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MINIMUM CONGESTION SPANNING TREES OF GRIDS AND DISCRETE TORUSES

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Abstract

The paper is devoted to estimates of the spanning tree congestion for grid graphs and discrete toruses of dimensions two and three.

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1. Introduction

In this paper we consider finite simple graphs. Our graph-theoretic terminology follows [3]. For a graph G by V_G and E_G we denote its vertex set and its edge set, respectively.

Let G be a graph and let T be a spanning tree in G (that is, T is both a tree and a spanning subgraph of G). For each edge e of T let A_e and B_e

be the vertex sets of the components of $T\backslash e$. By $e_G(A_e, B_e)$ we denote the number of edges with one end vertex in A_e and the other end vertex in B_e. We define the *edge congestion* of G in T by

$$\mathrm{ec}(G : T) = \max_{e \in E_T} e_G(A_e, B_e).$$

The name comes from the following analogy. Imagine that edges of G are roads, and that edges of T are those roads which are cleaned from snow after snowstorms. For each road $g \in E_G$ there exists a unique path P_g in T joining the end vertices of g, we call such path a *detour*, even in the case when $P_g = g$. For an edge h of T it is quite natural to define the *congestion* $c(h)$ as the number of times h is used in different detours $\{P_g\}_{g \in E_G}$. Then

$$\mathrm{ec}(G : T) = \max_{h \in E_T} c(h).$$

It is clear that for applications it is interesting to find a spanning tree which minimizes the congestion.

We define *the spanning tree congestion* of G by

$$s(G) = \min\{\mathrm{ec}(G : T) : \ T \text{ is a spanning tree of } G\}. \tag{1}$$

Each spanning tree T in G satisfying $\mathrm{ec}(G : T) = s(G)$ is called a *minimum congestion spanning tree* for G. The parameters $\mathrm{ec}(G : T)$ and $s(G)$ were introduced and studied in [7], see also [9]. These parameters are of interest in the study of Banach-space-theoretical properties of Sobolev spaces on graphs, see [8, Section 3.5.1]. An additional motivation for this study is that minimum congestion spanning trees can be considered as 'congestional' analogues of the well-known shortest (or minimal) spanning trees. See [11] for an account on shortest spanning trees, and on other minimality properties of spanning trees.

The purpose of this paper is to find exact values or estimates of $s(G)$ for some grids and discrete toruses.

2. Some General Remarks on the Spanning Tree Congestion

The definitions and observations recalled in this paragraph go back to C. Jordan [6], see [5, pp. 35–36]. Let u be a vertex of a tree T. If we delete all edges incident with u from T, we get a forest. The maximal number of vertices in components of the forest is called the *weight of* T *at* u. A vertex

v of T is called a *centroid vertex* if the weight of T at v is minimal. Each tree has one or two centroid vertices.

Denote by Δ_G the maximum degree of the graph G. Let T be an optimal spanning tree in G, that is, $\mathrm{ec}(G:T) = s(G)$. Let u be a centroid of T. Since T is a subgraph of G, there are at most Δ_G edges incident with u in T. Hence at least one of the components, we denote it by A, in the forest obtained from T after deletion of u has at least $o_G = \lceil \frac{|V_G|-1}{\Delta_G} \rceil$ vertices, and at most $\frac{|V_G|}{2}$ vertices. The edge connecting u with A is used in $e_G(A, \bar{A})$ detours. Therefore, any isoperimetric type inequality, estimating from below the number of edges in G joining a set containing at least o_G and at most $|V_G|/2$ vertices with its complement provides an estimate of $s(G)$ from below. In the next section we see that combining this observation with the known isoperimetric inequalities [2] for discrete toruses and for grids, we get close-to-optimal estimates from below for the spanning tree congestion of some of these graphs.

