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THE LIST LINEAR ARBORICITY OF PLANAR GRAPHS *

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Abstract

The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests which partition the edges of G. An and Wu introduce the notion of list linear arboricity $lla(G)$ of a graph G and conjecture that $lla(G) = la(G)$ for any graph G. We confirm that this conjecture is true for any planar graph having $\Delta \geqslant 13$, or for any planar graph with $\Delta \geqslant 7$ and without i-cycles for some $i \in \{3, 4, 5\}$. We also prove that $\lceil \frac{\Delta(G)}{2} \rceil \leqslant lla(G) \leqslant \lceil \frac{\Delta(G)+1}{2} \rceil$ for any planar graph having $\Delta \geqslant 9$.

Keywords: list coloring, linear arboricity, list linear arboricity, planar graph.

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1. Introduction

All graphs considered here are finite, undirected and simple. We refer to [4] for unexplained terminology and notations. For a real number x, $\lceil x \rceil$ is the least integer not less than x. Let $G = (V(G), E(G))$ be a graph. $|V(G)|$ and $|E(G)|$ are called the *order* and the *size* of G, respectively. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and the minimum degree of G, respectively. Let v be a vertex of G. The neighborhood of v, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. The degree of v, denoted

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by $d_G(v)$, is the number of edges incident with v in G. Since G is simple, $d_G(v) = |N_G(v)|$. If there is no confusion, we use $N(v)$ and $d(v)$ for the neighborhood and degree of v instead of $N_G(v)$ and $d_G(v)$, respectively. Let $N_k(v) = \{u|u \in N(v) \text{ and } d(u) = k\}$. The girth of G is the minimum length of cycles in G. A k- or k^+-vertex is a vertex of degree k, or at least k.

A *linear forest* is a graph in which each component is a path. A map φ from $E(G)$ to $\{1, 2, \dots, k\}$ is called a *k-linear coloring* if $(V(G), \varphi^{-1}(i))$ is a linear forest for $1 \leqslant i \leqslant k$. The *linear arboricity* $la(G)$ of a graph G, introduced by Harary [8], is the minimum number k for which G has a k-linear coloring. Akiyama, Exoo and Harary [1] conjectured that $la(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$ for any regular graph G. It is obvious that for a graph G, $la(G) \geqslant \lceil \frac{\Delta(G)}{2} \rceil$ and $la(G) \geqslant \lceil \frac{\Delta(G)+1}{2} \rceil$ when G is regular. So it is equivalent to the following conjecture, known as the linear arboricity conjecture.

Linear Arboricity Conjecture. For any graph G,

$$\left\lceil \frac{\Delta(G)}{2} \right\rceil \leqslant la(G) \leqslant \left\lceil \frac{\Delta(G)+1}{2} \right\rceil.$$

The linear arboricity has been determined for complete bipartite graphs [1], series-parallel graphs [10], and regular graphs with $\Delta = 3$ [1], 4 [2], 5, 6, 8 [6], 10 [7]. The LAC also has already been proved to be true for any planar graphs in [9] and [12]. In particular, the author proved that if G is a planar graph with $\Delta \geqslant 13$, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$. In [9] and [11], the authors showed that the same also holds for a planar graph with $\Delta \geqslant 7$ and without i-cycles for some $i \in \{3, 4, 5\}$.

A list assignment L to the edges of G is the assignment of a set $L(e) \subseteq N$ of colors to every edge e of G, where N is the set of natural numbers. If G has a coloring φ such that $\varphi(e) \in L(e)$ for every edge e and $(V(G), \varphi^{-1}(i))$ is a linear forest for any $i \in C_\varphi$, where $C_\varphi = \{\varphi(e)|e \in E(G)\}$, then we say that G is *linear L-colorable* and φ is a *linear L-coloring* of G. We say that G is *linear k-list colorable* if it is linear L-colorable for every list assignment L satisfying $|L(e)| = k$ for all edges e. The *list linear arboricity* $lla(G)$ of a graph G is the minimum number k for which G is linear k-list colorable. It is obvious that $la(G) \leqslant lla(G)$. In [3], the authors raised the following conjecture, and confirmed that it is true for any series-parallel graph.

