# DECOMPOSITIONS OF NEARLY COMPLETE DIGRAPHS INTO $t$ ISOMORPHIC PARTS 

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#### Abstract

An arc decomposition of the complete digraph $\mathcal{D} K_{n}$ into $t$ isomorphic subdigraphs is generalized to the case where the numerical divisibility condition is not satisfied. Two sets of nearly $t$ th parts are constructively proved to be nonempty. These are the floor $t$ th class $\left(\mathcal{D} K_{n}-R\right) / t$ and the ceiling $t$ th class $\left(\mathcal{D} K_{n}+S\right) / t$, where $R$ and $S$ comprise (possibly copies of) arcs whose number is the smallest possible. The existence of cyclically 1 -generated decompositions of $\mathcal{D} K_{n}$ into cycles $\vec{C}_{n-1}$ and into paths $\vec{P}_{n}$ is characterized.


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## 1. Introduction

Let $t$ be an integer, $t \geq 2$. We focus our considerations on decompositions into $t$ isomorphic parts. One of the most significant results in the graph decomposition theory is that a $t$ th part of (or one $t$ th of) the complete digraph exists whenever the size of the digraph is divisible by $t$. In case $t=2$ parts are halves and they are called self-complementary digraphs; their existence is proved by Read [15]. The relevant result for any $t$ is proved in Harary et al. [8]. However, the related problem of characterizing $t$ th parts remains open if the order of the complete digraph is large enough.

Given the complete digraph $\mathcal{D} K_{n}$ on $n$ vertices (with $n(n-1)$ arcs), the numerical divisibility condition $t \mid n(n-1)$ is also known [8] to ensure
that self-converse $t$ th parts of $\mathcal{D} K_{n}$ exist. Our first aim is to extend this classical result to the case where parts are to be self-converse oriented graphs. Secondly, if the numerical divisibility condition is not satisfied, we consider $t$ th parts of a corresponding nearly complete digraph obtained from $\mathcal{D} K_{n}$ either by adding a surplus $S$ or by deleting a remainder $R$. Then $S$ comprises copies of arcs and $R$ is a subset of arcs, both $S$ and $R$ are to have cardinalities as small as possible, $|R|=n(n-1) \bmod t$ and $|S|=(t-|R|) \bmod t$. Thus $R$ is a set and $S$ is possibly a multiset. Following Skupien [16], the classes of such $t$ th parts are denoted by $\left\lceil\mathcal{D} K_{n} / t\right\rceil_{S}:=\left(\mathcal{D} K_{n}+S\right) / t$ and $\left\lfloor\mathcal{D} K_{n} / t\right\rfloor_{R}:=$ $\left(\mathcal{D} K_{n}-R\right) / t$, and are called the ceiling tth class and the floor tth class, respectively. Call elements of those classes (also if $S=\emptyset=R$ ) to be neartth parts of $\mathcal{D} K_{n}$; more precisely, these are ceiling-S tth parts and floor- $R$ tth parts, respectively.

The proof of theorem on divisibility of $\mathcal{D} K_{n}$ by $t$ in Harary et al. [8] gives the following result.

Proposition 1. $\mathcal{D} K_{n} / 2$ contains a self-converse oriented graph, e.g. the transitive tournament $T_{n}$.

For $t=3 \leq n, R=\emptyset=S$ unless $n \equiv 2(\bmod 3)$ and then $|S|=1,|R|=2$ and there are five configurations of $R$, which we call admissible, three of them being self-converse. In $[11,12]$ we have proved the following three theorems on third parts.

Theorem 2 [11]. For each $n \geq 3$ and any admissible and self-converse $R$, the floor third class $\left\lfloor\mathcal{D} K_{n} / 3\right\rfloor_{R}$ contains a self-converse oriented graph unless either $n=3$ or possibly $n=8$ and $R$ induces a path $\overrightarrow{P_{3}}$.

A computer has not found such a member in case where $n=8$ and $R$ induces $\overrightarrow{P_{3}}$.

Theorem 3 [11]. If $k \in \mathbb{N}, n=3 k-1 \neq 5$, and $t=3$, then $|S|=1$ and the ceiling third class $\left\lceil\mathcal{D} K_{n} / 3\right\rceil_{S}$ contains a self-converse oriented graph.