3. GRIDS AND TORUSES

We use the notation from [2]. Namely, the *grid graph* is the graph with vertex set $[k]^n = \{0, 1, 2, 3, \ldots, k-1\}^n$ in which $x = (x_i)_1^n$ is joined to $y = (y_i)_1^n$ if and only if $|x_i - y_i| = 1$ for some i and $x_j = y_j$ for all $j \neq i$. Let $\mathbb{Z}_k$ be the quotient group $\mathbb{Z}/k\mathbb{Z}$. The *discrete torus* is the graph with the vertex set $\mathbb{Z}_k^n = (\mathbb{Z}_k)^n$ in which $x = (x_i)_1^n$ is joined to $y = (y_i)_1^n$ if and only if $x_i - y_i = \pm 1$ for some i and $x_j = y_j$ for all $j \neq i$.

3.1. Two-dimensional case

It turns out that for two-dimensional square grids and toruses the isoperimetric inequalities proved in [2] provide bounds for $s(G)$ from below which are achieved for easily constructed trees.

Theorem 1. *$s([k]^2) = k$ for $k \geq 2$ and $s(\mathbb{Z}_k^2) = 2k$ for $k \geq 3$.*

Proof. Observe that for both graphs the maximum degree is 4, therefore $o_{\mathbb{Z}_k^2} = o_{[k]^2} = \lceil \frac{k^2-1}{4} \rceil$ (o_G was defined in Section 2). We need

Theorem 2 ([2, Theorem 3])**.** *Let A be a subset of $[k]^n$ with $|A| \leq k^n/2$. Then*

$$e_G(A, \bar{A}) \geq \min\left\{ |A|^{1-1/r} r k^{(n/r)-1} : r = 1, \ldots, n \right\}. \tag{2}$$

Theorem 3 ([2, Theorem 8]). *Let A be a subset of $\mathbb{Z}_k^n$ with $|A| \leq k^n/2$. Then*

$$e_G(A, \bar{A}) \geq \min\left\{2|A|^{1-1/r} r k^{(n/r)-1} : r = 1, \dots, n\right\}. \tag{3}$$

Using Theorem 2 with $|A| = o_{[k]^2} = \lceil \frac{k^2-1}{4} \rceil$ and the observation from Section 2, we get $s([k]^2) \geq 2\sqrt{\lceil \frac{k^2-1}{4} \rceil}$. Hence $s([k]^2) \geq k$ for $k \geq 2$. (Observe that for even k we can use the second inequality from [2, Corollary 4].)

Using Theorem 3 with $|A| = o_{\mathbb{Z}_k^2} = \lceil \frac{k^2-1}{4} \rceil$ and the observation from Section 2, we get $s(\mathbb{Z}_k^2) \geq 4\sqrt{\lceil \frac{k^2-1}{4} \rceil}$. Hence $s(\mathbb{Z}_k^2) \geq 2k$ for $k \geq 3$.

It remains to show that there exist trees for which these bounds are attained. It turns out that if we consider $[k]^2$ as a subgraph of $\mathbb{Z}_k^2$, we can use the same tree for both of them.

Now we construct optimal spanning trees T_k in $[k]^2$. We construct them somewhat differently for odd and even k.

First let $k = 2r + 1$, $r \in \mathbb{N}$. In this case the edge set of T_k consists of

(A) 'Vertical' edges joining the vertices (r, y) and $(r, y + 1)$, $y \in \{0, \dots, k - 2\}$;

(B) 'Horizontal' edges joining the vertices (x, y) and $(x + 1, y)$, $x \in \{0, \dots, k - 2\}$, $y \in \{0, \dots, k - 1\}$.

The verification of the equality $\text{ec}([k]^2 : T_k) = k$ is straightforward (the reader is advised to sketch a picture).

Now let $k = 2r$, $r \in \mathbb{N}$. In this case the edge set of T_k consists of

(A) Two sets of 'vertical' edges:

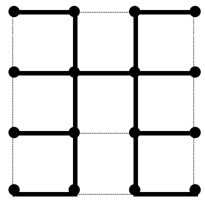
(1) edges joining the vertices (r, y) and $(r, y + 1)$, $y \in \{0, \dots, k - 2\}$;

(2) edges joining the vertices $(r - 1, y)$ and $(r - 1, y + 1)$, $y \in \{0, \dots, k - 2\}$;

(B)

(1) 'Horizontal' edges joining the vertices (x, y) and $(x + 1, y)$, $x \in \{0, \dots, k - 2\} \setminus \{r - 1\}$, $y \in \{0, \dots, k - 1\}$;

(2) A horizontal edge joining $(r - 1, r)$ and (r, r). (The enclosed figure shows the resulting graph for $k = 4$.)