List Linear Arboricity Conjecture. For any graph G,

$$\left\lceil \frac{\Delta(G)}{2} \right\rceil \leqslant la(G) = lla(G) \leqslant \left\lceil \frac{\Delta(G)+1}{2} \right\rceil.$$

Little was known for this conjecture. In this paper, we will prove that it is true for any planar graph having $\Delta \geqslant 13$, or for any planar graph with $\Delta \geqslant 7$ and without i-cycles for some $i \in \{3,4,5\}$. We also prove that $\lceil \frac{\Delta(G)}{2} \rceil \leqslant lla(G) \leqslant \lceil \frac{\Delta(G)+1}{2} \rceil$ for any planar graph having $\Delta \geqslant 9$.

2. Planar Graphs with $la(G) = lla(G)$

For convenience, we introduce two definitions. The *weight* $w(e)$ of an edge $e = uv$ is $d(u) + d(v)$. An even cycle $v_1 v_2 \cdots v_{2t} v_1$ is called *k-alternating* if $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = k$.

Let L be a list assignment of G, and φ be a coloring of G such that $\varphi(e) \in L(e)$ for any edge e of G. For a vertex $v \in V(G)$, we denote by $C_\varphi(v)$ the set of colors that appear on the edges incident with v in G.

$$C^i_\varphi(v) = \{j \mid \text{the color } j \text{ appears } i \text{ times at edges incident with } v\},$$

for any positive integer i. Observe that φ is a linear L-coloring of G if and only if G does not contain a monochromatic cycle under coloring φ and $|C^i_\varphi(v)| = 0$ for every vertex v of G and any $i \geqslant 3$. Thus, if φ is a linear L-coloring of G then $C_\varphi(v) = C^1_\varphi(v) \cup C^2_\varphi(v)$.

The following two lemmas can be found in [9].

Lemma 2.1. *Let G be a planar graph with $\delta(G) \geqslant 2$. Then either there is an edge e with $w(e) \leqslant 15$ or there is a 2-alternating cycle $v_0 v_1 \cdots v_{2n-1} v_0$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $\max_{0 \leqslant i < n} |N_2(v_{2i})| \geqslant 3$.*

Lemma 2.2. *Let G be a planar graph with girth at least g and maximum degree Δ, and assume that $\delta(G) \geqslant 2$. If $g = 4, 5$ or 6, then either there is an edge e with $w(e) \leqslant 17 - 2g$ or there is a 2-alternating cycle $v_0 v_1 \cdots v_{2n-1} v_0$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $\max_{0 \leqslant i < n} |N_2(v_{2i})| \geqslant 3$.*

Under the same conditions as given in the next theorem, Wu [9] proved that $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

Theorem 2.3. *Let G be a planar graph having girth at least g and maximum degree Δ. Then $la(G) = lla(G) = \lceil \frac{\Delta(G)}{2} \rceil$, provided that one of the following holds:*

$$(1)\ \Delta \geqslant 13, \quad (2)\ \Delta \geqslant 7 \text{ and } g \geqslant 4,$$
$$(3)\ \Delta \geqslant 5 \text{ and } g \geqslant 5, \quad (4)\ \Delta \geqslant 3 \text{ and } g \geqslant 6.$$

Proof. Since $\lceil \frac{\Delta(G)}{2} \rceil \leqslant la(G) \leqslant lla(G)$, we show (1) by proving somewhat a stronger statement: any planar graph G is linear k-list colorable for $k = \max\{7, \lceil \frac{\Delta(G)}{2} \rceil\}$.

We shall prove it by induction on $|E(G)|$. The result holds trivially if $|E(G)| \leqslant 7$. Next we assume G be a graph with $|E(G)| \geqslant 8$, and let L be a list assignment of G with $|L(e)| = k$ for any $e \in E(G)$.

Suppose that G has an edge xy such that $w(xy) \leqslant 2k + 1$. Then by induction hypothesis, $G^* = G - xy$ has a linear L-coloring φ. Let $C_\varphi = C^2_\varphi(x) \cup C^2_\varphi(y) \cup (C^1_\varphi(x) \cap C^1_\varphi(y))$. Since $2|C_\varphi| \leqslant d_{G^*}(x) + d_{G^*}(y) = w(xy) - 2 \leqslant 2k - 1$, $|C_\varphi| < k$. We can extend φ to a linear L-coloring of G by taking $\varphi(xy) \in L(xy) \backslash C_\varphi$.