Theorem 4 [12]. For $n=5$, there is no ceiling third part of the complete digraph $\mathcal{D} K_{5}$ which could be a self-converse oriented graph.

The main result of the paper provides a complete solution to the related existence problem for $t \geq 4$. It turns out that the problem we solve is not mentioned among unsolved problems listed in Harary and Robinson [7].

Theorem 5. For every $n \geq 2$ and every $t \geq 2$ there exist a remainder $R$ and surplus $S$ (both of the smallest possible cardinality) such that both the floor class $\left\lfloor\mathcal{D} K_{n} / t\right\rfloor_{R}$ and the ceiling class $\left\lceil\mathcal{D} K_{n} / t\right\rceil_{S}$ contain self-converse digraphs. If neither $n=5$ and $t=3$ in case of the ceiling class nor $n=t=3$ then the digraphs can be required to be self-converse oriented graphs.

## 2. Notation and Terminology

We use standard notation and terminology of graph theory [4, 5] unless otherwise stated.

Digraphs are loopless and without multiple arcs. Multidigraphs may have multiple arcs, loops being forbidden. A digraph without 2-cycle $\mathcal{D} K_{2}$ $\left(=\vec{C}_{2}\right)$ is called an oriented graph.

The ordered pair ( $v_{1}, v_{2}$ ) of vertices $v_{1}$ and $v_{2}$ (or the symbol $v_{1} \rightarrow v_{2}$ ) denotes the arc which goes from the tail $v_{1}$ to the head $v_{2}$. The converse of a multidigraph is obtained by reversal of each arc. A multidigraph is called self-converse if it is isomorphic to its converse.

The symbol $\cup$ when applied to multidigraphs stands for the vertexdisjoint union. Moreover, given a digraph $D$, the symbol $D+A^{\prime}$ denotes the spanning supermultidigraph of $D$ with the arc set $A(D) \cup A^{\prime}$, where $A^{\prime}$ is a set of (possibly copies of) arcs and $A^{\prime} \cap A(D)=\emptyset$. Similarly, $D-A^{\prime}$ denotes the spanning subdigraph obtained from $D$ by removal of $A^{\prime}$, where $A^{\prime}$ is to be a subset of $A(D)$. We write $D \pm A^{\prime}=D \pm a$ if $A^{\prime}=\{a\}$ and $a$ is an arc.

By a decomposition of a multidigraph $D$ we mean a family of arc-disjoint submultidigraphs of $D$ which include all arcs of $D$. Those substructures are called elements of a decomposition. By an $H$-decomposition we mean a decomposition of $D$ into $t$ elements all isomorphic to $H$; then we write $H \mid D$ or $t \mid D$. The isomorphism class of those $t$ pairwise isomorphic elements of a decomposition is called a tth part of $D$.

There are two non-self-converse digraphs of size two each, namely,

$$
\begin{aligned}
& P^{\vee}=\left(\left\{v_{1}, v_{2}, v_{3}\right\},\left\{\left(v_{1}, v_{2}\right),\left(v_{3}, v_{2}\right)\right\}\right), \text { a gutter, } \\
& P^{\wedge}=\left(\left\{v_{1}, v_{2}, v_{3}\right\},\left\{\left(v_{2}, v_{1}\right),\left(v_{2}, v_{3}\right)\right\}\right), \text { a roof. }
\end{aligned}
$$

Note that $\mathcal{D} K_{3} / 3$ comprises three digraphs, none of which is a self-converse oriented graph.

$$
\begin{equation*}
\mathcal{D} K_{3} / 3=\left\{\vec{C}_{2}, P^{\vee}, P^{\wedge}\right\} . \tag{1}
\end{equation*}
$$

A decomposition of $D$ is called to be 1-generated if there is a permutation $\gamma$ of $V(D)$ which generates the decomposition from any single decomposition element $H$ in the sense that, for $j=0, \ldots, t-1$, the image of $H$ under the $j$ th iteration $(\gamma)^{j}$ of $\gamma$, denoted by $(\gamma)^{j} H$, is one of $t$ decomposition parts, where $(\gamma)^{0}=\mathrm{id}$. Call $\gamma$ to be a placement-generating permutation for $H$. If, moreover, $\gamma$ is a cyclic permutation then the decomposition is called cyclically 1-generated (cf. cyclic decomposition in Chartrand and Lesniak [5], see also Bosák [4] for the equivalent notion of a decomposition according to a cyclic group).