The verification of the equality $\mathrm{ec}([k]^2 : T_k) = k$ is straightforward. ∎

3.2. Three-dimensional grids and toruses

It is much more difficult to find the exact values of $s([k]^3)$ and $s(\mathbb{Z}_k^3)$, except for very small values of k. We prove the following asymptotic estimates only.

Theorem 4.

$$\frac{2}{\sqrt{6}}k^2 - o(1) \le s([k]^3) \le \frac{7}{8}k^2 + O(k). \tag{4}$$

$$\frac{4}{\sqrt{6}}k^2 - o(1) \le s(\mathbb{Z}_k^3) \le \frac{7}{4}k^2 + O(k). \tag{5}$$

Proof. The estimates from below are proved in the same way as in the two-dimensional case. If $k \ge 3$, the maximum degree of $[k]^3$ is 6, therefore $o_{[k]^3} = \lceil \frac{k^3-1}{6} \rceil$ (o_G was defined in Section 2). Using Theorem 2 with $|A| = o_{[k]^3}$ we get

$$e_{[k]^3}(A, \bar{A}) \ge \min \left\{ k^2,\ 2\sqrt{\left\lceil \frac{k^3-1}{6} \right\rceil \cdot k},\ 3\left(\left\lceil \frac{k^3-1}{6} \right\rceil\right)^{\frac{2}{3}} \right\}. \tag{6}$$

A simple computation shows that for $k \ge 2$ the number

$$2\sqrt{\left\lceil \frac{k^3-1}{6} \right\rceil \cdot k}$$

is the minimum in (6). Using the observation from Section 2, we get the estimate from below in (4). Using Theorem 3 and the same computation as above, we get the estimate from below in (5).

Now we turn to the estimate from above. First we consider a solid cube of size $k \times k \times k$. We cut it into 6 rectangular pieces, first we cut a slice of height $\frac{3}{8}k$ at the top, and divide it vertically into two rectangular pieces of the same size. We get two rectangular parallelepipeds of height $\frac{3}{8}k$ whose bases are $k \times \frac{k}{2}$ rectangles. We cut the bottom slice of height $\frac{5}{8}k$ into four pieces, using vertical cuts, each of the 4 pieces from above is an $\frac{k}{2} \times \frac{k}{2}$ square. We get four rectangular parallelepipeds of height $\frac{5}{8}k$ whose bases are $\frac{k}{2} \times \frac{k}{2}$ squares. It is easy to check that for each of the 6 pieces the surface area of the intersection of cuts with the piece is $\frac{7}{8}k^2$. It is an interesting geometric question: is it possible to get less than $\frac{7}{8}k^2$ cutting the cube into 6 pieces in a different way? It is natural to expect that an answer to this question would allow to improve bounds in Theorem 4.

We partition $[k]^3$ into 6 pieces using discrete versions of the cuts described above. Of course we have to round the numbers $\frac{5}{8}k$ and $\frac{k}{2}$ to nearest integers. We introduce the notation: $A = \lceil \frac{k}{2} \rceil$, $B = \lceil \frac{k}{2} \rceil - k + 1$, $C = \lceil \frac{3}{8}k \rceil$, $D = \lceil \frac{3}{8}k \rceil - k + 1$. We consider the grid $[k]^3$ as the set of integer points $(x, y, z) \in \mathbb{R}^3$ satisfying the inequalities $B \le x \le A$, $B \le y \le A$, $D \le z \le C$.

