# THE LIST LINEAR ARBORICITY OF PLANAR GRAPHS * 

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#### Abstract

The linear arboricity $l a(G)$ of a graph $G$ is the minimum number of linear forests which partition the edges of $G$. An and Wu introduce the notion of list linear arboricity $l l a(G)$ of a graph $G$ and conjecture that $l l a(G)=l a(G)$ for any graph $G$. We confirm that this conjecture is true for any planar graph having $\Delta \geqslant 13$, or for any planar graph with $\Delta \geqslant 7$ and without $i$-cycles for some $i \in\{3,4,5\}$. We also prove that $\left\lceil\frac{\Delta(G)}{2}\right\rceil \leqslant l l a(G) \leqslant\left\lceil\frac{\Delta(G)+1}{2}\right\rceil$ for any planar graph having $\Delta \geqslant 9$.


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## 1. Introduction

All graphs considered here are finite, undirected and simple. We refer to [4] for unexplained terminology and notations. For a real number $x,\lceil x\rceil$ is the least integer not less than $x$. Let $G=(V(G), E(G))$ be a graph. $|V(G)|$ and $|E(G)|$ are called the order and the size of $G$, respectively. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and the minimum degree of $G$, respectively. Let $v$ be a vertex of $G$. The neighborhood of $v$, denoted by $N_{G}(v)$, is the set of vertices adjacent to $v$ in $G$. The degree of $v$, denoted

[^0]by $d_{G}(v)$, is the number of edges incident with $v$ in $G$. Since $G$ is simple, $d_{G}(v)=\left|N_{G}(v)\right|$. If there is no confusion, we use $N(v)$ and $d(v)$ for the neighborhood and degree of $v$ instead of $N_{G}(v)$ and $d_{G}(v)$, respectively. Let $N_{k}(v)=\{u \mid u \in N(v)$ and $d(u)=k\}$. The girth of $G$ is the minimum length of cycles in $G$. A $k$ - or $k^{+}$-vertex is a vertex of degree $k$, or at least $k$.

A linear forest is a graph in which each component is a path. A map $\varphi$ from $E(G)$ to $\{1,2, \ldots, k\}$ is called a $k$-linear coloring if $\left(V(G), \varphi^{-1}(i)\right)$ is a linear forest for $1 \leqslant i \leqslant k$. The linear arboricity la $(G)$ of a graph $G$, introduced by Harary [8], is the minimum number $k$ for which $G$ has a $k$-linear coloring. Akiyama, Exoo and Harary [1] conjectured that $l a(G)=$ $\left\lceil\frac{\Delta(G)+1}{2}\right\rceil$ for any regular graph $G$. It is obvious that for a graph $G, l a(G) \geqslant$ $\left\lceil\frac{\Delta(G)}{2}\right\rceil$ and $l a(G) \geqslant\left\lceil\frac{\Delta(G)+1}{2}\right\rceil$ when $G$ is regular. So it is equivalent to the following conjecture, known as the linear arboricity conjecture.

Linear Arboricity Conjecture. For any graph $G$,

$$
\left\lceil\frac{\Delta(G)}{2}\right\rceil \leqslant l a(G) \leqslant\left\lceil\frac{\Delta(G)+1}{2}\right\rceil .
$$

The linear arboricity has been determined for complete bipartite graphs [1], series-parallel graphs [10], and regular graphs with $\Delta=3$ [1], 4 [2], 5, 6, 8 [6], 10 [7]. The LAC also has already been proved to be true for any planar graphs in [9] and [12]. In particular, the author proved that if $G$ is a planar graph with $\Delta \geqslant 13$, then $l a(G)=\left\lceil\frac{\Delta(G)}{2}\right\rceil$. In [9] and [11], the authors showed that the same also holds for a planar graph with $\Delta \geqslant 7$ and without $i$-cycles for some $i \in\{3,4,5\}$.

A list assignment $L$ to the edges of $G$ is the assignment of a set $L(e) \subseteq$ $N$ of colors to every edge $e$ of $G$, where $N$ is the set of natural numbers. If $G$ has a coloring $\varphi$ such that $\varphi(e) \in L(e)$ for every edge $e$ and $\left(V(G), \varphi^{-1}(i)\right)$ is a linear forest for any $i \in C_{\varphi}$, where $C_{\varphi}=\{\varphi(e) \mid e \in$ $E(G)\}$, then we say that $G$ is linear L-colorable and $\varphi$ is a linear $L$ coloring of $G$. We say that $G$ is linear $k$-list colorable if it is linear $L$ colorable for every list assignment $L$ satisfying $|L(e)|=k$ for all edges $e$. The list linear arboricity lla $(G)$ of a graph $G$ is the minimum number $k$ for which $G$ is linear $k$-list colorable. It is obvious that $l a(G) \leqslant l l a(G)$. In [3], the authors raised the following conjecture, and confirmed that it is true for any series-parallel graph.

List Linear Arboricity Conjecture. For any graph $G$,

$$
\left\lceil\frac{\Delta(G)}{2}\right\rceil \leqslant l a(G)=l l a(G) \leqslant\left\lceil\frac{\Delta(G)+1}{2}\right\rceil .
$$

Little was known for this conjecture. In this paper, we will prove that it is true for any planar graph having $\Delta \geqslant 13$, or for any planar graph with $\Delta \geqslant 7$ and without $i$-cycles for some $i \in\{3,4,5\}$. We also prove that $\left\lceil\frac{\Delta(G)}{2}\right\rceil \leqslant l l a(G) \leqslant\left\lceil\frac{\Delta(G)+1}{2}\right\rceil$ for any planar graph having $\Delta \geqslant 9$.

## 2. Planar Graphs with $l a(G)=l l a(G)$

For convenience, we introduce two definitions. The weight $w(e)$ of an edge $e=u v$ is $d(u)+d(v)$. An even cycle $v_{1} v_{2} \cdots v_{2 t} v_{1}$ is called $k$-alternating if $d\left(v_{1}\right)=d\left(v_{3}\right)=\cdots=d\left(v_{2 t-1}\right)=k$.

