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# THE LIST LINEAR ARBORICITY OF PLANAR GRAPHS \*

XINHUI AN AND BAOYINDURENG WU

College of Mathematics and System Science Xinjiang University Urumqi 830046, P.R. China e-mail: xjaxh@xju.edu.cn, baoyin@xju.edu.cn

#### Abstract

The linear arboricity la(G) of a graph G is the minimum number of linear forests which partition the edges of G. An and Wu introduce the notion of list linear arboricity lla(G) of a graph G and conjecture that lla(G) = la(G) for any graph G. We confirm that this conjecture is true for any planar graph having  $\Delta \ge 13$ , or for any planar graph with  $\Delta \ge 7$  and without *i*-cycles for some  $i \in \{3, 4, 5\}$ . We also prove that  $\lceil \frac{\Delta(G)}{2} \rceil \le lla(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$  for any planar graph having  $\Delta \ge 9$ . **Keywords:** list coloring, linear arboricity, list linear arboricity, planar graph.

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## 1. INTRODUCTION

All graphs considered here are finite, undirected and simple. We refer to [4] for unexplained terminology and notations. For a real number x,  $\lceil x \rceil$  is the least integer not less than x. Let G = (V(G), E(G)) be a graph. |V(G)| and |E(G)| are called the *order* and the *size* of G, respectively. We use  $\Delta(G)$  and  $\delta(G)$  to denote the maximum degree and the minimum degree of G, respectively. Let v be a vertex of G. The neighborhood of v, denoted by  $N_G(v)$ , is the set of vertices adjacent to v in G. The degree of v, denoted

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by  $d_G(v)$ , is the number of edges incident with v in G. Since G is simple,  $d_G(v) = |N_G(v)|$ . If there is no confusion, we use N(v) and d(v) for the neighborhood and degree of v instead of  $N_G(v)$  and  $d_G(v)$ , respectively. Let  $N_k(v) = \{u | u \in N(v) \text{ and } d(u) = k\}$ . The girth of G is the minimum length of cycles in G. A k- or  $k^+$ -vertex is a vertex of degree k, or at least k.

A linear forest is a graph in which each component is a path. A map  $\varphi$  from E(G) to  $\{1, 2, \ldots, k\}$  is called a k-linear coloring if  $(V(G), \varphi^{-1}(i))$  is a linear forest for  $1 \leq i \leq k$ . The linear arboricity la(G) of a graph G, introduced by Harary [8], is the minimum number k for which G has a k-linear coloring. Akiyama, Exoo and Harary [1] conjectured that  $la(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$  for any regular graph G. It is obvious that for a graph G,  $la(G) \geq \lceil \frac{\Delta(G)+1}{2} \rceil$  and  $la(G) \geq \lceil \frac{\Delta(G)+1}{2} \rceil$  when G is regular. So it is equivalent to the following conjecture, known as the linear arboricity conjecture.

### **Linear Arboricity Conjecture.** For any graph G,

$$\left\lceil \frac{\Delta(G)}{2} \right\rceil \leqslant la(G) \leqslant \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil.$$

The linear arboricity has been determined for complete bipartite graphs [1], series-parallel graphs [10], and regular graphs with  $\Delta = 3$  [1], 4 [2], 5, 6, 8 [6], 10 [7]. The LAC also has already been proved to be true for any planar graphs in [9] and [12]. In particular, the author proved that if G is a planar graph with  $\Delta \ge 13$ , then  $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$ . In [9] and [11], the authors showed that the same also holds for a planar graph with  $\Delta \ge 7$  and without *i*-cycles for some  $i \in \{3, 4, 5\}$ .

A list assignment L to the edges of G is the assignment of a set  $L(e) \subseteq N$  of colors to every edge e of G, where N is the set of natural numbers. If G has a coloring  $\varphi$  such that  $\varphi(e) \in L(e)$  for every edge e and  $(V(G), \varphi^{-1}(i))$  is a linear forest for any  $i \in C_{\varphi}$ , where  $C_{\varphi} = \{\varphi(e) | e \in E(G)\}$ , then we say that G is *linear* L-colorable and  $\varphi$  is a *linear* L-colorable for every list assignment L satisfying |L(e)| = k for all edges e. The list linear arboricity lla(G) of a graph G is the minimum number k for which G is linear k-list colorable. It is obvious that  $la(G) \leq lla(G)$ . In [3], the authors raised the following conjecture, and confirmed that it is true for any series-parallel graph.

List Linear Arboricity Conjecture. For any graph G,

$$\left\lceil \frac{\Delta(G)}{2} \right\rceil \leqslant la(G) = lla(G) \leqslant \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil$$

Little was known for this conjecture. In this paper, we will prove that it is true for any planar graph having  $\Delta \ge 13$ , or for any planar graph with  $\Delta \ge 7$  and without *i*-cycles for some  $i \in \{3, 4, 5\}$ . We also prove that  $\lceil \frac{\Delta(G)}{2} \rceil \le lla(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$  for any planar graph having  $\Delta \ge 9$ .

2. Planar Graphs with la(G) = lla(G)

For convenience, we introduce two definitions. The weight w(e) of an edge e = uv is d(u) + d(v). An even cycle  $v_1v_2 \cdots v_{2t}v_1$  is called *k*-alternating if  $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = k$ .