Hence, we assume that $w(xy) > 2k + 1$ for any edge $xy \in E(G)$. Since $k = \max\{7, \lceil \frac{\Delta(G)}{2} \rceil\}$, we have $\delta(G) \geqslant 2$ and $2k + 1 \geqslant 15$. Therefore, for any edge $xy \in E(G)$, $w(xy) > 15$. By Lemma 2.1, G contains a 2-alternating cycle $C = v_0 v_1 \cdots v_{2n-1} v_0$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $\max\limits_{0 \leqslant i < n} |N_2(v_{2i})| \geqslant 3$.

Without loss of generality, let $|N_2(v_0)| \geqslant 3$. Let $u \in N_2(v_0) \backslash \{v_{2n-1}, v_1\}$ and $v \in N(u) \backslash \{v_0\}$. By induction hypothesis, $G^* = G - \{v_1, v_3, \ldots, v_{2n-1}\} - v_0 u$ has a linear L-coloring σ. Next, we shall extend σ to a linear L-coloring φ of G by setting $\varphi(e) = \sigma(e)$ for each $e \in E(G^*)$, and assigning some appropriate colors for the remaining edges as follows. We consider two cases.

Case 1. $|C_\sigma(v_0)| < k$.
Since $2|C^2_\sigma(v_0)| \leqslant d_{G^*}(v_0) = d(v_0) - 3 \leqslant \Delta(G) - 3 \leqslant 2k - 3$, we have $|C^2_\sigma(v_0)| \leqslant k - 2$.

Subcase 1.1. $|C_\sigma(v_{2j})| < k$ for each $2j$ with $j \in \{1, 2, \ldots, n-1\}$.
We take
$\varphi(v_0 u) \in L(v_0 u) \backslash C_\sigma(v_0)$,
$\varphi(v_0 v_1) \in L(v_0 v_1) \backslash C_\sigma(v_0)$,
$\varphi(v_0 v_{2n-1}) \in L(v_0 v_{2n-1}) \backslash (C^2_\sigma(v_0) \cup \{\varphi(v_0 v_1)\})$, and furthermore

$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash\, C_\sigma(v_{2j})$ for any $j \in \{1, 2, \ldots, n-1\}$.

To check that φ is a linear L-coloring of G, we need to show that there exists no monochromatic cycle containing at least one edge of $E(C) \cup \{v_0u\}$ in G and $|C^i_\varphi(x)| = 0$ for any vertex $x \in V(C) \cup \{u\}$ and any $i \geqslant 3$.

First note that if there is a monochromatic cycle C' in G, then C' does not contain any edges of C since $\varphi(v_0v_{2n-1}) \neq \varphi(v_0v_1)$, $\varphi(v_{2j-1}v_{2j}) \notin C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \notin C_\sigma(v_{2j})$ for each $j \in \{1, 2, \ldots, n-1\}$. Thus C' must contain the edges v_0u and uv. However, since $\varphi(v_0u) \notin C_\sigma(v_0)$, C' cannot be monochromatic.

Now let $x \in V(C) \cup \{u\}$ and i be an integer at least 3. We show that $|C^i_\varphi(x)| = 0$. Since $d(u) = 2$ and $d(v_{2j-1}) = 2$ for each $j \in \{1, 2, \ldots, n-1\}$, the result is trivially true when $x \in \{u, v_1, v_3, \cdots v_{2n-1}\}$. Since $\varphi(v_{2j-1}v_{2j}) \notin C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \notin C_\sigma(v_{2j})$, we have $|C^i_\varphi(v_{2j})| = 0$ for any $j \in \{1, 2, \ldots, n-1\}$. The selection of colors for v_0u, v_0v_1 and v_0v_{2n-1} ensure that $|C^i_\varphi(v_0)| = 0$.