Proposition 6. Each decomposition of the complete digraph $\mathcal{D} K_{n}$ into two isomorphic halves is 1-generated.

Proof. Note that these halves are self-complementary digraphs. The result follows from the known characterization of complementing permutations for those halves, cf. Bosák [4, Ch. 14].

Given a self-converse multidigraph $D$ on $n$ vertices, we use the symbol $\varphi$ $\left(=\varphi_{n}\right)$ to denote a conversing permutation, that is, a permutation of $V(D)$ such that $\varphi D$ is the converse multidigraph of $D$.

## 3. Cyclic Decompositions Into $n$ Parts

Bermond and Faber prove [3] that the complete digraph $\mathcal{D} K_{n}$ is decomposable into cycles $\vec{C}_{n-1}$ of length $n-1$. It can be noted that $\vec{C}_{n-1^{-}}$ decomposition of $\mathcal{D} K_{n}$, presented in [3] as well as in [1], is not 1-generated. We are going to improve this result by characterizing cyclically 1-generated decompositions of $\mathcal{D} K_{n}$ into ( $n-1$ )-cycles. Namely, a cyclically 1-generated $\vec{C}_{n-1}$-decomposition exists precisely if $n$ is odd. Additionally, a cyclically 1-generated decomposition of $\mathcal{D} K_{n}$ into hamiltonian paths for even $n$ follows from the widely known construction presented in Berge [2, p. 232] and also in Lucas [10, Ch. 6] (who attributes this result to Walecki) by passing on from $K_{n}$ to $\mathcal{D} K_{n}$. It is worth noting that just this cyclically 1-generated $\vec{P}_{n}$-decomposition of $\mathcal{D} K_{n}$ is presented in [10, Remark on p. 176] in terms of designing a set of single file walks for $n$ children so that each child once is the first, once the last, and no ordered pair of neighbours in a file is repeated among the files in the set. We prove that this decomposition exists precisely if $n$ is even (Theorem 7). In either case cyclically 1-generated decomposition
plays a crucial role in the proof of the main result since those decompositions enable a recursive construction in proofs of Lemma 8 and Theorem 5.

Let $V\left(\mathcal{D} K_{n}\right)=\mathbb{Z}_{n}$, the cyclic group of order $n$. Let $W_{0}$ be a sequence of (possibly repeating) vertices of the digraph $\mathcal{D} K_{n}$, say $W_{0}=<x_{1}, x_{2}$, $\ldots, x_{k}>$. In what follows we use the convention that $W_{0}$ refers to the walk whose subsequence of vertices is $W_{0}$. Moreover, the symbol $\left\langle W_{0}\right\rangle$ stands for the graph induced by the arc set of the walk $W_{0}$.

Definition 1. Assume that $n \geq 3$. Define the vertex sequence, which depends on the parity of $n$ and is denoted by $W_{0}(n)$ or $W_{0}$, as follows.
(i) For odd $\left.n \geq 3, W_{0}=<0, \frac{n-3}{2}, \ldots, \frac{n+1}{2}, 0\right\rangle$, which represents a cycle in which $\frac{n-1}{2}$ is the only vertex which is omitted. If $n=3$ then $W_{0}:=$ $<0,2,0\rangle$. If $n \geq 5$, we assume that the cycle $W_{0}$ comprises the following arcs, where $k$ stands for an integer:

$$
\begin{array}{rlrl}
k \rightarrow & \frac{n-3}{2}-k, & & 0 \leq k \leq \frac{n-5}{4}, \\
& \frac{n-3}{2}-k \rightarrow k+1, & & 0 \leq k \leq \frac{n-7}{4}, \\
\frac{n+3}{2}+k \rightarrow & n-1-k, & & 0 \leq k \leq \frac{n-7}{4}, \\
n-1-k \rightarrow \frac{n+1}{2}+k, & 0 & 0 & \leq \frac{n-5}{4}, \tag{5}
\end{array}
$$

and also two arcs $\frac{n+1}{2} \rightarrow 0,\left\lfloor\frac{n-1}{4}\right\rfloor \rightarrow\left\lfloor\frac{3 n+1}{4}\right\rfloor$.