Let L be a list assignment of G, and  $\varphi$  be a coloring of G such that  $\varphi(e) \in L(e)$  for any edge e of G. For a vertex  $v \in V(G)$ , we denote by  $C_{\varphi}(v)$  the set of colors that appear on the edges incident with v in G.

 $C^i_{\varphi}(v) = \{j \mid \text{the color } j \text{ appears } i \text{ times at edges incident with } v\},\$ 

for any positive integer *i*. Observe that  $\varphi$  is a linear *L*-coloring of *G* if and only if *G* does not contain a monochromatic cycle under coloring  $\varphi$  and  $|C_{\varphi}^{i}(v)| = 0$  for every vertex *v* of *G* and any  $i \ge 3$ . Thus, if  $\varphi$  is a linear *L*-coloring of *G* then  $C_{\varphi}(v) = C_{\varphi}^{1}(v) \cup C_{\varphi}^{2}(v)$ .

The following two lemmas can be found in [9].

**Lemma 2.1.** Let G be a planar graph with  $\delta(G) \ge 2$ . Then either there is an edge e with  $w(e) \le 15$  or there is a 2-alternating cycle  $v_0v_1 \cdots v_{2n-1}v_0$ such that  $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$  and  $\max_{0 \le i < n} |N_2(v_{2i})| \ge 3$ .

**Lemma 2.2.** Let G be a planar graph with girth at least g and maximum degree  $\Delta$ , and assume that  $\delta(G) \ge 2$ . If g = 4, 5 or 6, then either there is an edge e with  $w(e) \le 17 - 2g$  or there is a 2-alternating cycle  $v_0v_1 \cdots v_{2n-1}v_0$  such that  $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$  and  $\max_{0 \le i \le n} |N_2(v_{2i})| \ge 3$ .

Under the same conditions as given in the next theorem, Wu [9] proved that  $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$ .

**Theorem 2.3.** Let G be a planar graph having girth at least g and maximum degree  $\Delta$ . Then  $la(G) = lla(G) = \lceil \frac{\Delta(G)}{2} \rceil$ , provided that one of the following holds:

(1)  $\Delta \ge 13$ , (2)  $\Delta \ge 7$  and  $g \ge 4$ , (3)  $\Delta \ge 5$  and  $g \ge 5$ , (4)  $\Delta \ge 3$  and  $g \ge 6$ .

**Proof.** Since  $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq lla(G)$ , we show (1) by proving somewhat a stronger statement: any planar graph G is linear k-list colorable for  $k = \max\{7, \lceil \frac{\Delta(G)}{2} \rceil\}$ .

We shall prove it by induction on |E(G)|. The result holds trivially if  $|E(G)| \leq 7$ . Next we assume G be a graph with  $|E(G)| \geq 8$ , and let L be a list assignment of G with |L(e)| = k for any  $e \in E(G)$ .

Suppose that G has an edge xy such that  $w(xy) \leq 2k + 1$ . Then by induction hypothesis,  $G^* = G - xy$  has a linear L-coloring  $\varphi$ . Let  $C_{\varphi} = C_{\varphi}^2(x) \cup C_{\varphi}^2(y) \cup (C_{\varphi}^1(x) \cap C_{\varphi}^1(y))$ . Since  $2|C_{\varphi}| \leq d_{G^*}(x) + d_{G^*}(y) = w(xy) - 2 \leq 2k - 1$ ,  $|C_{\varphi}| < k$ . We can extend  $\varphi$  to a linear L-coloring of G by taking  $\varphi(xy) \in L(xy) \setminus C_{\varphi}$ .

Hence, we assume that w(xy) > 2k + 1 for any edge  $xy \in E(G)$ . Since  $k = \max\{7, \lceil \frac{\Delta(G)}{2} \rceil\}$ , we have  $\delta(G) \ge 2$  and  $2k + 1 \ge 15$ . Therefore, for any edge  $xy \in E(G)$ , w(xy) > 15. By Lemma 2.1, G contains a 2-alternating cycle  $C = v_0v_1 \cdots v_{2n-1}v_0$  such that  $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$  and  $\max_{0 \le i < n} |N_2(v_{2i})| \ge 3$ .

Without loss of generality, let  $|N_2(v_0)| \ge 3$ . Let  $u \in N_2(v_0) \setminus \{v_{2n-1}, v_1\}$ and  $v \in N(u) \setminus \{v_0\}$ . By induction hypothesis,  $G^* = G - \{v_1, v_3, \ldots, v_{2n-1}\} - v_0 u$  has a linear *L*-coloring  $\sigma$ . Next, we shall extend  $\sigma$  to a linear *L*-coloring  $\varphi$  of *G* by setting  $\varphi(e) = \sigma(e)$  for each  $e \in E(G^*)$ , and assigning some appropriate colors for the remaining edges as follows. We consider two cases.

Case 1.  $|C_{\sigma}(v_0)| < k$ . Since  $2|C_{\sigma}^2(v_0)| \leq d_{G^*}(v_0) = d(v_0) - 3 \leq \Delta(G) - 3 \leq 2k - 3$ , we have  $|C_{\sigma}^2(v_0)| \leq k - 2$ .