Our partition of the vertex set of $[k]^3$ is the following:

$$\begin{array}{llll}
P_1: & B \le x \le A, & 1 \le y \le A, & 1 \le z \le C,\\
P_2: & B \le x \le A, & B \le y \le 0, & 1 \le z \le C,\\
P_3: & 1 \le x \le A, & 1 \le y \le A, & D \le z \le 0,\\
P_4: & B \le x \le 0, & 1 \le y \le A, & D \le z \le 0,\\
P_5: & 1 \le x \le A, & B \le y \le 0, & D \le z \le 0,\\
P_6: & B \le x \le 0, & B \le y \le 0, & D \le z \le 0.
\end{array}$$

We construct a spanning tree T whose centroid is the vertex $O = (0, 0, 0)$. All edges incident with O are in T. Therefore O has six neighbors in T. We denote them by $\{n_i\}_{i=1}^6$, where $n_1 = (0, 1, 0)$, $n_2 = (0, 0, 1)$, $n_3 = (0, 0, -1)$, $n_4 = (-1, 0, 0)$, $n_5 = (1, 0, 0)$, $n_6 = (0, -1, 0)$. The indices of $\{n_i\}_{i=1}^6$ are chosen in such a way that n_i is on the path from O to P_i ($i = 1, \dots, 6$) in the following sense: If we delete the edge joining O and n_i from T, the vertex set of one of the obtained components is, up to certain small and independent of k amount of vertices, the vertex set of P_i. Together with the estimates for the boundaries of P_i, this leads to the desired estimates for the numbers of detours in which each of these 6 edges is used.

Now we construct the rest of T. We start by construction of a tree which is a subgraph of $[k]^3$ and whose vertex set is P_1. This tree is constructed as follows: each 'rectangle' $R_1(x)$ in P_1 consisting of vertices (x, y, z) with fixed x is joined with the rectangle $R_1(x + 1)$ ($B \le x \le A - 1$) with exactly one edge, and this edge is the edge joining the vertices $(x, 1, 1)$ and $(x+1, 1, 1)$. In each of $R_1(x)$ we construct a tree in the following way. It contains 'vertical' edges joining $(x, 1, z)$ and $(x, 1, z+1)$ for $1 \le z \le C-1$ and 'horizontal' edges joining (x, y, z) with $(x, y+1, z)$ for $1 \le z \le C$ and $1 \le y \le A-1$.

We also add to T the edge between n_1 and the vertex $(0, 1, 1)$. There are no other edges in T having one of their end vertices in P_1.

The part of T corresponding to P_2 is constructed similarly. One of the differences is that we do not need an additional edge mentioned in the previous paragraph: the vertex n_2 is already in P_2. Because of lack of symmetry and for the sake of completeness we describe the rest of the construction. Each 'rectangle' $R_2(x)$ in P_2 consisting of vertices (x, y, z) with fixed x is joined with the rectangle $R_2(x+1)$ $(B \le x \le A-1)$ with exactly one edge, and this edge is the edge joining the vertices $(x, 0, 1)$ and $(x+1, 0, 1)$. In each of $R_2(x)$ we construct a tree in the following way. It contains 'vertical' edges joining $(x, 0, z)$ and $(x, 0, z+1)$ for $1 \le z \le C-1$ and 'horizontal' edges joining (x, y, z) with $(x, y+1, z)$ for $1 \le z \le C$ and $B \le y \le -1$.

The trees corresponding to P_i, $i = 3, 4, 5, 6$ are somewhat different. Unfortunately, the structure we consider does not have enough symmetry for the uniform description of the trees. Nevertheless, the trees corresponding to P_i, $i = 3, 4, 5, 6$ are similar to each other.

First we describe paths from O to each of P_i, $i = 3, 4, 5, 6$.

- Path to P_3 goes through $n_3 = (0, 0, -1)$, and the vertex $(0, 1, -1)$. It reaches P_3 at the vertex $p_3 = (1, 1, -1)$.
- Path to P_4 goes through $n_4 = (-1, 0, 0)$, and reaches P_4 at the vertex $p_4 = (-1, 1, 0)$.
- Path to P_5 is trivial: n_5 is already in P_5, we let $p_5 = n_5$.
- Path to P_6 is also trivial: n_6 is already in P_6, we let $p_6 = n_6$.