Hence

$$
W_{0}= \begin{cases}<0,1,4,3,0>, & n=5 \\ <0,3,1,2,7,6,8,5,0>, & n=9 \\ <0,2,1,5,6,4,0>, & n=7 \\ <0,4,1,3,2,8,9,7,10,6,0> & n=11\end{cases}
$$

Figure $1(n=7,9)$ shows the difference between cases $n \equiv 1,3(\bmod 4)$.
(ii) For even $n \geq 4, W_{0}=<0,1, n-1, \ldots, \frac{n}{2}>$, which represents a hamiltonian path of $\mathcal{D} K_{n}$. It is assumed that the path includes the following arcs:

$$
k \xrightarrow{(6)} n-k \xrightarrow{(7)} k+1, \quad 1 \leq k \leq \frac{n-2}{2},
$$

and the initial arc $0 \rightarrow 1$, see Figure 2 wherein $n=8$.
Define the length of the arc $(i, j)$ to be $(j-i) \bmod n$.


Figure 1. $n=7,9$


Figure 2. $n=8$
Theorem 7. For $n \geq 4$, the cyclic permutation $\gamma_{0}:=[0,1,2, \ldots, n-1]$ is a placement-generating permutation for the self-converse oriented graph $<W_{0}>$ in $\mathcal{D} K_{n}$, that is, $\mathcal{D} K_{n}$ has a cyclically 1-generated $<W_{0}>$-decomposition into $n$ parts which are either the cycle $\vec{C}_{n-1}$ or path $\vec{P}_{n}$ according as $n$ is odd or even. Moreover, if a 1-generated $\vec{C}_{n-1}$-decomposition [1-generated $\vec{P}_{n}$-decomposition] of $\mathcal{D} K_{n}$ exists then $n$ is odd $[n$ is even $]$.

Proof. Using the list of arcs of $W_{0}$ in Definition 1 one can see that arcs of $W_{0}$ have mutually distinct lengths. Namely, if $n \geq 4$ is odd and $\left.<W_{0}\right\rangle=$ $\vec{C}_{n-1}$, the lengths are $\frac{n-3}{2}-2 k$ for arcs listed in (2), $\frac{n-5}{2}-2 k$ in (4), and similarly $\frac{n+5}{2}+2 k$ for arcs in (3), and $\frac{n+3}{2}+2 k$ in (5), for increasing $k$ starting at $k=0$. Two additional arcs $\frac{n+1}{2} \rightarrow 0,\left\lfloor\frac{n-1}{4}\right\rfloor \rightarrow\left\lfloor\frac{3 n+1}{4}\right\rfloor$ have lengths $\frac{n-1}{2}$, $\frac{n+1}{2}$, respectively. Analogously, if $n$ is even and $\left\langle W_{0}\right\rangle=\vec{P}_{n}$, the lengths are even, $n-2 k$, for arcs with label (6) and odd, $2 k+1$, with label (7), where $k=1,2, \ldots, \frac{n-2}{2}$. The initial arc $0 \rightarrow 1$ has length 1 .

Note that a noncyclic permutation of $n$ vertices cannot generate $n$ elements of a decomposition because then there is an edge which has less than
$n$ different images under iterations of the permutation. Suppose that a cyclic permutation generates a decomposition of $\mathcal{D} K_{n}$ into $n$ subdigraphs. It is easy to see that an $(n-1)$-cycle if $n$ is even, as well as a path on $n$ vertices for odd $n$, with all their arcs of different lengths do not exist. Namely, otherwise the sum of lengths would be 0 modulo $n$ for the cycle (i.e., for even $n$ ) and non-zero for the path (i.e., for odd $n$ ). However, just the opposite is true. Namely, the sum of lengths is $s=\sum_{i=1}^{n-1} i=\frac{n(n-1)}{2}$ and therefore $s \not \equiv 0(\bmod n)$ if $n$ is even but $s \equiv 0(\bmod n)$ if $n$ is odd.
Note that none of third parts of $\mathcal{D} K_{3}$ is a self-converse oriented graph, cf. (1), and therefore the case $n=3$ is excluded in Theorem 7 .