Subcase 1.1.  $|C_{\sigma}(v_{2j})| < k$  for each 2j with  $j \in \{1, 2, \dots, n-1\}$ . We take

$$\begin{split} &\varphi(v_0u) \in L(v_0u) \backslash C_{\sigma}(v_0), \\ &\varphi(v_0v_1) \in L(v_0v_1) \backslash C_{\sigma}(v_0), \\ &\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1}) \backslash (C_{\sigma}^2(v_0) \cup \{\varphi(v_0v_1)\}), \text{ and furthermore} \end{split}$$

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 $\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j}) \setminus C_{\sigma}(v_{2j}) \text{ and } \varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1}) \setminus C_{\sigma}(v_{2j})$ for any  $j \in \{1, 2, \dots, n-1\}.$ 

To check that  $\varphi$  is a linear *L*-coloring of *G*, we need to show that there exists no monochromatic cycle containing at least one edge of  $E(C) \cup \{v_0u\}$  in *G* and  $|C^i_{\varphi}(x)| = 0$  for any vertex  $x \in V(C) \cup \{u\}$  and any  $i \ge 3$ .

First note that if there is a monochromatic cycle C' in G, then C'does not contain any edges of C since  $\varphi(v_0v_{2n-1}) \neq \varphi(v_0v_1), \varphi(v_{2j-1}v_{2j}) \notin C_{\sigma}(v_{2j})$  and  $\varphi(v_{2j}v_{2j+1}) \notin C_{\sigma}(v_{2j})$  for each  $j \in \{1, 2, \ldots, n-1\}$ . Thus C'must contain the edges  $v_0u$  and uv. However, since  $\varphi(v_0u) \notin C_{\sigma}(v_0), C'$ cannot be monochromatic.

Now let  $x \in V(C) \cup \{u\}$  and *i* be an integer at least 3. We show that  $|C_{\varphi}^{i}(x)| = 0$ . Since d(u) = 2 and  $d(v_{2j-1}) = 2$  for each  $j \in \{1, 2, \ldots, n-1\}$ , the result is trivially true when  $x \in \{u, v_1, v_3, \cdots, v_{2n-1}\}$ . Since  $\varphi(v_{2j-1}v_{2j}) \notin C_{\sigma}(v_{2j})$  and  $\varphi(v_{2j}v_{2j+1}) \notin C_{\sigma}(v_{2j})$ , we have  $|C_{\varphi}^{i}(v_{2j})| = 0$  for any  $j \in \{1, 2, \ldots, n-1\}$ . The selection of colors for  $v_0u, v_0v_1$  and  $v_0v_{2n-1}$  ensure that  $|C_{\varphi}^{i}(v_0)| = 0$ .

Subcase 1.2.  $|C_{\sigma}(v_{2j})| \ge k$  for some 2j with  $j \in \{1, 2, \dots, n-1\}$ . We take

 $\varphi(v_0u) \in L(v_0u) \setminus (C^2_{\sigma}(v_0) \cup \{\sigma(uv)\}), \\
\varphi(v_0v_1) \in L(v_0v_1) \setminus (C^2_{\sigma}(v_0) \cup \{\varphi(v_0u)\}), \\
\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1}) \setminus C_{\sigma}(v_0).$ 

For  $j \in \{1, 2, \cdots, n-1\}$ , if  $|C_{\sigma}(v_{2j})| < k$ , we take  $\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j}) \setminus C_{\sigma}(v_{2j})$  and  $\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1}) \setminus C_{\sigma}(v_{2j})$ ; otherwise,

$$\begin{aligned} \varphi(v_{2j-1}v_{2j}) &\in L(v_{2j-1}v_{2j}) \setminus (C^2_{\sigma}(v_{2j}) \cup \{\varphi(v_{2j-2}v_{2j-1})\}) \text{ and } \\ \varphi(v_{2j}v_{2j+1}) &\in L(v_{2j}v_{2j+1}) \setminus (C^2_{\sigma}(v_{2j}) \cup \{\varphi(v_{2j-1}v_{2j})\}). \\ \text{Note that } |C^2_{\sigma}(v_{2j})| &\leq k-2 \text{ since } k + |C^2_{\sigma}(v_{2j})| \leq |C^1_{\sigma}(v_{2j})| + 2|C^2_{\sigma}(v_{2j})| = \\ d(v_{2j}) - 2 &\leq 2k-2. \end{aligned}$$

We can check that  $|C_{\varphi}^{i}(x)| = 0$  for any vertex  $x \in V(C) \cup \{u\}$  and any  $i \geq 3$  by a similar argument as in Subcase 1.1. Now, suppose that there is a monochromatic cycle C' in G. Clearly, C' cannot contain the edge  $v_0 u$  since  $\varphi(v_0 u) \neq \sigma(uv)$ . Thus C' must contain the edges of C. Since there exist some 2j such that  $\varphi(v_{2j-1}v_{2j}) \neq \varphi(v_{2j-2}v_{2j-1}), C' \neq C$ . Then C' must contain the path  $v_{2l}v_{2l+1}v_{2l+2}\cdots v_{2r-1}v_{2r}$  of C since  $\varphi(v_{2l-2}v_{2l-1})$  and  $\varphi(v_0v_{2n-1}) \notin C_{\sigma}(v_0)$ , where  $2 \leq 2l < 2r \leq 2n-2$  and  $\min\{|C_{\sigma}(v_{2l})|, |C_{\sigma}(v_{2r})|\} \geq k$ . But  $\varphi(v_{2r}v_{2r-1}) \neq \varphi(v_{2r-1}v_{2r-2})$  leads to the contradiction that C' is monochromatic. Thus  $\varphi$  is a linear L-coloring of G.