We exclude from each of the P_i vertices used in the described above paths to other pieces, and denote the obtained sets by $\widetilde{P}_i$. We let $(x_3, y_3) = (1, 1)$, $(x_4, y_4) = (-1, 1)$, $(x_5, y_5) = (1, 0)$, $(x_6, y_6) = (0, -1)$. Observe that these pairs satisfy the conditions: (a) They are the pairs of the (x, y)-coordinates of p_3, p_4, p_5, and p_6, respectively; (b) All points of the form (x_i, y_i, z), $D \le z \le 0$ are in $\widetilde{P}_i$ (this means that they are not among the vertices excluded from P_i according to the first sentence of this paragraph).

In each $\widetilde{P}_i$ $(i = 3, 4, 5)$ we construct a tree in the following way: it contains.

- Edges joining (x_i, y_i, z) and $(x_i, y_i, z+1)$ for $D \le z \le -1$.
- Edges joining (x_i, y, z) and $(x_i, y+1, z)$ in the cases when both points are in $\widetilde{P}_i$.

- Edges joining (x, y, z) and $(x + 1, y, z)$ in the cases when both points are in $\widetilde{P}_i$.

For $\widetilde{P}_6$ we construct the tree slightly differently. The tree contains

- Edges joining $(0, -1, z)$ and $(0, -1, z + 1)$ for $D \leq z \leq -1$.
- Edges joining $(x, -1, z)$ and $(x + 1, -1, z)$ in the cases when both points are in $\widetilde{P}_6$.
- Edges joining (x, y, z) and $(x, y + 1, z)$ in the cases when both points are in $\widetilde{P}_6$.

For each z satisfying $D \leq z \leq 0$ we denote the set of the points of the form (x, y, z) which are in $\widetilde{P}_i$ $(i = 3, 4, 5, 6)$ by $R_i(z)$.

Now we estimate the congestion on each of the edges of T. It is easy to see that, up to a small constant number of edges, the congestion on the edge from O to n_1 is equal to the number of edges joining P_1 with its complement, and this number is $\leq \frac{7}{8}k^2 + O(k)$. The same can be said about the edge joining n_1 and $(0, 1, 1)$. Similarly we estimate the congestion on the edge joining O and each point of the set $\{n_i\}_{i=2}^6$ as well as the congestion on the other edges from paths joining O and p_3, and O and p_4.

Now we consider other edges. The tree T was constructed in such a way that each of the edges is either an edge between different rectangles R_i, or an edge inside one of R_i.

For an edge between different rectangles R_i, the congestion on it is equal to the 'surface area' of the piece which is separated by the removal of such edge. Such pieces are discrete rectangular parallelepipeds and their 'surface area' can be easily estimated from above by $\frac{7}{8}k^2 + O(k)$ in all possible cases.

If we consider an edge inside one of the rectangles R_i, the congestion on it can be estimated from above by the $2M + B + 4$, where M is the number of vertices in the sub-rectangle of $R_i(z)$ (or $R_i(x)$) separated by the removal of the edge, and B is the number points on the boundary of the sub-rectangle. It is clear that this number is $\leq \frac{7}{8}k^2 + O(k)$.

The same tree can be used to estimate $s(\mathbb{Z}_k^3)$ from above. It is easy to verify that the congestion on each edge is at most doubled when we pass from $[k]^3$ to $\mathbb{Z}_k^3$. ∎

4. Final Remarks

It looks plausible that in order to obtain estimates for the spanning tree congestion of d-dimensional, $d \geq 3$, cubic grids and toruses up to $O(k)$ it suffices to solve the following isoperimetric problem.

Problem. How to cut $[k]^d$ (or $\mathbb{Z}_k^d$) into $2d$ pieces in such a way that each piece contains at most half of vertices of $[k]^d$ ($\mathbb{Z}_k^d$), and the maximum edge boundary of pieces is minimized?

This problem, with $2d$ replaced by a general number I is well-known in computer science, see [4, p. 358] and [10]. The main focus in the mentioned papers is somewhat different, and the obtained results do not provide estimates, which are precise enough for the problem above.

R. Ahlswede and S. L. Bezrukov [1] obtained versions of results of [2] for rectangular grids. These results could help to generalize results of the present paper to the rectangular case.

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