Let $L$ be a list assignment of $G$, and $\varphi$ be a coloring of $G$ such that $\varphi(e) \in L(e)$ for any edge $e$ of $G$. For a vertex $v \in V(G)$, we denote by $C_{\varphi}(v)$ the set of colors that appear on the edges incident with $v$ in $G$.

$$
C_{\varphi}^{i}(v)=\{j \mid \text { the color } j \text { appears } i \text { times at edges incident with } v\},
$$

for any positive integer $i$. Observe that $\varphi$ is a linear $L$-coloring of $G$ if and only if $G$ does not contain a monochromatic cycle under coloring $\varphi$ and $\left|C_{\varphi}^{i}(v)\right|=0$ for every vertex $v$ of $G$ and any $i \geqslant 3$. Thus, if $\varphi$ is a linear $L$-coloring of $G$ then $C_{\varphi}(v)=C_{\varphi}^{1}(v) \cup C_{\varphi}^{2}(v)$.
The following two lemmas can be found in [9].
Lemma 2.1. Let $G$ be a planar graph with $\delta(G) \geqslant 2$. Then either there is an edge $e$ with $w(e) \leqslant 15$ or there is a 2 -alternating cycle $v_{0} v_{1} \cdots v_{2 n-1} v_{0}$ such that $d\left(v_{1}\right)=d\left(v_{3}\right)=\cdots=d\left(v_{2 n-1}\right)=2$ and $\max _{0 \leqslant i<n}\left|N_{2}\left(v_{2 i}\right)\right| \geqslant 3$.

Lemma 2.2. Let $G$ be a planar graph with girth at least $g$ and maximum degree $\Delta$, and assume that $\delta(G) \geqslant 2$. If $g=4,5$ or 6 , then either there is an edge $e$ with $w(e) \leqslant 17-2 g$ or there is a 2-alternating cycle $v_{0} v_{1} \cdots v_{2 n-1} v_{0}$ such that $d\left(v_{1}\right)=d\left(v_{3}\right)=\cdots=d\left(v_{2 n-1}\right)=2$ and $\max _{0 \leqslant i<n}\left|N_{2}\left(v_{2 i}\right)\right| \geqslant 3$.

Under the same conditions as given in the next theorem, Wu [9] proved that $l a(G)=\left\lceil\frac{\Delta(G)}{2}\right\rceil$.

Theorem 2.3. Let $G$ be a planar graph having girth at least $g$ and maximum degree $\Delta$. Then la $(G)=$ lla $(G)=\left\lceil\frac{\Delta(G)}{2}\right\rceil$, provided that one of the following holds:
(1) $\Delta \geqslant 13$,
(2) $\Delta \geqslant 7$ and $g \geqslant 4$,
(3) $\Delta \geqslant 5$ and $g \geqslant 5$,
(4) $\Delta \geqslant 3$ and $g \geqslant 6$.

Proof. Since $\left\lceil\frac{\Delta(G)}{2}\right\rceil \leqslant l a(G) \leqslant l l a(G)$, we show (1) by proving somewhat a stronger statement: any planar graph $G$ is linear $k$-list colorable for $k=$ $\max \left\{7,\left\lceil\frac{\Delta(G)}{2}\right\rceil\right\}$.

We shall prove it by induction on $|E(G)|$. The result holds trivially if $|E(G)| \leqslant 7$. Next we assume $G$ be a graph with $|E(G)| \geqslant 8$, and let $L$ be a list assignment of $G$ with $|L(e)|=k$ for any $e \in E(G)$.

Suppose that $G$ has an edge $x y$ such that $w(x y) \leqslant 2 k+1$. Then by induction hypothesis, $G^{*}=G-x y$ has a linear $L$-coloring $\varphi$. Let $C_{\varphi}=$ $C_{\varphi}^{2}(x) \cup C_{\varphi}^{2}(y) \cup\left(C_{\varphi}^{1}(x) \cap C_{\varphi}^{1}(y)\right)$. Since $2\left|C_{\varphi}\right| \leqslant d_{G^{*}}(x)+d_{G^{*}}(y)=w(x y)-2 \leqslant$ $2 k-1,\left|C_{\varphi}\right|<k$. We can extend $\varphi$ to a linear $L$-coloring of $G$ by taking $\varphi(x y) \in L(x y) \backslash C_{\varphi}$.

Hence, we assume that $w(x y)>2 k+1$ for any edge $x y \in E(G)$. Since $k=\max \left\{7,\left\lceil\frac{\Delta(G)}{2}\right\rceil\right\}$, we have $\delta(G) \geqslant 2$ and $2 k+1 \geqslant 15$. Therefore, for any edge $x y \in E(G), w(x y)>15$. By Lemma 2.1, $G$ contains a 2 -alternating cycle $C=v_{0} v_{1} \cdots v_{2 n-1} v_{0}$ such that $d\left(v_{1}\right)=d\left(v_{3}\right)=\cdots=d\left(v_{2 n-1}\right)=2$ and $\max _{0 \leqslant i<n}\left|N_{2}\left(v_{2 i}\right)\right| \geqslant 3$.

Without loss of generality, let $\left|N_{2}\left(v_{0}\right)\right| \geqslant 3$. Let $u \in N_{2}\left(v_{0}\right) \backslash\left\{v_{2 n-1}, v_{1}\right\}$ and $v \in N(u) \backslash\left\{v_{0}\right\}$. By induction hypothesis, $G^{*}=G-\left\{v_{1}, v_{3}, \ldots, v_{2 n-1}\right\}-$ $v_{0} u$ has a linear $L$-coloring $\sigma$. Next, we shall extend $\sigma$ to a linear $L$-coloring $\varphi$ of $G$ by setting $\varphi(e)=\sigma(e)$ for each $e \in E\left(G^{*}\right)$, and assigning some appropriate colors for the remaining edges as follows. We consider two cases.

Case 1. $\left|C_{\sigma}\left(v_{0}\right)\right|<k$.
Since $2\left|C_{\sigma}^{2}\left(v_{0}\right)\right| \leqslant d_{G^{*}}\left(v_{0}\right)=d\left(v_{0}\right)-3 \leqslant \Delta(G)-3 \leqslant 2 k-3$, we have $\left|C_{\sigma}^{2}\left(v_{0}\right)\right| \leqslant k-2$.

Subcase 1.1. $\left|C_{\sigma}\left(v_{2 j}\right)\right|<k$ for each $2 j$ with $j \in\{1,2, \ldots, n-1\}$.
We take

$$
\begin{aligned}
& \varphi\left(v_{0} u\right) \in L\left(v_{0} u\right) \backslash C_{\sigma}\left(v_{0}\right) \\
& \varphi\left(v_{0} v_{1}\right) \in L\left(v_{0} v_{1}\right) \backslash C_{\sigma}\left(v_{0}\right), \\
& \varphi\left(v_{0} v_{2 n-1}\right) \in L\left(v_{0} v_{2 n-1}\right) \backslash\left(C_{\sigma}^{2}\left(v_{0}\right) \cup\left\{\varphi\left(v_{0} v_{1}\right)\right\}\right), \text { and furthermore }
\end{aligned}
$$

$\varphi\left(v_{2 j-1} v_{2 j}\right) \in L\left(v_{2 j-1} v_{2 j}\right) \backslash C_{\sigma}\left(v_{2 j}\right)$ and $\varphi\left(v_{2 j} v_{2 j+1}\right) \in L\left(v_{2 j} v_{2 j+1}\right) \backslash C_{\sigma}\left(v_{2 j}\right)$ for any $j \in\{1,2, \ldots, n-1\}$.