Case 2.  $|C_{\sigma}(v_0)| \ge k$ .

Since  $k + |C_{\sigma}^{2}(v_{0})| \leq |C_{\sigma}^{1}(v_{0})| + 2|C_{\sigma}^{2}(v_{0})| = d(v_{0}) - 3 \leq 2k - 3$ , we have  $|C^2_{\sigma}(v_0)| \leqslant k - 3.$ 

Subcase 2.1.  $L(v_0v_1)\backslash C^2_{\sigma}(v_0) \nsubseteq L(v_0u)\backslash C^2_{\sigma}(v_0).$ We take  $\varphi(v_0v_1) \in L(v_0v_1) \setminus (C^2_{\sigma}(v_0) \cup L(v_0u))$ . Furthermore, for any j = $\{1, 2, \ldots, n-1\}$ , we take

 $\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j}) \setminus C_{\sigma}(v_{2j}) \text{ and } \varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1}) \setminus C_{\sigma}(v_{2j})$ if  $|C_{\sigma}(v_{2j})| < k$ ; otherwise,

 $\begin{array}{l} \varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j}) \backslash (C^2_{\sigma}(v_{2j}) \cup \{\varphi(v_{2j-2}v_{2j-1})\}) \text{ and} \\ \varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1}) \backslash (C^2_{\sigma}(v_{2j}) \cup \{\varphi(v_{2j-1}v_{2j})\}), \text{ and finally} \end{array}$  $\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1}) \setminus (C^2_{\sigma}(v_0) \cup \{\varphi(v_0v_1), \varphi(v_{2n-1}v_{2n-2})\})$  and  $\varphi(v_0u) \in L(v_0u) \setminus (C^2_{\sigma}(v_0) \cup \{\varphi(v_0v_{2n-1}), \sigma(uv)\}).$ 

Subcase 2.2.  $L(v_0v_1)\backslash C^2_{\sigma}(v_0) \subseteq L(v_0u)\backslash C^2_{\sigma}(v_0).$ Since  $|C^2_{\sigma}(v_0)| \leq k-3$ , we have  $|L(v_0u)\backslash C^2_{\sigma}(v_0)| \geq |L(v_0v_1)\backslash C^2_{\sigma}(v_0)| \geq 3.$ We take  $\varphi(v_0v_1) = \sigma(uv)$  if  $\sigma(uv) \in L(v_0v_1) \setminus C^2_{\sigma}(v_0)$ , and  $\varphi(v_0v_1) \in$ 

 $L(v_0v_1)\setminus C^2_{\sigma}(v_0)$ , otherwise. For  $j \in \{1, 2, \dots, n-1\}$ , we assign a color  $v_{2j-1}v_{2j}$  and  $v_{2j}v_{2j+1}$  by the way as described in Subcase 2.1.

And then  $\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1}) \setminus (C^2_{\sigma}(v_0) \cup \{\varphi(v_{2n-1}v_{2n-2}), \varphi(v_0v_1)\}).$ If  $\sigma(uv) \in L(v_0u) \setminus C^2_{\sigma}(v_0)$ , but  $\sigma(uv) \notin L(v_0v_1) \setminus C^2_{\sigma}(v_0)$ , then  $|L(v_0u) \setminus C^2_{\sigma}(v_0)|$  $\geq 4$ . So, we take

 $\varphi(v_0u) \in L(v_0u) \setminus (C^2_{\sigma}(v_0) \cup \{\varphi(v_0v_{2n-1}), \varphi(v_0v_1), \sigma(uv)\}); \text{ otherwise,}$  $\varphi(v_0u) \in L(v_0u) \setminus (C^2_{\sigma}(v_0) \cup \{\varphi(v_0v_{2n-1}), \varphi(v_0v_1)\}).$ 

It is easy to check that  $\varphi$  is a linear L-coloring of G both in Subcase 2.1 and Subcase 2.2 by a similar argument as in Subcase 1.2. So we complete the proof of (1).

By using Lemma 2.2, one can similarly prove (2), (3), and (4).

For a plane graph G, F(G) denotes the set of faces of G. The degree of a face f, denote by d(f), is the number of edges incident with it, where each cut edge is counted twice. A k-face is a face of degree k.

**Theorem 2.4.** Let G be a planar graph with maximum degree  $\Delta \ge 7$  and without *i*-cycle for some  $i \in \{4, 5\}$ . Then  $la(G) = lla(G) = \lceil \frac{\Delta(G)}{2} \rceil$ .

**Proof.** We prove the theorem by contradiction. Let G = (V, E) be a counterexample with the minimum size to the theorem, and be embedded in the plane. Set  $k = \lceil \frac{\Delta(G)}{2} \rceil$ . Then  $k \ge 4$  since  $\Delta \ge 7$ . By a similar argument as in proof of Theorem 2.3, we can obtain the following claims.