Subcase 1.2. $|C_\sigma(v_{2j})| \geqslant k$ for some $2j$ with $j \in \{1, 2, \ldots, n-1\}$.
We take

$\varphi(v_0u) \in L(v_0u)\backslash(C^2_\sigma(v_0) \cup \{\sigma(uv)\})$,

$\varphi(v_0v_1) \in L(v_0v_1)\backslash(C^2_\sigma(v_0) \cup \{\varphi(v_0u)\})$,

$\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1})\backslash\, C_\sigma(v_0)$.

For $j \in \{1, 2, \cdots, n-1\}$, if $|C_\sigma(v_{2j})| < k$, we take
$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash\, C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash\, C_\sigma(v_{2j})$; otherwise,

$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash(C^2_\sigma(v_{2j}) \cup \{\varphi(v_{2j-2}v_{2j-1})\})$ and

$\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash(C^2_\sigma(v_{2j}) \cup \{\varphi(v_{2j-1}v_{2j})\})$.

Note that $|C^2_\sigma(v_{2j})| \leqslant k-2$ since $k+|C^2_\sigma(v_{2j})| \leqslant |C^1_\sigma(v_{2j})|+2|C^2_\sigma(v_{2j})| = d(v_{2j}) - 2 \leqslant 2k-2$.

We can check that $|C^i_\varphi(x)| = 0$ for any vertex $x \in V(C) \cup \{u\}$ and any $i \geqslant 3$ by a similar argument as in Subcase 1.1. Now, suppose that there is a monochromatic cycle C' in G. Clearly, C' cannot contain the edge v_0u since $\varphi(v_0u) \neq \sigma(uv)$. Thus C' must contain the edges of C. Since there exist some $2j$ such that $\varphi(v_{2j-1}v_{2j}) \neq \varphi(v_{2j-2}v_{2j-1})$, $C' \neq C$. Then C' must contain the path $v_{2l}v_{2l+1}v_{2l+2}\cdots v_{2r-1}v_{2r}$ of C since $\varphi(v_{2l-1}v_{2l}) \neq \varphi(v_{2l-2}v_{2l-1})$ and $\varphi(v_0v_{2n-1}) \notin C_\sigma(v_0)$, where $2 \leqslant 2l < 2r \leqslant 2n-2$ and $min\{|C_\sigma(v_{2l})|, |C_\sigma(v_{2r})|\} \geqslant k$. But $\varphi(v_{2r}v_{2r-1}) \neq \varphi(v_{2r-1}v_{2r-2})$ leads to the contradiction that C' is monochromatic. Thus φ is a linear L-coloring of G.

Case 2. $|C_\sigma(v_0)| \geqslant k$.
Since $k + |C^2_\sigma(v_0)| \leqslant |C^1_\sigma(v_0)| + 2|C^2_\sigma(v_0)| = d(v_0) - 3 \leqslant 2k - 3$, we have $|C^2_\sigma(v_0)| \leqslant k - 3$.

Subcase 2.1. $L(v_0v_1)\backslash C^2_\sigma(v_0) \nsubseteq L(v_0u)\backslash C^2_\sigma(v_0)$.
We take $\varphi(v_0v_1) \in L(v_0v_1)\backslash(C^2_\sigma(v_0) \cup L(v_0u))$. Furthermore, for any $j = \{1, 2, \ldots, n-1\}$, we take

$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash C_\sigma(v_{2j})$ and $\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash C_\sigma(v_{2j})$

if $|C_\sigma(v_{2j})| < k$; otherwise,

$\varphi(v_{2j-1}v_{2j}) \in L(v_{2j-1}v_{2j})\backslash(C^2_\sigma(v_{2j}) \cup \{\varphi(v_{2j-2}v_{2j-1})\})$ and
$\varphi(v_{2j}v_{2j+1}) \in L(v_{2j}v_{2j+1})\backslash(C^2_\sigma(v_{2j}) \cup \{\varphi(v_{2j-1}v_{2j})\})$, and finally
$\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1})\backslash(C^2_\sigma(v_0) \cup \{\varphi(v_0v_1), \varphi(v_{2n-1}v_{2n-2})\})$ and
$\varphi(v_0u) \in L(v_0u)\backslash(C^2_\sigma(v_0) \cup \{\varphi(v_0v_{2n-1}), \sigma(uv)\})$.