To check that $\varphi$ is a linear $L$-coloring of $G$, we need to show that there exists no monochromatic cycle containing at least one edge of $E(C) \cup\left\{v_{0} u\right\}$ in $G$ and $\left|C_{\varphi}^{i}(x)\right|=0$ for any vertex $x \in V(C) \cup\{u\}$ and any $i \geqslant 3$.

First note that if there is a monochromatic cycle $C^{\prime}$ in $G$, then $C^{\prime}$ does not contain any edges of $C$ since $\varphi\left(v_{0} v_{2 n-1}\right) \neq \varphi\left(v_{0} v_{1}\right), \varphi\left(v_{2 j-1} v_{2 j}\right) \notin$ $C_{\sigma}\left(v_{2 j}\right)$ and $\varphi\left(v_{2 j} v_{2 j+1}\right) \notin C_{\sigma}\left(v_{2 j}\right)$ for each $j \in\{1,2, \ldots, n-1\}$. Thus $C^{\prime}$ must contain the edges $v_{0} u$ and $u v$. However, since $\varphi\left(v_{0} u\right) \notin C_{\sigma}\left(v_{0}\right), C^{\prime}$ cannot be monochromatic.

Now let $x \in V(C) \cup\{u\}$ and $i$ be an integer at least 3. We show that $\left|C_{\varphi}^{i}(x)\right|=0$. Since $d(u)=2$ and $d\left(v_{2 j-1}\right)=2$ for each $j \in\{1,2, \ldots, n-1\}$, the result is trivially true when $x \in\left\{u, v_{1}, v_{3}, \cdots v_{2 n-1}\right\}$. Since $\varphi\left(v_{2 j-1} v_{2 j}\right) \notin$ $C_{\sigma}\left(v_{2 j}\right)$ and $\varphi\left(v_{2 j} v_{2 j+1}\right) \notin C_{\sigma}\left(v_{2 j}\right)$, we have $\left|C_{\varphi}^{i}\left(v_{2 j}\right)\right|=0$ for any $j \in$ $\{1,2, \ldots, n-1\}$. The selection of colors for $v_{0} u, v_{0} v_{1}$ and $v_{0} v_{2 n-1}$ ensure that $\left|C_{\varphi}^{i}\left(v_{0}\right)\right|=0$.

Subcase 1.2. $\left|C_{\sigma}\left(v_{2 j}\right)\right| \geqslant k$ for some $2 j$ with $j \in\{1,2, \ldots, n-1\}$. We take

$$
\begin{aligned}
& \varphi\left(v_{0} u\right) \in L\left(v_{0} u\right) \backslash\left(C_{\sigma}^{2}\left(v_{0}\right) \cup\{\sigma(u v)\}\right), \\
& \varphi\left(v_{0} v_{1}\right) \in L\left(v_{0} v_{1}\right) \backslash\left(C_{\sigma}^{2}\left(v_{0}\right) \cup\left\{\varphi\left(v_{0} u\right)\right\}\right), \\
& \varphi\left(v_{0} v_{2 n-1}\right) \in L\left(v_{0} v_{2 n-1}\right) \backslash C_{\sigma}\left(v_{0}\right) .
\end{aligned}
$$

For $j \in\{1,2, \cdots, n-1\}$, if $\left|C_{\sigma}\left(v_{2 j}\right)\right|<k$, we take $\varphi\left(v_{2 j-1} v_{2 j}\right) \in L\left(v_{2 j-1} v_{2 j}\right) \backslash C_{\sigma}\left(v_{2 j}\right)$ and $\varphi\left(v_{2 j} v_{2 j+1}\right) \in L\left(v_{2 j} v_{2 j+1}\right) \backslash C_{\sigma}\left(v_{2 j}\right)$; otherwise,
$\varphi\left(v_{2 j-1} v_{2 j}\right) \in L\left(v_{2 j-1} v_{2 j}\right) \backslash\left(C_{\sigma}^{2}\left(v_{2 j}\right) \cup\left\{\varphi\left(v_{2 j-2} v_{2 j-1}\right)\right\}\right)$ and
$\varphi\left(v_{2 j} v_{2 j+1}\right) \in L\left(v_{2 j} v_{2 j+1}\right) \backslash\left(C_{\sigma}^{2}\left(v_{2 j}\right) \cup\left\{\varphi\left(v_{2 j-1} v_{2 j}\right)\right\}\right)$.
Note that $\left|C_{\sigma}^{2}\left(v_{2 j}\right)\right| \leqslant k-2$ since $k+\left|C_{\sigma}^{2}\left(v_{2 j}\right)\right| \leqslant\left|C_{\sigma}^{1}\left(v_{2 j}\right)\right|+2\left|C_{\sigma}^{2}\left(v_{2 j}\right)\right|=$ $d\left(v_{2 j}\right)-2 \leqslant 2 k-2$.

We can check that $\left|C_{\varphi}^{i}(x)\right|=0$ for any vertex $x \in V(C) \cup\{u\}$ and any $i \geqslant 3$ by a similar argument as in Subcase 1.1. Now, suppose that there is a monochromatic cycle $C^{\prime}$ in $G$. Clearly, $C^{\prime}$ cannot contain the edge $v_{0} u$ since $\varphi\left(v_{0} u\right) \neq \sigma(u v)$. Thus $C^{\prime}$ must contain the edges of $C$. Since there exist some $2 j$ such that $\varphi\left(v_{2 j-1} v_{2 j}\right) \neq \varphi\left(v_{2 j-2} v_{2 j-1}\right), C^{\prime} \neq C$. Then $C^{\prime}$ must contain the path $v_{2 l} v_{2 l+1} v_{2 l+2} \cdots v_{2 r-1} v_{2 r}$ of $C$ since $\varphi\left(v_{2 l-1} v_{2 l}\right) \neq$ $\varphi\left(v_{2 l-2} v_{2 l-1}\right)$ and $\varphi\left(v_{0} v_{2 n-1}\right) \notin C_{\sigma}\left(v_{0}\right)$, where $2 \leqslant 2 l<2 r \leqslant 2 n-2$ and $\min \left\{\left|C_{\sigma}\left(v_{2 l}\right)\right|,\left|C_{\sigma}\left(v_{2 r}\right)\right|\right\} \geqslant k$. But $\varphi\left(v_{2 r} v_{2 r-1}\right) \neq \varphi\left(v_{2 r-1} v_{2 r-2}\right)$ leads to the contradiction that $C^{\prime}$ is monochromatic. Thus $\varphi$ is a linear $L$-coloring of $G$.

