns out that the problem we solve is not mentioned among unsolved problems listed in Harary and Robinson [7].

Theorem 5. *For every $n \geq 2$ and every $t \geq 2$ there exist a remainder R and surplus S (both of the smallest possible cardinality) such that both the floor class $\lfloor \mathcal{DK}_n/t \rfloor_R$ and the ceiling class $\lceil \mathcal{DK}_n/t \rceil_S$ contain self-converse digraphs. If neither $n = 5$ and $t = 3$ in case of the ceiling class nor $n = t = 3$ then the digraphs can be required to be self-converse oriented graphs.*

2. NOTATION AND TERMINOLOGY

We use standard notation and terminology of graph theory [4, 5] unless otherwise stated.

Digraphs are loopless and without multiple arcs. Multidigraphs may have multiple arcs, loops being forbidden. A digraph without 2-cycle \mathcal{DK}_2 ($= \vec{C}_2$) is called an *oriented graph*.

The ordered pair (v_1, v_2) of vertices v_1 and v_2 (or the symbol $v_1 \rightarrow v_2$) denotes the arc which goes from the *tail* v_1 to the *head* v_2 . The *converse* of a multidigraph is obtained by reversal of each arc. A multidigraph is called *self-converse* if it is isomorphic to its converse.

The symbol \cup when applied to multidigraphs stands for the *vertex-disjoint* union. Moreover, given a digraph D , the symbol $D + A'$ denotes the spanning supermultidigraph of D with the arc set $A(D) \cup A'$, where A' is a set of (possibly copies of) arcs and $A' \cap A(D) = \emptyset$. Similarly, $D - A'$ denotes the spanning subdigraph obtained from D by removal of A' , where A' is to be a subset of $A(D)$. We write $D \pm A' = D \pm a$ if $A' = \{a\}$ and a is an arc.

By a *decomposition* of a multidigraph D we mean a family of arc-disjoint submultidigraphs of D which include all arcs of D . Those substructures are called *elements of a decomposition*. By an *H-decomposition* we mean a decomposition of D into t elements all isomorphic to H ; then we write $H|D$ or $t|D$. The isomorphism class of those t pairwise isomorphic elements of a decomposition is called a *tth part* of D .

There are two non-self-converse digraphs of size two each, namely,

$$P^\vee = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_3, v_2)\}), \text{ a gutter,}$$

$$P^\wedge = (\{v_1, v_2, v_3\}, \{(v_2, v_1), (v_2, v_3)\}), \text{ a roof.}$$

Note that $\mathcal{DK}_3/3$ comprises three digraphs, none of which is a self-converse oriented graph.

$$(1) \quad \mathcal{DK}_3/3 = \{\vec{C}_2, P^\vee, P^\wedge\}.$$

A decomposition of D is called to be *1-generated* if there is a permutation γ of $V(D)$ which *generates* the decomposition from any single decomposition element H in the sense that, for $j = 0, \dots, t - 1$, the image of H under the j th iteration $(\gamma)^j$ of γ , denoted by $(\gamma)^j H$, is one of t decomposition parts, where $(\gamma)^0 = \text{id}$. Call γ to be a *placement-generating permutation* for H . If, moreover, γ is a cyclic permutation then the decomposition is called *cyclically 1-generated* (cf. cyclic decomposition in Chartrand and Lesniak [5], see also Bosák [4] for the equivalent notion of a decomposition according to a cyclic group).

Proposition 6. *Each decomposition of the complete digraph $\mathcal{D}K_n$ into two isomorphic halves is 1-generated.*

Proof. Note that these halves are self-complementary digraphs. The result follows from the known characterization of complementing permutations for those halves, cf. Bosák [4, Ch. 14]. ■

Given a self-converse multidigraph D on n vertices, we use the symbol φ ($= \varphi_n$) to denote a *conversing permutation*, that is, a permutation of $V(D)$ such that φD is the converse multidigraph of D .