Subcase 2.2. $L(v_0v_1)\backslash C^2_\sigma(v_0) \subseteq L(v_0u)\backslash C^2_\sigma(v_0)$.
Since $|C^2_\sigma(v_0)| \leqslant k - 3$, we have $|L(v_0u)\backslash C^2_\sigma(v_0)| \geqslant |L(v_0v_1)\backslash C^2_\sigma(v_0)| \geqslant 3$.

We take $\varphi(v_0v_1) = \sigma(uv)$ if $\sigma(uv) \in L(v_0v_1)\backslash C^2_\sigma(v_0)$, and $\varphi(v_0v_1) \in L(v_0v_1)\backslash C^2_\sigma(v_0)$, otherwise. For $j \in \{1, 2, \ldots, n-1\}$, we assign a color $v_{2j-1}v_{2j}$ and $v_{2j}v_{2j+1}$ by the way as described in Subcase 2.1.

And then $\varphi(v_0v_{2n-1}) \in L(v_0v_{2n-1})\backslash(C^2_\sigma(v_0) \cup \{\varphi(v_{2n-1}v_{2n-2}), \varphi(v_0v_1)\})$. If $\sigma(uv) \in L(v_0u)\backslash C^2_\sigma(v_0)$, but $\sigma(uv) \notin L(v_0v_1)\backslash C^2_\sigma(v_0)$, then $|L(v_0u)\backslash C^2_\sigma(v_0)| \geqslant 4$. So, we take

$\varphi(v_0u) \in L(v_0u)\backslash(C^2_\sigma(v_0) \cup \{\varphi(v_0v_{2n-1}), \varphi(v_0v_1), \sigma(uv)\})$; otherwise,
$\varphi(v_0u) \in L(v_0u)\backslash(C^2_\sigma(v_0) \cup \{\varphi(v_0v_{2n-1}), \varphi(v_0v_1)\})$.

It is easy to check that φ is a linear L-coloring of G both in Subcase 2.1 and Subcase 2.2 by a similar argument as in Subcase 1.2. So we complete the proof of (1).

By using Lemma 2.2, one can similarly prove (2), (3), and (4). ∎

For a plane graph G, $F(G)$ denotes the set of faces of G. The degree of a face f, denote by $d(f)$, is the number of edges incident with it, where each cut edge is counted twice. A k-face is a face of degree k.

Theorem 2.4. *Let G be a planar graph with maximum degree $\Delta \geqslant 7$ and without i-cycle for some $i \in \{4, 5\}$. Then $la(G) = lla(G) = \lceil \frac{\Delta(G)}{2} \rceil$.*

Proof. We prove the theorem by contradiction. Let $G = (V, E)$ be a counterexample with the minimum size to the theorem, and be embedded in the plane. Set $k = \lceil \frac{\Delta(G)}{2} \rceil$. Then $k \geqslant 4$ since $\Delta \geqslant 7$. By a similar argument as in proof of Theorem 2.3, we can obtain the following claims.

Claim 1. For any edge $xy \in E(G)$, $w(xy) \geqslant 2k+2$.

Claim 2. G has no even cycle $v_0v_1 \cdots v_{2n-1}v_0$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $\max\limits_{0 \leqslant i < n} |N_2(v_{2i})| \geqslant 3$.

Let G' be the subgraph induced by edges incident with 2-vertices. Since G does not contain two adjacent 2-vertices by Claim 1, G' does not contain any odd cycle. So it follows from Claim 2 that any component of G' is either an even cycle or a tree. So it is easy to find a matching M in G saturating all 2-vertices. Thus if $xy \in M$ and $d(x) = 2$, y is called a 2-master of x. Note that every 2-vertex has a 2-master.

We define a weight function ch on $V(G) \cup F(G)$ by letting $ch(v) = 2d(v) - 6$ for each $v \in V(G)$ and $ch(f) = d(f) - 6$ for each $f \in F(G)$. Applying Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we have

$$\sum_{x \in V(G) \cup F(G)} ch(x) = \sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

In the following, we will reassign a new weight $ch'(x)$ to each $x \in V(G) \cup F(G)$ according to some discharging rules. Since we discharge weight from one element to another, the total weight is kept fixed during the discharging. Thus

$$\sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12.$$

We shall show that $ch'(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$, a contradiction, completing the proof.