Case 2. $\left|C_{\sigma}\left(v_{0}\right)\right| \geqslant k$.
Since $k+\left|C_{\sigma}^{2}\left(v_{0}\right)\right| \leqslant\left|C_{\sigma}^{1}\left(v_{0}\right)\right|+2\left|C_{\sigma}^{2}\left(v_{0}\right)\right|=d\left(v_{0}\right)-3 \leqslant 2 k-3$, we have $\left|C_{\sigma}^{2}\left(v_{0}\right)\right| \leqslant k-3$.

Subcase 2.1. $L\left(v_{0} v_{1}\right) \backslash C_{\sigma}^{2}\left(v_{0}\right) \nsubseteq L\left(v_{0} u\right) \backslash C_{\sigma}^{2}\left(v_{0}\right)$.
We take $\varphi\left(v_{0} v_{1}\right) \in L\left(v_{0} v_{1}\right) \backslash\left(C_{\sigma}^{2}\left(v_{0}\right) \cup L\left(v_{0} u\right)\right)$. Furthermore, for any $j=$ $\{1,2, \ldots, n-1\}$, we take
$\varphi\left(v_{2 j-1} v_{2 j}\right) \in L\left(v_{2 j-1} v_{2 j}\right) \backslash C_{\sigma}\left(v_{2 j}\right)$ and $\varphi\left(v_{2 j} v_{2 j+1}\right) \in L\left(v_{2 j} v_{2 j+1}\right) \backslash C_{\sigma}\left(v_{2 j}\right)$ if $\left|C_{\sigma}\left(v_{2 j}\right)\right|<k$; otherwise,
$\varphi\left(v_{2 j-1} v_{2 j}\right) \in L\left(v_{2 j-1} v_{2 j}\right) \backslash\left(C_{\sigma}^{2}\left(v_{2 j}\right) \cup\left\{\varphi\left(v_{2 j-2} v_{2 j-1}\right)\right\}\right)$ and
$\varphi\left(v_{2 j} v_{2 j+1}\right) \in L\left(v_{2 j} v_{2 j+1}\right) \backslash\left(C_{\sigma}^{2}\left(v_{2 j}\right) \cup\left\{\varphi\left(v_{2 j-1} v_{2 j}\right)\right\}\right)$, and finally
$\varphi\left(v_{0} v_{2 n-1}\right) \in L\left(v_{0} v_{2 n-1}\right) \backslash\left(C_{\sigma}^{2}\left(v_{0}\right) \cup\left\{\varphi\left(v_{0} v_{1}\right), \varphi\left(v_{2 n-1} v_{2 n-2}\right)\right\}\right)$ and
$\varphi\left(v_{0} u\right) \in L\left(v_{0} u\right) \backslash\left(C_{\sigma}^{2}\left(v_{0}\right) \cup\left\{\varphi\left(v_{0} v_{2 n-1}\right), \sigma(u v)\right\}\right)$.
Subcase 2.2. $L\left(v_{0} v_{1}\right) \backslash C_{\sigma}^{2}\left(v_{0}\right) \subseteq L\left(v_{0} u\right) \backslash C_{\sigma}^{2}\left(v_{0}\right)$.
Since $\left|C_{\sigma}^{2}\left(v_{0}\right)\right| \leqslant k-3$, we have $\left|L\left(v_{0} u\right) \backslash C_{\sigma}^{2}\left(v_{0}\right)\right| \geqslant\left|L\left(v_{0} v_{1}\right) \backslash C_{\sigma}^{2}\left(v_{0}\right)\right| \geqslant 3$.
We take $\varphi\left(v_{0} v_{1}\right)=\sigma(u v)$ if $\sigma(u v) \in L\left(v_{0} v_{1}\right) \backslash C_{\sigma}^{2}\left(v_{0}\right)$, and $\varphi\left(v_{0} v_{1}\right) \in$ $L\left(v_{0} v_{1}\right) \backslash C_{\sigma}^{2}\left(v_{0}\right)$, otherwise. For $j \in\{1,2, \ldots, n-1\}$, we assign a color $v_{2 j-1} v_{2 j}$ and $v_{2 j} v_{2 j+1}$ by the way as described in Subcase 2.1.

And then $\varphi\left(v_{0} v_{2 n-1}\right) \in L\left(v_{0} v_{2 n-1}\right) \backslash\left(C_{\sigma}^{2}\left(v_{0}\right) \cup\left\{\varphi\left(v_{2 n-1} v_{2 n-2}\right), \varphi\left(v_{0} v_{1}\right)\right\}\right)$. If $\sigma(u v) \in L\left(v_{0} u\right) \backslash C_{\sigma}^{2}\left(v_{0}\right)$, but $\sigma(u v) \notin L\left(v_{0} v_{1}\right) \backslash C_{\sigma}^{2}\left(v_{0}\right)$, then $\left|L\left(v_{0} u\right) \backslash C_{\sigma}^{2}\left(v_{0}\right)\right|$ $\geqslant 4$. So, we take
$\varphi\left(v_{0} u\right) \in L\left(v_{0} u\right) \backslash\left(C_{\sigma}^{2}\left(v_{0}\right) \cup\left\{\varphi\left(v_{0} v_{2 n-1}\right), \varphi\left(v_{0} v_{1}\right), \sigma(u v)\right\}\right)$; otherwise, $\varphi\left(v_{0} u\right) \in L\left(v_{0} u\right) \backslash\left(C_{\sigma}^{2}\left(v_{0}\right) \cup\left\{\varphi\left(v_{0} v_{2 n-1}\right), \varphi\left(v_{0} v_{1}\right)\right\}\right)$.
It is easy to check that $\varphi$ is a linear $L$-coloring of $G$ both in Subcase 2.1 and Subcase 2.2 by a similar argument as in Subcase 1.2. So we complete the proof of (1).

By using Lemma 2.2, one can similarly prove (2), (3), and (4).
For a plane graph $G, F(G)$ denotes the set of faces of $G$. The degree of a face $f$, denote by $d(f)$, is the number of edges incident with it, where each cut edge is counted twice. A $k$-face is a face of degree $k$.