3. CYCLIC DECOMPOSITIONS INTO n PARTS

Bermond and Faber prove [3] that the complete digraph $\mathcal{D}K_n$ is decomposable into cycles \vec{C}_{n-1} of length $n - 1$. It can be noted that \vec{C}_{n-1} -decomposition of $\mathcal{D}K_n$, presented in [3] as well as in [1], is not 1-generated. We are going to improve this result by characterizing cyclically 1-generated decompositions of $\mathcal{D}K_n$ into $(n - 1)$ -cycles. Namely, a cyclically 1-generated \vec{C}_{n-1} -decomposition exists precisely if n is odd. Additionally, a cyclically 1-generated decomposition of $\mathcal{D}K_n$ into hamiltonian paths for even n follows from the widely known construction presented in Berge [2, p. 232] and also in Lucas [10, Ch. 6] (who attributes this result to Walecki) by passing on from K_n to $\mathcal{D}K_n$. It is worth noting that just this cyclically 1-generated \vec{P}_n -decomposition of $\mathcal{D}K_n$ is presented in [10, Remark on p. 176] in terms of designing a set of single file walks for n children so that each child once is the first, once the last, and no ordered pair of neighbours in a file is repeated among the files in the set. We prove that this decomposition exists precisely if n is even (Theorem 7). In either case cyclically 1-generated decomposition

plays a crucial role in the proof of the main result since those decompositions enable a recursive construction in proofs of Lemma 8 and Theorem 5.

Let $V(\mathcal{DK}_n) = \mathbb{Z}_n$, the cyclic group of order n . Let W_0 be a sequence of (possibly repeating) vertices of the digraph \mathcal{DK}_n , say $W_0 = \langle x_1, x_2, \dots, x_k \rangle$. In what follows we use the convention that W_0 refers to the walk whose subsequence of vertices is W_0 . Moreover, the symbol $\langle W_0 \rangle$ stands for the graph induced by the arc set of the walk W_0 .

Definition 1. Assume that $n \geq 3$. Define the vertex sequence, which depends on the parity of n and is denoted by $W_0(n)$ or W_0 , as follows.

(i) For odd $n \geq 3$, $W_0 = \langle 0, \frac{n-3}{2}, \dots, \frac{n+1}{2}, 0 \rangle$, which represents a cycle in which $\frac{n-1}{2}$ is the only vertex which is omitted. If $n = 3$ then $W_0 := \langle 0, 2, 0 \rangle$. If $n \geq 5$, we assume that the cycle W_0 comprises the following arcs, where k stands for an integer:

$$\begin{aligned}
 (2) \quad & k \rightarrow \frac{n-3}{2} - k, & 0 \leq k \leq \frac{n-5}{4}, \\
 (3) \quad & \frac{n-3}{2} - k \rightarrow k + 1, & 0 \leq k \leq \frac{n-7}{4}, \\
 (4) \quad & \frac{n+3}{2} + k \rightarrow n - 1 - k, & 0 \leq k \leq \frac{n-7}{4}, \\
 (5) \quad & n - 1 - k \rightarrow \frac{n+1}{2} + k, & 0 \leq k \leq \frac{n-5}{4},
 \end{aligned}$$

and also two arcs $\frac{n+1}{2} \rightarrow 0, \lfloor \frac{n-1}{4} \rfloor \rightarrow \lfloor \frac{3n+1}{4} \rfloor$.

Hence
$$W_0 = \begin{cases} \langle 0, 1, 4, 3, 0 \rangle, & n = 5, \\ \langle 0, 3, 1, 2, 7, 6, 8, 5, 0 \rangle, & n = 9, \\ \langle 0, 2, 1, 5, 6, 4, 0 \rangle, & n = 7, \\ \langle 0, 4, 1, 3, 2, 8, 9, 7, 10, 6, 0 \rangle, & n = 11. \end{cases}$$

Figure 1 ($n = 7, 9$) shows the difference between cases $n \equiv 1, 3 \pmod{4}$.