Claim 1. For any edge  $xy \in E(G)$ ,  $w(xy) \ge 2k + 2$ .

**Claim 2.** *G* has no even cycle  $v_0v_1 \cdots v_{2n-1}v_0$  such that  $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$  and  $\max_{0 \le i \le n} |N_2(v_{2i})| \ge 3$ .

Let G' be the subgraph induced by edges incident with 2-vertices. Since G does not contain two adjacent 2-vertices by Claim 1, G' does not contain any odd cycle. So it follows from Claim 2 that any component of G' is either an even cycle or a tree. So it is easy to find a matching M in G saturating all 2-vertices. Thus if  $xy \in M$  and d(x) = 2, y is called a 2-master of x. Note that every 2-vertex has a 2-master.

We define a weight function ch on  $V(G) \cup F(G)$  by letting ch(v) = 2d(v) - 6 for each  $v \in V(G)$  and ch(f) = d(f) - 6 for each  $f \in F(G)$ . Applying Euler's formula |V(G)| - |E(G)| + |F(G)| = 2, we have

$$\sum_{x \in V(G) \cup F(G)} ch(x) = \sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

In the following, we will reassign a new weight ch'(x) to each  $x \in V(G) \cup F(G)$  according to some discharging rules. Since we discharge weight from one element to another, the total weight is kept fixed during the discharging. Thus

$$\sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12.$$

We shall show that  $ch'(x) \ge 0$  for each  $x \in V(G) \cup F(G)$ , a contradiction, completing the proof.

If G contains no 4-cycles, then we give the following discharging rules.

R1-1. Each 2-vertex receives 2 from its 2-master.

**R1-2.** Each 3-face f receives  $\frac{3}{2}$  from each of its incident 5<sup>+</sup>-vertex.

**R1-3.** Each 5-face f receives  $\frac{1}{3}$  from each of its incident 5<sup>+</sup>-vertex.

We can obtain that  $ch'(x) \ge 0$  for each  $x \in V(G) \cup F(G)$  by using the same argument in [11]. This complete the proof of the case that G contains no 4-cycles.

Now assume that G contains no 5-cycles. The discharging rules are defined as follows.

**R2-1.** Each 2-vertex receives 2 from its 2-master.

**R2-2.** For a 3-face f and its incident vertex v, f receives  $\frac{1}{2}$  from v if d(v) = 4, 1 if d(v) = 5,  $\frac{5}{4}$  if d(v) = 6 and  $\frac{3}{2}$  if  $d(v) \ge 7$ .

**R2-3.** For a 4-face f and its incident vertex v, f receives  $\frac{1}{2}$  from v if  $4 \leq d(v) \leq 6, 1$  if  $d(v) \geq 7$ .

By the same argument in [11],  $ch'(x) \ge 0$  for each  $x \in V(G) \cup F(G)$ . Hence, the proof was done for the case that G contains no 5-cycles.

## 3. Planar Graphs with $\Delta \ge 9$

**Lemma 3.1** ([5], Lemma 1). Let G be a planar graph with  $\delta(G) \ge 3$ . Then there is either an edge  $e \in E(G)$  with  $w(e) \le 11$  or a 3-alternating 4-cycle.

**Theorem 3.2.** Let G be a planar graph with  $\Delta(G) \ge 9$ . Then  $\lceil \frac{\Delta(G)}{2} \rceil \le la(G) \le lla(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$ .

**Proof.** We prove the theorem by proving somewhat a stronger statement that any planar graph G is linear k-list colorable for  $k = \max\{5, \lceil \frac{\Delta(G)+1}{2} \rceil\}$ .

We shall prove it by induction on |E(G)|. Let L be a list assignment of G with |L(e)| = k for any  $e \in E(G)$ . Clearly, the result is true when  $|E(G)| \leq 5$ . Next we assume  $|E(G)| \geq 6$ .

Suppose that G has an edge xy such that  $w(xy) \leq 2k + 1$ . Then by induction hypothesis, G - xy has a linear L-coloring  $\varphi$ . Let  $C_{\varphi} = C_{\varphi}^2(x) \cup C_{\varphi}^2(y) \cup (C_{\varphi}^1(x) \cap C_{\varphi}^1(y))$ . Since  $2|C_{\varphi}| \leq d_{G-xy}(x) + d_{G-xy}(y) = w(xy) - 2 \leq 2k - 1$ ,  $|C_{\varphi}| < k$ . We can extend  $\varphi$  to a linear L-coloring of G by setting  $\varphi(xy) \in L(xy) \setminus C_{\varphi}$ .