Theorem 2.4. Let $G$ be a planar graph with maximum degree $\Delta \geqslant 7$ and without $i$-cycle for some $i \in\{4,5\}$. Then la $(G)=l l a(G)=\left\lceil\frac{\Delta(G)}{2}\right\rceil$.

Proof. We prove the theorem by contradiction. Let $G=(V, E)$ be a counterexample with the minimum size to the theorem, and be embedded in the plane. Set $k=\left\lceil\frac{\Delta(G)}{2}\right\rceil$. Then $k \geqslant 4$ since $\Delta \geqslant 7$. By a similar argument as in proof of Theorem 2.3, we can obtain the following claims.

Claim 1. For any edge $x y \in E(G), w(x y) \geqslant 2 k+2$.
Claim 2. $G$ has no even cycle $v_{0} v_{1} \cdots v_{2 n-1} v_{0}$ such that $d\left(v_{1}\right)=d\left(v_{3}\right)=$ $\cdots=d\left(v_{2 n-1}\right)=2$ and $\max _{0 \leqslant i<n}\left|N_{2}\left(v_{2 i}\right)\right| \geqslant 3$.

Let $G^{\prime}$ be the subgraph induced by edges incident with 2 -vertices. Since $G$ does not contain two adjacent 2 -vertices by Claim $1, G^{\prime}$ does not contain any odd cycle. So it follows from Claim 2 that any component of $G^{\prime}$ is either an even cycle or a tree. So it is easy to find a matching $M$ in $G$ saturating all 2-vertices. Thus if $x y \in M$ and $d(x)=2, y$ is called a 2-master of $x$. Note that every 2 -vertex has a 2 -master.

We define a weight function $c h$ on $V(G) \cup F(G)$ by letting $\operatorname{ch}(v)=$ $2 d(v)-6$ for each $v \in V(G)$ and $\operatorname{ch}(f)=d(f)-6$ for each $f \in F(G)$. Applying Euler's formula $|V(G)|-|E(G)|+|F(G)|=2$, we have

$$
\sum_{x \in V(G) \cup F(G)} c h(x)=\sum_{v \in V(G)}(2 d(v)-6)+\sum_{f \in F(G)}(d(f)-6)=-12
$$

In the following, we will reassign a new weight $c h^{\prime}(x)$ to each $x \in V(G) \cup$ $F(G)$ according to some discharging rules. Since we discharge weight from one element to another, the total weight is kept fixed during the discharging. Thus

$$
\sum_{x \in V(G) \cup F(G)} c h^{\prime}(x)=\sum_{x \in V(G) \cup F(G)} c h(x)=-12 .
$$

We shall show that $\operatorname{ch}^{\prime}(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$, a contradiction, completing the proof.

If $G$ contains no 4 -cycles, then we give the following discharging rules.
R1-1. Each 2-vertex receives 2 from its 2-master.
R1-2. Each 3 -face $f$ receives $\frac{3}{2}$ from each of its incident $5^{+}$-vertex.
R1-3. Each 5-face $f$ receives $\frac{1}{3}$ from each of its incident $5^{+}$-vertex.
We can obtain that $c h^{\prime}(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$ by using the same argument in [11]. This complete the proof of the case that $G$ contains no 4-cycles.

Now assume that $G$ contains no 5 -cycles. The discharging rules are defined as follows.

R2-1. Each 2-vertex receives 2 from its 2-master.

R2-2. For a 3-face $f$ and its incident vertex $v, f$ receives $\frac{1}{2}$ from $v$ if $d(v)=4,1$ if $d(v)=5, \frac{5}{4}$ if $d(v)=6$ and $\frac{3}{2}$ if $d(v) \geqslant 7$.
$\mathbf{R 2 - 3}$. For a 4 -face $f$ and its incident vertex $v, f$ receives $\frac{1}{2}$ from $v$ if $4 \leqslant d(v) \leqslant 6,1$ if $d(v) \geqslant 7$.
By the same argument in [11], $c h^{\prime}(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$. Hence, the proof was done for the case that $G$ contains no 5 -cycles.

## 3. Planar Graphs with $\Delta \geqslant 9$

Lemma 3.1 ([5], Lemma 1). Let $G$ be a planar graph with $\delta(G) \geqslant 3$. Then there is either an edge $e \in E(G)$ with $w(e) \leqslant 11$ or a 3-alternating 4-cycle.

Theorem 3.2. Let $G$ be a planar graph with $\Delta(G) \geqslant 9$. Then $\left\lceil\frac{\Delta(G)}{2}\right\rceil \leqslant$ $l a(G) \leqslant l l a(G) \leqslant\left\lceil\frac{\Delta(G)+1}{2}\right\rceil$.

Proof. We prove the theorem by proving somewhat a stronger statement that any planar graph $G$ is linear $k$-list colorable for $k=\max \left\{5,\left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$.

We shall prove it by induction on $|E(G)|$. Let $L$ be a list assignment of $G$ with $|L(e)|=k$ for any $e \in E(G)$. Clearly, the result is true when $|E(G)| \leqslant 5$. Next we assume $|E(G)| \geqslant 6$.

Suppose that $G$ has an edge $x y$ such that $w(x y) \leqslant 2 k+1$. Then by induction hypothesis, $G-x y$ has a linear $L$-coloring $\varphi$. Let $C_{\varphi}=C_{\varphi}^{2}(x) \cup$ $C_{\varphi}^{2}(y) \cup\left(C_{\varphi}^{1}(x) \cap C_{\varphi}^{1}(y)\right)$. Since $2\left|C_{\varphi}\right| \leqslant d_{G-x y}(x)+d_{G-x y}(y)=w(x y)-2 \leqslant$ $2 k-1,\left|C_{\varphi}\right|<k$. We can extend $\varphi$ to a linear $L$-coloring of $G$ by setting $\varphi(x y) \in L(x y) \backslash C_{\varphi}$.

Hence, we assume that $w(x y)>2 k+1$ for any edge $x y \in E(G)$ as follows. Since $k=\max \left\{5,\left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$, we have $\delta(G) \geqslant 3$ and $2 k+1 \geqslant 11$. Thus for any edge $x y \in E(G), w(x y)>11$. By Lemma 3.1, there is a 4-cycle $v_{1} v_{2} v_{3} v_{4} v_{1}$ of $G$ such that $d\left(v_{1}\right)=d\left(v_{3}\right)=3$. Let $\{u\}=N\left(v_{1}\right) \backslash\left\{v_{2}, v_{4}\right\}$ and $\{w\}=N\left(v_{3}\right) \backslash\left\{v_{2}, v_{4}\right\}$. Note that $u$ and $w$ might be the same vertex. By induction hypothesis, $G^{*}=G-\left\{v_{1}, v_{3}\right\}$ has a linear $L$-coloring $\sigma$. Next, we shall extend $\sigma$ to a linear $L$-coloring $\varphi$ of $G$. To do this, set $\varphi(e)=\sigma(e)$ for each $e \in E\left(G^{*}\right)$, and we consider three cases.