(ii) For even $n \geq 4$, $W_0 = \langle 0, 1, n - 1, \dots, \frac{n}{2} \rangle$, which represents a hamiltonian path of \mathcal{DK}_n . It is assumed that the path includes the following arcs:

$$(6) \quad k \rightarrow n - k \quad (7) \quad \rightarrow k + 1, \quad 1 \leq k \leq \frac{n-2}{2},$$

and the initial arc $0 \rightarrow 1$, see Figure 2 wherein $n = 8$.

Define the length of the arc (i, j) to be $(j - i) \pmod{n}$.

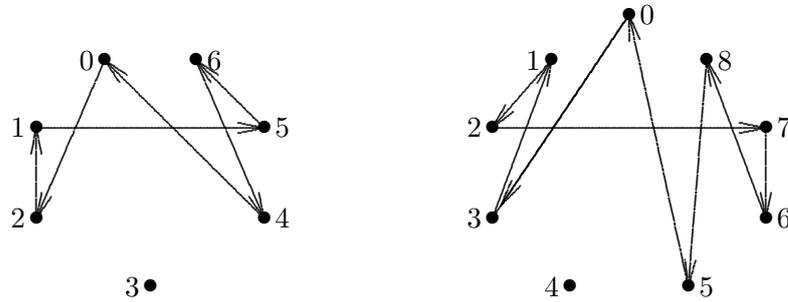


Figure 1. $n = 7, 9$

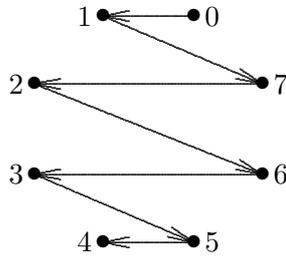


Figure 2. $n = 8$

Theorem 7. For $n \geq 4$, the cyclic permutation $\gamma_0 := [0, 1, 2, \dots, n - 1]$ is a placement-generating permutation for the self-converse oriented graph $\langle W_0 \rangle$ in \mathcal{DK}_n , that is, \mathcal{DK}_n has a cyclically 1-generated $\langle W_0 \rangle$ -decomposition into n parts which are either the cycle \vec{C}_{n-1} or path \vec{P}_n according as n is odd or even. Moreover, if a 1-generated \vec{C}_{n-1} -decomposition [1-generated \vec{P}_n -decomposition] of \mathcal{DK}_n exists then n is odd [n is even].

Proof. Using the list of arcs of W_0 in Definition 1 one can see that arcs of W_0 have mutually distinct lengths. Namely, if $n \geq 4$ is odd and $\langle W_0 \rangle = \vec{C}_{n-1}$, the lengths are $\frac{n-3}{2} - 2k$ for arcs listed in (2), $\frac{n-5}{2} - 2k$ in (4), and similarly $\frac{n+5}{2} + 2k$ for arcs in (3), and $\frac{n+3}{2} + 2k$ in (5), for increasing k starting at $k = 0$. Two additional arcs $\frac{n+1}{2} \rightarrow 0$, $\lfloor \frac{n-1}{4} \rfloor \rightarrow \lfloor \frac{3n+1}{4} \rfloor$ have lengths $\frac{n-1}{2}$, $\frac{n+1}{2}$, respectively. Analogously, if n is even and $\langle W_0 \rangle = \vec{P}_n$, the lengths are even, $n - 2k$, for arcs with label (6) and odd, $2k + 1$, with label (7), where $k = 1, 2, \dots, \frac{n-2}{2}$. The initial arc $0 \rightarrow 1$ has length 1.