Hence, we assume that w(xy) > 2k + 1 for any edge  $xy \in E(G)$  as follows. Since  $k = \max\{5, \lceil \frac{\Delta(G)+1}{2} \rceil\}$ , we have  $\delta(G) \ge 3$  and  $2k + 1 \ge 11$ . Thus for any edge  $xy \in E(G)$ , w(xy) > 11. By Lemma 3.1, there is a 4-cycle  $v_1v_2v_3v_4v_1$  of G such that  $d(v_1) = d(v_3) = 3$ . Let  $\{u\} = N(v_1) \setminus \{v_2, v_4\}$  and  $\{w\} = N(v_3) \setminus \{v_2, v_4\}$ . Note that u and w might be the same vertex. By induction hypothesis,  $G^* = G - \{v_1, v_3\}$  has a linear L-coloring  $\sigma$ . Next, we shall extend  $\sigma$  to a linear L-coloring  $\varphi$  of G. To do this, set  $\varphi(e) = \sigma(e)$  for each  $e \in E(G^*)$ , and we consider three cases.

Case 1.  $\max\{|C_{\sigma}(v_2)|, |C_{\sigma}(v_4)|\} < k$ . Since  $2|C_{\sigma}^2(v_2)| \leq d_{G^*}(v_2) = d(v_2) - 2 \leq \Delta(G) - 2 \leq 2k - 3$ , we have  $|C_{\sigma}^2(v_2)| \leq k - 2$ , and similarly  $|C_{\sigma}^2(v_4)| \leq k - 2$ . We take  $\varphi(v_1v_2) \in L(v_1v_2) \setminus C_{\sigma}(v_2),$   $\varphi(v_3v_4) \in L(v_3v_4) \setminus C_{\sigma}(v_4),$   $\varphi(v_2v_3) \in L(v_2v_3) \setminus (C_{\sigma}^2(v_2) \cup \{\varphi(v_3v_4)\}) \text{ and }$  $\varphi(v_1v_4) \in L(v_1v_4) \setminus (C_{\sigma}^2(v_4) \cup \{\varphi(v_1v_2)\}).$ 

Subcase 1.1.  $u \neq w$ . If  $|C_{\sigma}(w)| \geq k$  then  $k + |C_{\sigma}^2(w)| \leq |C_{\sigma}^1(w)| + 2|C_{\sigma}^2(w)| = d(w) - 1 \leq 2k - 2$ , and so  $|C_{\sigma}^2(w)| \leq k - 2$ . Then we assign  $v_3w$  a color

- $\varphi(v_3w) \in L(v_3w) \setminus (C^2_{\sigma}(w) \cup \{\varphi(v_2v_3)\})$  if  $|C_{\sigma}(w)| \ge k$ , and  $\varphi(v_3w) \in L(v_3w) \setminus C_{\sigma}(w)$ , otherwise. Finally,
- $\varphi(v_1u) \in L(v_1u) \setminus (C^2_{\sigma}(u) \cup \{\varphi(v_1v_4)\})$  if  $|C_{\sigma}(u)| \ge k$ , and
- $\varphi(v_1u) \in L(v_1u) \setminus (C_{\sigma}(u) \cup \{\varphi(v_1v_4)\}) \text{ if } |C_{\sigma}(u)| \ge r$
- $\varphi(v_1u) \in L(v_1u) \setminus C_{\sigma}(u)$ , otherwise.

To see that  $\varphi$  is a linear *L*-coloring of *G*, we shall check that  $|C_{\varphi}^{i}(x)| = 0$ for any vertex  $x \in \{v_{1}, v_{2}, v_{3}, v_{4}, u, w\}$  and any  $i \ge 3$ , and there exists no monochromatic cycle containing at least one edge of  $\{v_{1}v_{2}, v_{2}v_{3}, v_{3}v_{4}, v_{4}v_{1}, v_{1}u, v_{3}w\}$ .

Since  $d(v_1) = d(v_3) = 3$ ,  $\varphi(v_1v_4) \neq \varphi(v_1v_2)$  and  $\varphi(v_2v_3) \neq \varphi(v_3v_4)$ ,  $|C^i_{\varphi}(x)| = 0$  for  $x \in \{v_1, v_3\}$  and any  $i \ge 3$ .  $|C^i_{\varphi}(v_2)| = 0$  for any  $i \ge 3$  since  $\varphi(v_1v_2) \notin C_{\sigma}(v_2)$  and  $\varphi(v_2v_3) \notin C^2_{\sigma}(v_2)$ . Similarly,  $|C^i_{\varphi}(v_4)| = 0$  for any  $i \ge 3$ . Since  $\varphi(v_1u) \notin C^2_{\sigma}(u)$  and  $\varphi(v_3w) \notin C^2_{\sigma}(w)$ ,  $|C^i_{\varphi}(u)| = |C^i_{\varphi}(w)| = 0$ for any  $i \ge 3$ .

By contradiction, suppose C is a monochromatic cycle in G. Since  $\varphi(v_4v_1) \neq \varphi(v_1v_2)$  and  $\varphi(v_4v_1) \neq \varphi(v_1u)$  or  $\varphi(v_1u) \notin C_{\sigma}(u)$ , C cannot contain the edge  $v_4v_1$ . Similarly, C cannot contain the edge  $v_2v_3$ . Thus C must contain the path  $uv_1v_2$  or the path  $wv_3v_4$ . However, since  $\varphi(v_1v_2) \notin C_{\sigma}(v_2)$  and  $\varphi(v_3v_4) \notin C_{\sigma}(v_4)$ , C cannot be monochromatic.

Subcase 1.2. u = w.