Case 1. $\max \left\{\left|C_{\sigma}\left(v_{2}\right)\right|,\left|C_{\sigma}\left(v_{4}\right)\right|\right\}<k$.
Since $2\left|C_{\sigma}^{2}\left(v_{2}\right)\right| \leqslant d_{G^{*}}\left(v_{2}\right)=d\left(v_{2}\right)-2 \leqslant \Delta(G)-2 \leqslant 2 k-3$, we have $\left|C_{\sigma}^{2}\left(v_{2}\right)\right| \leqslant k-2$, and similarly $\left|C_{\sigma}^{2}\left(v_{4}\right)\right| \leqslant k-2$. We take

$$
\begin{aligned}
& \varphi\left(v_{1} v_{2}\right) \in L\left(v_{1} v_{2}\right) \backslash C_{\sigma}\left(v_{2}\right) \\
& \varphi\left(v_{3} v_{4}\right) \in L\left(v_{3} v_{4}\right) \backslash C_{\sigma}\left(v_{4}\right) \\
& \varphi\left(v_{2} v_{3}\right) \in L\left(v_{2} v_{3}\right) \backslash\left(C_{\sigma}^{2}\left(v_{2}\right) \cup\left\{\varphi\left(v_{3} v_{4}\right)\right\}\right) \text { and } \\
& \varphi\left(v_{1} v_{4}\right) \in L\left(v_{1} v_{4}\right) \backslash\left(C_{\sigma}^{2}\left(v_{4}\right) \cup\left\{\varphi\left(v_{1} v_{2}\right)\right\}\right)
\end{aligned}
$$

Subcase 1.1. $u \neq w$.
If $\left|C_{\sigma}(w)\right| \geqslant k$ then $k+\left|C_{\sigma}^{2}(w)\right| \leqslant\left|C_{\sigma}^{1}(w)\right|+2\left|C_{\sigma}^{2}(w)\right|=d(w)-1 \leqslant 2 k-2$, and so $\left|C_{\sigma}^{2}(w)\right| \leqslant k-2$. Then we assign $v_{3} w$ a color
$\varphi\left(v_{3} w\right) \in L\left(v_{3} w\right) \backslash\left(C_{\sigma}^{2}(w) \cup\left\{\varphi\left(v_{2} v_{3}\right)\right\}\right)$ if $\left|C_{\sigma}(w)\right| \geqslant k$, and
$\varphi\left(v_{3} w\right) \in L\left(v_{3} w\right) \backslash C_{\sigma}(w)$, otherwise. Finally,
$\varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash\left(C_{\sigma}^{2}(u) \cup\left\{\varphi\left(v_{1} v_{4}\right)\right\}\right)$ if $\left|C_{\sigma}(u)\right| \geqslant k$, and
$\varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash C_{\sigma}(u)$, otherwise.
To see that $\varphi$ is a linear $L$-coloring of $G$, we shall check that $\left|C_{\varphi}^{i}(x)\right|=0$ for any vertex $x \in\left\{v_{1}, v_{2}, v_{3}, v_{4}, u, w\right\}$ and any $i \geqslant 3$, and there exists no monochromatic cycle containing at least one edge of $\left\{v_{1} v_{2}, v_{2} v_{3}, v_{3} v_{4}\right.$, $\left.v_{4} v_{1}, v_{1} u, v_{3} w\right\}$.

Since $d\left(v_{1}\right)=d\left(v_{3}\right)=3, \varphi\left(v_{1} v_{4}\right) \neq \varphi\left(v_{1} v_{2}\right)$ and $\varphi\left(v_{2} v_{3}\right) \neq \varphi\left(v_{3} v_{4}\right)$, $\left|C_{\varphi}^{i}(x)\right|=0$ for $x \in\left\{v_{1}, v_{3}\right\}$ and any $i \geqslant 3$. $\left|C_{\varphi}^{i}\left(v_{2}\right)\right|=0$ for any $i \geqslant 3$ since $\varphi\left(v_{1} v_{2}\right) \notin C_{\sigma}\left(v_{2}\right)$ and $\varphi\left(v_{2} v_{3}\right) \notin C_{\sigma}^{2}\left(v_{2}\right)$. Similarly, $\left|C_{\varphi}^{i}\left(v_{4}\right)\right|=0$ for any $i \geqslant 3$. Since $\varphi\left(v_{1} u\right) \notin C_{\sigma}^{2}(u)$ and $\varphi\left(v_{3} w\right) \notin C_{\sigma}^{2}(w),\left|C_{\varphi}^{i}(u)\right|=\left|C_{\varphi}^{i}(w)\right|=0$ for any $i \geqslant 3$.

By contradiction, suppose $C$ is a monochromatic cycle in $G$. Since $\varphi\left(v_{4} v_{1}\right) \neq \varphi\left(v_{1} v_{2}\right)$ and $\varphi\left(v_{4} v_{1}\right) \neq \varphi\left(v_{1} u\right)$ or $\varphi\left(v_{1} u\right) \notin C_{\sigma}(u), C$ cannot contain the edge $v_{4} v_{1}$. Similarly, $C$ cannot contain the edge $v_{2} v_{3}$. Thus $C$ must contain the path $u v_{1} v_{2}$ or the path $w v_{3} v_{4}$. However, since $\varphi\left(v_{1} v_{2}\right) \notin$ $C_{\sigma}\left(v_{2}\right)$ and $\varphi\left(v_{3} v_{4}\right) \notin C_{\sigma}\left(v_{4}\right), C$ cannot be monochromatic.

Subcase 1.2. $u=w$.
Since $2\left|C_{\sigma}^{2}(u)\right| \leqslant d(u)-2 \leqslant 2 k-3$, we have $\left|C_{\sigma}^{2}(u)\right| \leqslant k-2$.
Assign $v_{3} u$ a color $\varphi\left(v_{3} u\right) \in L\left(v_{3} u\right) \backslash\left(C_{\sigma}^{2}(u) \cup\left\{\varphi\left(v_{2} v_{3}\right)\right\}\right)$. A choice for a color for $v_{1} u$ is somewhat complicated.