Note that a noncyclic permutation of n vertices cannot generate n elements of a decomposition because then there is an edge which has less than

n different images under iterations of the permutation. Suppose that a cyclic permutation generates a decomposition of \mathcal{DK}_n into n subdigraphs. It is easy to see that an $(n - 1)$ -cycle if n is even, as well as a path on n vertices for odd n , with all their arcs of different lengths do not exist. Namely, otherwise the sum of lengths would be 0 modulo n for the cycle (i.e., for even n) and non-zero for the path (i.e., for odd n). However, just the opposite is true. Namely, the sum of lengths is $s = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ and therefore $s \not\equiv 0 \pmod{n}$ if n is even but $s \equiv 0 \pmod{n}$ if n is odd. ■

Note that none of third parts of \mathcal{DK}_3 is a self-converse oriented graph, cf. (1), and therefore the case $n = 3$ is excluded in Theorem 7.

4. DECOMPOSITIONS INTO t PARTS

Harary *et al.* [8] prove that if the numerical divisibility condition $t|n(n-1)$ is satisfied then the complete digraph \mathcal{DK}_n is decomposable into t isomorphic self-converse parts. If $t|n$ or $t|(n-1)$, we extend this result to the case where parts are to be self-converse oriented graphs and a decomposition is to be 1-generated.

Lemma 8. *For every $n \geq 2$, if $t|n$ or $t|(n-1)$ then there exists a 1-generated decomposition of the complete digraph \mathcal{DK}_n into t isomorphic self-converse parts which are oriented graphs unless $n = t = 3$.*

Proof. For $t = 2$ and $t = 3$, Lemma is proved in [12] and [11], respectively. Namely, if $t = 2$, one of halves of \mathcal{DK}_n with any n is the transitive tournament (see Propositions 1 and 6). For $t = 3$ and $n = 3t$ or $n = 3t + 1$, decompositions of \mathcal{DK}_n constructed in [11] can be seen to be 1-generated. More precisely, oriented graphs D_i for $i = 4, 6, 7, 9$ constructed in the proof of Lemma 1.1 in [11] as well as an oriented graph D_n in the proof of Theorem 1 in [11] are required third parts of a decomposition, existence of which is a part of the assertion of Theorem 2. If $t = n (> 3)$, the result follows immediately from the above Theorem 7.

Consider the case $n = kt$, $k \in \mathbb{N}$, and $t \geq 4$. We proceed by induction on k . Theorem 7 can be viewed as the first step for $k = 1$, namely, the self-converse oriented graph $\langle W_0 \rangle \in \mathcal{DK}_t/t$ and the vertex set $V(\langle W_0 \rangle) = \{0, 1, \dots, t-1\}$. Assume now that Lemma is true for $k-1$, that is, for a fixed admissible self-converse oriented graph $D_{n-t} \in \mathcal{DK}_{n-t}/t$ where $V(D_{n-t}) = \{t, t + 1, \dots, n - 1\}$. We construct a self-converse oriented graph $D_n \in$

\mathcal{DK}_n/t . Let $V(D_n) = V(D_{n-t}) \cup V(\langle W_0 \rangle)$. The construction of D_n takes the advantage of the structure of $\langle W_0 \rangle$. Namely, one can easily find a conversing permutation φ_t of $\langle W_0 \rangle$ in \mathcal{DK}_t , which moves the vertex 0 into $\lceil \frac{t}{2} \rceil$ (cf. Definition 1). Then we define D_n to be the following digraph.

$$D_n = \langle W_0 \rangle \cup D_{n-t} + \left\{ (0, i), \left(i, \left\lceil \frac{t}{2} \right\rceil \right) : t \leq i \leq n - 1 \right\}.$$

One can see that D_n is a self-converse oriented graph with conversing permutation $\varphi_n = (\varphi_t, \varphi_{n-t}) := \varphi_t \varphi_{n-t}$, where φ_{n-t} is a conversing permutation of D_{n-t} . Moreover, $D_n \in \mathcal{DK}_n/t$ with placement-generating permutation $\gamma_n = (\gamma_0, \gamma_{n-t})$, where γ_{n-t} is a placement-generating permutation of D_{n-t} in \mathcal{DK}_{n-t} (see Theorem 7 for γ_0).