Since  $2|C_{\sigma}^{2}(u)| \leq d(u) - 2 \leq 2k - 3$ , we have  $|C_{\sigma}^{2}(u)| \leq k - 2$ .

Assign  $v_3u$  a color  $\varphi(v_3u) \in L(v_3u) \setminus (C^2_{\sigma}(u) \cup \{\varphi(v_2v_3)\})$ . A choice for a color for  $v_1u$  is somewhat complicated.

If  $\varphi(v_3 u) = \varphi(v_3 v_4) = \varphi(v_1 v_4)$  then  $\varphi(v_1 u) \in L(v_1 u) \setminus (C^2_{\sigma}(u) \cup \{\varphi(v_3 u)\})$ . If it is not,  $\varphi(v_1 u) \in L(v_1 u) \setminus C_{\sigma}(u)$  when  $|C_{\sigma}(u)| < k$ . For the case

 $|C_{\sigma}(u)| \ge k$ , we have  $k + |C_{\sigma}^2(u)| \le |C_{\sigma}^1(u)| + 2|C_{\sigma}^2(u)| = d(u) - 2 \le 2k - 3$ , and thus  $|C_{\sigma}^2(u)| \le k - 3$ . Then assign a color  $\varphi(v_1u) \in L(v_1u) \setminus (C_{\sigma}^2(u) \cup \{\varphi(v_1v_4), \varphi(v_3u)\})$  for  $v_1u$ .

To see  $\varphi$  is a linear *L*-coloring of *G*, we verify that  $|C_{\varphi}^{i}(x)| = 0$  for any vertex  $x \in \{v_1, v_2, v_3, v_4, u\}$  and any  $i \ge 3$ , and and show that there exists no

monochromatic cycle containing at least one edge of  $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1u, v_3u\}$ . We can check that  $|C_{\varphi}^i(x)| = 0$  for any vertex  $x \in \{v_1, v_2, v_3, v_4\}$  and any  $i \ge 3$  by a similar argument as Subcase 1.1. The selection of colors for  $v_1u$  and  $v_3u$  ensure that  $|C_{\varphi}^i(u)| = 0$  for any  $i \ge 3$ . By contradiction, suppose G contains a monochromatic cycle C. One can see that C cannot contain the edge  $v_2v_3$  since  $\varphi(v_2v_3) \neq \varphi(v_3u)$  and  $\varphi(v_2v_3) \neq \varphi(v_3v_4)$ . Clearly,  $C \neq v_1uv_3v_4v_1$  by the choice of the color of  $v_1u$ . Since  $\varphi(v_1v_2) \notin C_{\sigma}(v_2)$  and  $\varphi(v_3v_4) \notin C_{\sigma}(v_4)$ , C cannot contain the edges  $v_1v_2$  and  $v_3v_4$ . Thus C must contain the path  $v_4v_1u$ , but  $\varphi(v_1u) \notin C_{\sigma}(u)$  or  $\varphi(v_1u) \neq \varphi(v_1v_4)$ , C is not monochromatic.

Case 2.  $|C_{\sigma}(v_i)| < k$  and  $|C_{\sigma}(v_j)| \ge k$  for  $\{i, j\} = \{2, 4\}$ .

By the symmetry of the roles of  $v_2$  and  $v_4$ , assume  $|C_{\sigma}(v_2)| < k$  and  $|C_{\sigma}(v_4)| \ge k$ . By the similar argument as in proof of Case 1, we have  $|C^2_{\sigma}(v_2)| \le k-2$  and  $|C^2_{\sigma}(v_4)| \le k-3$ . We take

 $\varphi(v_1v_2) \in L(v_1v_2) \backslash C_{\sigma}(v_2),$ 

 $\varphi(v_2v_3) \in L(v_2v_3) \setminus (C^2_{\sigma}(v_2) \cup \{\varphi(v_1v_2)\}),$ 

 $\varphi(v_3w) \in L(v_3w) \setminus C_{\sigma}(w)$  if  $|C_{\sigma}(w)| < k$ , and  $\varphi(v_3w) \in L(v_3w) \setminus (C_{\sigma}^2(w) \cup \{\varphi(v_2v_3)\})$ , otherwise. Then we successively take

 $\varphi(v_3v_4) \in L(v_3v_4) \setminus (C^2_{\sigma}(v_4) \cup \{\varphi(v_2v_3), \varphi(v_3w)\})$  and

 $\varphi(v_1v_4) \in L(v_1v_4) \setminus (C^2_{\sigma}(v_4) \cup \{\varphi(v_3v_4), \varphi(v_1v_2)\}).$ 

Finally we assign a color for  $v_1u$  as follows. If  $|C_{\sigma}(u)| < k$ ,  $\varphi(v_1u) \in L(v_1u) \setminus C_{\sigma}(u)$ . If  $|C_{\sigma}(u)| \ge k$ ,  $\varphi(v_1u) \in L(v_1u) \setminus (C_{\sigma}^2(u) \cup \{\varphi(v_1v_4)\})$  if  $u \ne w$ ;  $\varphi(v_1u) \in L(v_1u) \setminus (C_{\sigma}^2(u) \cup \{\varphi(v_1v_4), \varphi(v_3w)\})$ , otherwise.