If $\varphi\left(v_{3} u\right)=\varphi\left(v_{3} v_{4}\right)=\varphi\left(v_{1} v_{4}\right)$ then $\varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash\left(C_{\sigma}^{2}(u) \cup\left\{\varphi\left(v_{3} u\right)\right\}\right)$.
If it is not, $\varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash C_{\sigma}(u)$ when $\left|C_{\sigma}(u)\right|<k$. For the case $\left|C_{\sigma}(u)\right| \geqslant k$, we have $k+\left|C_{\sigma}^{2}(u)\right| \leqslant\left|C_{\sigma}^{1}(u)\right|+2\left|C_{\sigma}^{2}(u)\right|=d(u)-2 \leqslant 2 k-3$, and thus $\left|C_{\sigma}^{2}(u)\right| \leqslant k-3$. Then assign a color $\varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash\left(C_{\sigma}^{2}(u) \cup\right.$ $\left.\left\{\varphi\left(v_{1} v_{4}\right), \varphi\left(v_{3} u\right)\right\}\right)$ for $v_{1} u$.

To see $\varphi$ is a linear $L$-coloring of $G$, we verify that $\left|C_{\varphi}^{i}(x)\right|=0$ for any vertex $x \in\left\{v_{1}, v_{2}, v_{3}, v_{4}, u\right\}$ and any $i \geqslant 3$, and and show that there exists no
monochromatic cycle containing at least one edge of $\left\{v_{1} v_{2}, v_{2} v_{3}, v_{3} v_{4}, v_{4} v_{1}\right.$, $\left.v_{1} u, v_{3} u\right\}$. We can check that $\left|C_{\varphi}^{i}(x)\right|=0$ for any vertex $x \in\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and any $i \geqslant 3$ by a similar argument as Subcase 1.1. The selection of colors for $v_{1} u$ and $v_{3} u$ ensure that $\left|C_{\varphi}^{i}(u)\right|=0$ for any $i \geqslant 3$. By contradiction, suppose $G$ contains a monochromatic cycle $C$. One can see that $C$ cannot contain the edge $v_{2} v_{3}$ since $\varphi\left(v_{2} v_{3}\right) \neq \varphi\left(v_{3} u\right)$ and $\varphi\left(v_{2} v_{3}\right) \neq \varphi\left(v_{3} v_{4}\right)$. Clearly, $C \neq v_{1} u v_{3} v_{4} v_{1}$ by the choice of the color of $v_{1} u$. Since $\varphi\left(v_{1} v_{2}\right) \notin C_{\sigma}\left(v_{2}\right)$ and $\varphi\left(v_{3} v_{4}\right) \notin C_{\sigma}\left(v_{4}\right), C$ cannot contain the edges $v_{1} v_{2}$ and $v_{3} v_{4}$. Thus $C$ must contain the path $v_{4} v_{1} u$, but $\varphi\left(v_{1} u\right) \notin C_{\sigma}(u)$ or $\varphi\left(v_{1} u\right) \neq \varphi\left(v_{1} v_{4}\right), C$ is not monochromatic.

Case 2. $\left|C_{\sigma}\left(v_{i}\right)\right|<k$ and $\left|C_{\sigma}\left(v_{j}\right)\right| \geqslant k$ for $\{i, j\}=\{2,4\}$.
By the symmetry of the roles of $v_{2}$ and $v_{4}$, assume $\left|C_{\sigma}\left(v_{2}\right)\right|<k$ and $\left|C_{\sigma}\left(v_{4}\right)\right| \geqslant k$. By the similar argument as in proof of Case 1, we have $\left|C_{\sigma}^{2}\left(v_{2}\right)\right| \leqslant k-2$ and $\left|C_{\sigma}^{2}\left(v_{4}\right)\right| \leqslant k-3$. We take
$\varphi\left(v_{1} v_{2}\right) \in L\left(v_{1} v_{2}\right) \backslash C_{\sigma}\left(v_{2}\right)$,
$\varphi\left(v_{2} v_{3}\right) \in L\left(v_{2} v_{3}\right) \backslash\left(C_{\sigma}^{2}\left(v_{2}\right) \cup\left\{\varphi\left(v_{1} v_{2}\right)\right\}\right)$,
$\varphi\left(v_{3} w\right) \in L\left(v_{3} w\right) \backslash C_{\sigma}(w)$ if $\left|C_{\sigma}(w)\right|<k$, and $\varphi\left(v_{3} w\right) \in L\left(v_{3} w\right) \backslash\left(C_{\sigma}^{2}(w) \cup\right.$ $\left.\left\{\varphi\left(v_{2} v_{3}\right)\right\}\right)$, otherwise. Then we successively take
$\varphi\left(v_{3} v_{4}\right) \in L\left(v_{3} v_{4}\right) \backslash\left(C_{\sigma}^{2}\left(v_{4}\right) \cup\left\{\varphi\left(v_{2} v_{3}\right), \varphi\left(v_{3} w\right)\right\}\right)$ and $\varphi\left(v_{1} v_{4}\right) \in L\left(v_{1} v_{4}\right) \backslash\left(C_{\sigma}^{2}\left(v_{4}\right) \cup\left\{\varphi\left(v_{3} v_{4}\right), \varphi\left(v_{1} v_{2}\right)\right\}\right)$.
Finally we assign a color for $v_{1} u$ as follows. If $\left|C_{\sigma}(u)\right|<k, \varphi\left(v_{1} u\right) \in$ $L\left(v_{1} u\right) \backslash C_{\sigma}(u)$. If $\left|C_{\sigma}(u)\right| \geqslant k, \varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash\left(C_{\sigma}^{2}(u) \cup\left\{\varphi\left(v_{1} v_{4}\right)\right\}\right)$ if $u \neq w ; \varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash\left(C_{\sigma}^{2}(u) \cup\left\{\varphi\left(v_{1} v_{4}\right), \varphi\left(v_{3} w\right)\right\}\right)$, otherwise.

It is easy to check that $\varphi$ is a linear $L$-coloring of $G$ by a similar argument as in proof of Case 1.

Case 3. $\left|C_{\sigma}\left(v_{i}\right)\right| \geqslant k$ for each $i \in\{2,4\}$.
Then $\left|C_{\sigma}^{2}\left(v_{2}\right)\right| \leqslant k-3$ and $\left|C_{\sigma}^{2}\left(v_{4}\right)\right| \leqslant k-3$. We take $\varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash C_{\sigma}(u)$ if $\left|C_{\sigma}(u)\right|<k$, and $\varphi\left(v_{3} w\right) \in L\left(v_{3} w\right) \backslash C_{\sigma}(w)$ if $\left|C_{\sigma}(w)\right|<k$. Next we suppose that $\left|C_{\sigma}(u)\right| \geqslant k$ and $\left|C_{\sigma}(w)\right| \geqslant k$.