## 4. Decompositions Into $t$ Parts

Harary et al. [8] prove that if the numerical divisibility condition $t \mid n(n-1)$ is satisfied then the complete digraph $\mathcal{D} K_{n}$ is decomposable into $t$ isomorphic self-converse parts. If $t \mid n$ or $t \mid(n-1)$, we extend this result to the case where parts are to be self-converse oriented graphs and a decomposition is to be 1 -generated.

Lemma 8. For every $n \geq 2$, if $t \mid n$ or $t \mid(n-1)$ then there exists a 1-generated decomposition of the complete digraph $\mathcal{D} K_{n}$ into $t$ isomorphic self-converse parts which are oriented graphs unless $n=t=3$.

Proof. For $t=2$ and $t=3$, Lemma is proved in [12] and [11], respectively. Namely, if $t=2$, one of halves of $\mathcal{D} K_{n}$ with any $n$ is the transitive tournament (see Propositions 1 and 6 ). For $t=3$ and $n=3 t$ or $n=3 t+1$, decompositions of $\mathcal{D} K_{n}$ constructed in [11] can be seen to be 1-generated. More precisely, oriented graphs $D_{i}$ for $i=4,6,7,9$ constructed in the proof of Lemma 1.1 in [11] as well as an oriented graph $D_{n}$ in the proof of Theorem 1 in [11] are required third parts of a decomposition, existence of which is a part of the assertion of Theorem 2. If $t=n(>3)$, the result follows immediately from the above Theorem 7.

Consider the case $n=k t, k \in \mathbb{N}$, and $t \geq 4$. We proceed by induction on $k$. Theorem 7 can be viewed as the first step for $k=1$, namely, the selfconverse oriented graph $\left\langle W_{0}\right\rangle \in \mathcal{D} K_{t} / t$ and the vertex set $\left.V\left(<W_{0}\right\rangle\right)=$ $\{0,1, \ldots, t-1\}$. Assume now that Lemma is true for $k-1$, that is, for a fixed admissible self-converse oriented graph $D_{n-t} \in \mathcal{D} K_{n-t} / t$ where $V\left(D_{n-t}\right)=$ $\{t, t+1, \ldots, n-1\}$. We construct a self-converse oriented graph $D_{n} \in$
$\mathcal{D} K_{n} / t$. Let $\left.V\left(D_{n}\right)=V\left(D_{n-t}\right) \cup V\left(<W_{0}\right\rangle\right)$. The construction of $D_{n}$ takes the advantage of the structure of $\left\langle W_{0}\right\rangle$. Namely, one can easily find a conversing permutation $\varphi_{t}$ of $\left\langle W_{0}\right\rangle$ in $\mathcal{D} K_{t}$, which moves the vertex 0 into $\left\lceil\frac{t}{2}\right\rceil$ (cf. Definition 1). Then we define $D_{n}$ to be the following digraph.

$$
D_{n}=\left\langle W_{0}>\cup D_{n-t}+\left\{(0, i),\left(i,\left\lceil\frac{t}{2}\right\rceil\right): t \leq i \leq n-1\right\} .\right.
$$

One can see that $D_{n}$ is a self-converse oriented graph with conversing permutation $\varphi_{n}=\left(\varphi_{t}, \varphi_{n-t}\right):=\varphi_{t} \varphi_{n-t}$, where $\varphi_{n-t}$ is a conversing permutation of $D_{n-t}$. Moreover, $D_{n} \in \mathcal{D} K_{n} / t$ with placement-generating permutation $\gamma_{n}=\left(\gamma_{0}, \gamma_{n-t}\right)$, where $\gamma_{n-t}$ is a placement-generating permutation of $D_{n-t}$ in $\mathcal{D} K_{n-t}$ (see Theorem 7 for $\gamma_{0}$ ).