It is easy to check that  $\varphi$  is a linear *L*-coloring of *G* by a similar argument as in proof of Case 1.

Case 3.  $|C_{\sigma}(v_i)| \ge k$  for each  $i \in \{2, 4\}$ . Then  $|C_{\sigma}^2(v_2)| \le k-3$  and  $|C_{\sigma}^2(v_4)| \le k-3$ . We take  $\varphi(v_1u) \in L(v_1u) \setminus C_{\sigma}(u)$  if  $|C_{\sigma}(u)| < k$ , and  $\varphi(v_3w) \in L(v_3w) \setminus C_{\sigma}(w)$  if  $|C_{\sigma}(w)| < k$ . Next we suppose that  $|C_{\sigma}(u)| \ge k$  and  $|C_{\sigma}(w)| \ge k$ .

If  $L(v_1v_2)\backslash C^2_{\sigma}(v_2) \nsubseteq C^1_{\sigma}(v_2) \cap C^1_{\sigma}(v_4)$ , we take  $\varphi(v_1v_2) \in L(v_1v_2)\backslash (C^2_{\sigma}(v_2) \cup (C^1_{\sigma}(v_2) \cap C^1_{\sigma}(v_4)))$ , and then  $\varphi(v_1u) \in L(v_1u)\backslash (C^2_{\sigma}(u) \cup \{\varphi(v_1v_2)\})$ ,  $\varphi(v_3w) \in L(v_3w)\backslash (C^2_{\sigma}(w) \cup \{\varphi(v_1u)\})$ ,  $\varphi(v_2v_3) \in L(v_2v_3)\backslash (C^2_{\sigma}(v_2) \cup \{\varphi(v_1v_2), \varphi(v_3w)\})$ ,  $\varphi(v_3v_4) \in L(v_3v_4)\backslash (C^2_{\sigma}(v_4) \cup \{\varphi(v_2v_3), \varphi(v_3w)\})$ ,  $\varphi(v_1v_4) \in L(v_1v_4)\backslash (C^2_{\sigma}(v_4) \cup \{\varphi(v_3v_4), \varphi(v_1u)\})$ . By the similar argument as in the proof of Case 1, one can show that  $\varphi$  is a linear *L*-coloring of *G*.

By symmetry, we consider that  $L(v_1v_2)\setminus C^2_{\sigma}(v_2)$ ,  $L(v_2v_3)\setminus C^2_{\sigma}(v_2)$ ,  $L(v_3v_4)\setminus C^2_{\sigma}(v_4)$  and  $L(v_4v_1)\setminus C^2_{\sigma}(v_4)$  are all contained in  $C^1_{\sigma}(v_2)\cap C^1_{\sigma}(v_4)$ .

We claim that  $(L(v_1v_2)\backslash C^2_{\sigma}(v_2)) \cap (L(v_3v_4)\backslash C^2_{\sigma}(v_4)) = \emptyset$ . Suppose it is false, and  $|C^2_{\sigma}(v_2)| \ge |C^2_{\sigma}(v_4)|$ , without loss of generality. Therefore,

$$2k - 2|C_{\sigma}^{2}(v_{2})| \leq k - |C_{\sigma}^{2}(v_{2})| + k - |C_{\sigma}^{2}(v_{4})|$$

$$\leq |L(v_{1}v_{2}) \setminus C_{\sigma}^{2}(v_{2})| + |L(v_{3}v_{4}) \setminus C_{\sigma}^{2}(v_{4})|$$

$$\leq |C_{\sigma}^{1}(v_{2}) \cap C_{\sigma}^{1}(v_{4})|$$

$$\leq |C_{\sigma}^{1}(v_{2})|$$

$$\leq d(v_{2}) - 2|C_{\sigma}^{2}(v_{2})|$$

$$\leq 2k - 1 - 2|C_{\sigma}^{2}(v_{2})|.$$

It follows that  $2k \leq 2k - 1$ , a contradiction. Thus we take

$$\begin{split} \varphi(v_1v_2) &= \varphi(v_3v_4) \in (L(v_1v_2) \backslash C^2_{\sigma}(v_2)) \cap (L(v_3v_4) \backslash C^2_{\sigma}(v_4)) \text{ and} \\ \varphi(v_1u) \in L(v_1u) \backslash (C^2_{\sigma}(u) \cup \{\varphi(v_1v_2)\}). \text{ And then} \\ \varphi(v_3w) \in L(v_3w) \backslash (C^2_{\sigma}(w) \cup \{\varphi(v_3v_4)\}) \text{ if } u \neq w; \\ \varphi(v_3w) \in L(v_3w) \backslash (C^2_{\sigma}(w) \cup \{\varphi(v_3v_4), \varphi(v_1u)\}), \text{ otherwise. Finally,} \\ \varphi(v_2v_3) \in L(v_2v_3) \backslash (C^2_{\sigma}(v_2) \cup \{\varphi(v_3v_4), \varphi(v_3w)\}) \text{ and} \\ \varphi(v_1v_4) \in L(v_1v_4) \backslash (C^2_{\sigma}(v_4) \cup \{\varphi(v_3v_4), \varphi(v_1u)\}). \end{split}$$
One can verify that  $\varphi$  is a linear *L*-coloring of *G*.

The proof is complete.

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