In the case $n = kt + 1$, $k \in \mathbb{N}$, note first that the self-converse oriented graph $D'_{t+1} = \langle W_0 \rangle \cup [t] + \{(0, t), (t, \lceil \frac{t}{2} \rceil)\}$, where $[t] := K_1$ with $V(K_1) = \{t\}$, is the t th part of \mathcal{DK}_{t+1} with placement-generating permutation $\gamma_{t+1} = (\gamma_0, (t))$ and conversing permutation $\varphi_{t+1} = (\varphi_t, (t))$. For $n = kt + 1$ where $t \geq 4$ and $k \geq 2$, we construct a self-converse oriented graph $D'_n \in \mathcal{DK}_n/t$ analogously to the case above, using in the induction step digraphs D'_{t+1} and D_{n-t-1} . ■

In [13] we have proved the following result on decompositions of the complete digraph into nonhamiltonian paths, which is useful in proving Theorem 5.

Theorem 9 [13]. *For any $n \geq 3$, the complete digraph \mathcal{DK}_n is decomposable into paths of arbitrarily prescribed lengths ($\leq n - 2$) provided that the lengths sum up to the size $n(n - 1)$ of \mathcal{DK}_n .*

Now we are ready to prove the main result of the paper.

Proof of Theorem 5. If $t \leq 3$, Theorem is true by Proposition 1 for $t = 2$ and by (1) and Theorems 2, 3, 4 for $t = 3$. It is so, too, if $t = n$, by Theorem 7. The result is easily seen for $n \leq 3$ and $t > 3$. In particular, S can be chosen so that just \vec{P}_3 (the only possible candidate) is the ceiling- S t th part of \mathcal{DK}_3 for $t = 4, 5$.

Consider the case $t > n \geq 4$. Note that $\lfloor n(n - 1)/t \rfloor \leq n - 2$ whence, in case of the floor class, the result is true by Theorem 9, elements of a decomposition as well as $\langle R \rangle$ are oriented paths. In case of the ceiling class, if $t = n + 1$ then $\lceil n(n - 1)/t \rceil = n - 1$, $|S| = n - 1$ and, by Theorem 7, the result is true. For $t \geq n + 2$, $\lceil n(n - 1)/t \rceil \leq n - 2$ and analogously

the result is true by Theorem 9, i.e., elements of a decomposition and $\langle S \rangle$ are paths of prescribed lengths.

It remains to consider the case $4 \leq t < n$. Let $r = n \bmod t$. Then $r \leq n - t$ and $(n(n-1) \bmod t) = (r(r-1) \bmod t)$. Applying similar induction as in the proof of Lemma 8 one can easily construct required near- t th parts of \mathcal{DK}_n . Namely, in the induction step we use self-converse oriented graphs D_t (see proof of Lemma 8 for a construction) and D_{n-t} (construction follows from the case $t > n \geq 2$ above and by induction) which are t th and corresponding near- t th parts of \mathcal{DK}_t and \mathcal{DK}_{n-t} , respectively. ■

5. CONCLUDING REMARKS

Note that the above relatively short proof of the main result is based on Theorem 9 (the proof of which is nontrivial) and on characterizations of special cyclically 1-generated decompositions, see Theorem 7.

It is an open problem to determine all possible cyclically 1-generated n th parts of \mathcal{DK}_n , which are self-converse oriented graphs. Similar problem concerns t th parts as considered in Lemma 8 above.

Another open problem is related to the notion of a t th universal floor part, say F , of \mathcal{DK}_n . The meaning of ‘universal’ is that packings of t copies of F into \mathcal{DK}_n should leave remainders R of all possible shapes. The open problem is motivated by a conjecture of the second author, stated several years ago (see [17]), that universal floor parts of complete (undirected) graphs exist. Supporting results in [18, 9] cover infinitely many pairs (n, t) . Moreover, Plantholt’s deep theorem [14] on the chromatic index is equivalent to the truth of the conjecture for $t = n - 1$ with n being odd. On the other hand, there is no universal third part of \mathcal{DK}_5 , see [6].

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