In the case $n=k t+1, k \in \mathbb{N}$, note first that the self-converse oriented graph $D_{t+1}^{\prime}=\left\langle W_{0}\right\rangle \cup[t]+\left\{(0, t),\left(t,\left\lceil\frac{t}{2}\right\rceil\right)\right\}$, where $[t]:=K_{1}$ with $V\left(K_{1}\right)=$ $\{t\}$, is the $t$ th part of $\mathcal{D} K_{t+1}$ with placement-generating permutation $\gamma_{t+1}=$ $\left(\gamma_{0},(t)\right)$ and conversing permutation $\varphi_{t+1}=\left(\varphi_{t},(t)\right)$. For $n=k t+1$ where $t \geq 4$ and $k \geq 2$, we construct a self-converse oriented graph $D_{n}^{\prime} \in \mathcal{D} K_{n} / t$ analogously to the case above, using in the induction step digraphs $D_{t+1}^{\prime}$ and $D_{n-t-1}$.
In [13] we have proved the following result on decompositions of the complete digraph into nonhamiltonian paths, which is useful in proving Theorem 5.

Theorem 9 [13]. For any $n \geq 3$, the complete digraph $\mathcal{D} K_{n}$ is decomposable into paths of arbitrarily prescribed lengths $(\leq n-2)$ provided that the lengths sum up to the size $n(n-1)$ of $\mathcal{D} K_{n}$.

Now we are ready to prove the main result of the paper.
Proof of Theorem 5. If $t \leq 3$, Theorem is true by Proposition 1 for $t=2$ and by (1) and Theorems $2,3,4$ for $t=3$. It is so, too, if $t=n$, by Theorem 7. The result is easily seen for $n \leq 3$ and $t>3$. In particular, $S$ can be chosen so that just $\vec{P}_{3}$ (the only possible candidate) is the ceiling- $S$ th part of $\mathcal{D} K_{3}$ for $t=4,5$.

Consider the case $t>n \geq 4$. Note that $\lfloor n(n-1) / t\rfloor \leq n-2$ whence, in case of the floor class, the result is true by Theorem 9, elements of a decomposition as well as $\langle R\rangle$ are oriented paths. In case of the ceiling class, if $t=n+1$ then $\lceil n(n-1) / t\rceil=n-1,|S|=n-1$ and, by Theorem 7 , the result is true. For $t \geq n+2,\lceil n(n-1) / t\rceil \leq n-2$ and analogously
the result is true by Theorem 9, i.e., elements of a decomposition and $\langle S\rangle$ are paths of prescribed lengths.

It remains to consider the case $4 \leq t<n$. Let $r=n \bmod t$. Then $r \leq n-t$ and $(n(n-1) \bmod t)=(r(r-1) \bmod t)$. Applying similar induction as in the proof of Lemma 8 one can easily construct required near- $t$ th parts of $\mathcal{D} K_{n}$. Namely, in the induction step we use self converse oriented graphs $D_{t}$ (see proof of Lemma 8 for a construction) and $D_{n-t}$ (construction follows from the case $t>n \geq 2$ above and by induction) which are $t$ th and corresponding near- $t$ th parts of $\mathcal{D} K_{t}$ and $\mathcal{D} K_{n-t}$, respectively.

## 5. Concluding Remarks

Note that the above relatively short proof of the main result is based on Theorem 9 (the proof of which is nontrivial) and on characterizations of special cyclically 1 -generated decompositions, see Theorem 7.

It is an open problem to determine all possible cyclically 1-generated $n$th parts of $\mathcal{D} K_{n}$, which are self-converse oriented graphs. Similar problem concerns $t$ th parts as considered in Lemma 8 above.

Another open problem is related to the notion of a $t$ th universal floor part, say $F$, of $\mathcal{D} K_{n}$. The meaning of 'universal' is that packings of $t$ copies of $F$ into $\mathcal{D} K_{n}$ should leave remainders $R$ of all possible shapes. The open problem is motivated by a conjecture of the second author, stated several years ago (see [17]), that universal floor parts of complete (undirected) graphs exist. Supporting results in $[18,9]$ cover infinitely many pairs $(n, t)$. Moreover, Plantholt's deep theorem [14] on the chromatic index is equivalent to the truth of the conjecture for $t=n-1$ with $n$ being odd. On the other hand, there is no universal third part of $\mathcal{D} K_{5}$, see [6].

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