If $L\left(v_{1} v_{2}\right) \backslash C_{\sigma}^{2}\left(v_{2}\right) \nsubseteq C_{\sigma}^{1}\left(v_{2}\right) \cap C_{\sigma}^{1}\left(v_{4}\right)$, we take
$\varphi\left(v_{1} v_{2}\right) \in L\left(v_{1} v_{2}\right) \backslash\left(C_{\sigma}^{2}\left(v_{2}\right) \cup\left(C_{\sigma}^{1}\left(v_{2}\right) \cap C_{\sigma}^{1}\left(v_{4}\right)\right)\right)$, and then
$\varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash\left(C_{\sigma}^{2}(u) \cup\left\{\varphi\left(v_{1} v_{2}\right)\right\}\right)$,
$\varphi\left(v_{3} w\right) \in L\left(v_{3} w\right) \backslash\left(C_{\sigma}^{2}(w) \cup\left\{\varphi\left(v_{1} u\right)\right\}\right)$,
$\varphi\left(v_{2} v_{3}\right) \in L\left(v_{2} v_{3}\right) \backslash\left(C_{\sigma}^{2}\left(v_{2}\right) \cup\left\{\varphi\left(v_{1} v_{2}\right), \varphi\left(v_{3} w\right)\right\}\right)$,
$\varphi\left(v_{3} v_{4}\right) \in L\left(v_{3} v_{4}\right) \backslash\left(C_{\sigma}^{2}\left(v_{4}\right) \cup\left\{\varphi\left(v_{2} v_{3}\right), \varphi\left(v_{3} w\right)\right\}\right)$,
$\varphi\left(v_{1} v_{4}\right) \in L\left(v_{1} v_{4}\right) \backslash\left(C_{\sigma}^{2}\left(v_{4}\right) \cup\left\{\varphi\left(v_{3} v_{4}\right), \varphi\left(v_{1} u\right)\right\}\right)$.

By the similar argument as in the proof of Case 1 , one can show that $\varphi$ is a linear $L$-coloring of $G$.

By symmetry, we consider that $L\left(v_{1} v_{2}\right) \backslash C_{\sigma}^{2}\left(v_{2}\right), L\left(v_{2} v_{3}\right) \backslash C_{\sigma}^{2}\left(v_{2}\right), L\left(v_{3} v_{4}\right) \backslash$ $C_{\sigma}^{2}\left(v_{4}\right)$ and $L\left(v_{4} v_{1}\right) \backslash C_{\sigma}^{2}\left(v_{4}\right)$ are all contained in $C_{\sigma}^{1}\left(v_{2}\right) \cap C_{\sigma}^{1}\left(v_{4}\right)$.

We claim that $\left(L\left(v_{1} v_{2}\right) \backslash C_{\sigma}^{2}\left(v_{2}\right)\right) \cap\left(L\left(v_{3} v_{4}\right) \backslash C_{\sigma}^{2}\left(v_{4}\right)\right)=\emptyset$. Suppose it is false, and $\left|C_{\sigma}^{2}\left(v_{2}\right)\right| \geqslant\left|C_{\sigma}^{2}\left(v_{4}\right)\right|$, without loss of generality. Therefore,

$$
\begin{aligned}
2 k-2\left|C_{\sigma}^{2}\left(v_{2}\right)\right| & \leqslant k-\left|C_{\sigma}^{2}\left(v_{2}\right)\right|+k-\left|C_{\sigma}^{2}\left(v_{4}\right)\right| \\
& \leqslant\left|L\left(v_{1} v_{2}\right) \backslash C_{\sigma}^{2}\left(v_{2}\right)\right|+\left|L\left(v_{3} v_{4}\right) \backslash C_{\sigma}^{2}\left(v_{4}\right)\right| \\
& \leqslant\left|C_{\sigma}^{1}\left(v_{2}\right) \cap C_{\sigma}^{1}\left(v_{4}\right)\right| \\
& \leqslant\left|C_{\sigma}^{1}\left(v_{2}\right)\right| \\
& \leqslant d\left(v_{2}\right)-2\left|C_{\sigma}^{2}\left(v_{2}\right)\right| \\
& \leqslant 2 k-1-2\left|C_{\sigma}^{2}\left(v_{2}\right)\right| .
\end{aligned}
$$

It follows that $2 k \leqslant 2 k-1$, a contradiction.
Thus we take

$$
\begin{aligned}
& \varphi\left(v_{1} v_{2}\right)=\varphi\left(v_{3} v_{4}\right) \in\left(L\left(v_{1} v_{2}\right) \backslash C_{\sigma}^{2}\left(v_{2}\right)\right) \cap\left(L\left(v_{3} v_{4}\right) \backslash C_{\sigma}^{2}\left(v_{4}\right)\right) \text { and } \\
& \varphi\left(v_{1} u\right) \in L\left(v_{1} u\right) \backslash\left(C_{\sigma}^{2}(u) \cup\left\{\varphi\left(v_{1} v_{2}\right)\right\}\right) \text {. And then } \\
& \varphi\left(v_{3} w\right) \in L\left(v_{3} w\right) \backslash\left(C_{\sigma}^{2}(w) \cup\left\{\varphi\left(v_{3} v_{4}\right)\right\}\right) \text { if } u \neq w ; \\
& \varphi\left(v_{3} w\right) \in L\left(v_{3} w\right) \backslash\left(C_{\sigma}^{2}(w) \cup\left\{\varphi\left(v_{3} v_{4}\right), \varphi\left(v_{1} u\right)\right\}\right) \text {, otherwise. Finally, } \\
& \varphi\left(v_{2} v_{3}\right) \in L\left(v_{2} v_{3}\right) \backslash\left(C_{\sigma}^{2}\left(v_{2}\right) \cup\left\{\varphi\left(v_{3} v_{4}\right), \varphi\left(v_{3} w\right)\right\}\right) \text { and } \\
& \varphi\left(v_{1} v_{4}\right) \in L\left(v_{1} v_{4}\right) \backslash\left(C_{\sigma}^{2}\left(v_{4}\right) \cup\left\{\varphi\left(v_{3} v_{4}\right), \varphi\left(v_{1} u\right)\right\}\right) .
\end{aligned}
$$

One can verify that $\varphi$ is a linear $L$-coloring of $G$.
The proof is complete.

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