If G contains no 4-cycles, then we give the following discharging rules.

R1-1. Each 2-vertex receives 2 from its 2-master.

R1-2. Each 3-face f receives $\frac{3}{2}$ from each of its incident 5^+-vertex.

R1-3. Each 5-face f receives $\frac{1}{3}$ from each of its incident 5^+-vertex.

We can obtain that $ch'(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$ by using the same argument in [11]. This complete the proof of the case that G contains no 4-cycles.

Now assume that G contains no 5-cycles. The discharging rules are defined as follows.

R2-1. Each 2-vertex receives 2 from its 2-master.

R2-2. For a 3-face f and its incident vertex v, f receives $\frac{1}{2}$ from v if $d(v) = 4$, 1 if $d(v) = 5$, $\frac{5}{4}$ if $d(v) = 6$ and $\frac{3}{2}$ if $d(v) \geqslant 7$.

R2-3. For a 4-face f and its incident vertex v, f receives $\frac{1}{2}$ from v if $4 \leqslant d(v) \leqslant 6$, 1 if $d(v) \geqslant 7$.

By the same argument in [11], $ch'(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$. Hence, the proof was done for the case that G contains no 5-cycles. ■

3. Planar Graphs with $\Delta \geqslant 9$

Lemma 3.1 ([5], Lemma 1)**.** *Let G be a planar graph with $\delta(G) \geqslant 3$. Then there is either an edge $e \in E(G)$ with $w(e) \leqslant 11$ or a 3-alternating 4-cycle.*

Theorem 3.2. *Let G be a planar graph with $\Delta(G) \geqslant 9$. Then $\lceil \frac{\Delta(G)}{2} \rceil \leqslant la(G) \leqslant lla(G) \leqslant \lceil \frac{\Delta(G)+1}{2} \rceil$.*

Proof. We prove the theorem by proving somewhat a stronger statement that any planar graph G is linear k-list colorable for $k = \max\{5, \lceil \frac{\Delta(G)+1}{2} \rceil\}$.

We shall prove it by induction on $|E(G)|$. Let L be a list assignment of G with $|L(e)| = k$ for any $e \in E(G)$. Clearly, the result is true when $|E(G)| \leqslant 5$. Next we assume $|E(G)| \geqslant 6$.

Suppose that G has an edge xy such that $w(xy) \leqslant 2k + 1$. Then by induction hypothesis, $G - xy$ has a linear L-coloring φ. Let $C_\varphi = C^2_\varphi(x) \cup C^2_\varphi(y) \cup (C^1_\varphi(x) \cap C^1_\varphi(y))$. Since $2|C_\varphi| \leqslant d_{G-xy}(x) + d_{G-xy}(y) = w(xy) - 2 \leqslant 2k - 1$, $|C_\varphi| < k$. We can extend φ to a linear L-coloring of G by setting $\varphi(xy) \in L(xy) \backslash C_\varphi$.

Hence, we assume that $w(xy) > 2k + 1$ for any edge $xy \in E(G)$ as follows. Since $k = \max\{5, \lceil \frac{\Delta(G)+1}{2} \rceil\}$, we have $\delta(G) \geqslant 3$ and $2k + 1 \geqslant 11$. Thus for any edge $xy \in E(G)$, $w(xy) > 11$. By Lemma 3.1, there is a 4-cycle $v_1v_2v_3v_4v_1$ of G such that $d(v_1) = d(v_3) = 3$. Let $\{u\} = N(v_1)\backslash\{v_2, v_4\}$ and $\{w\} = N(v_3)\backslash\{v_2, v_4\}$. Note that u and w might be the same vertex. By induction hypothesis, $G^* = G - \{v_1, v_3\}$ has a linear L-coloring σ. Next, we shall extend σ to a linear L-coloring φ of G. To do this, set $\varphi(e) = \sigma(e)$ for each $e \in E(G^*)$, and we consider three cases.

Case 1. $\max\{|C_\sigma(v_2)|, |C_\sigma(v_4)|\} < k$.

Since $2|C^2_\sigma(v_2)| \leqslant d_{G^*}(v_2) = d(v_2) - 2 \leqslant \Delta(G) - 2 \leqslant 2k - 3$, we have $|C^2_\sigma(v_2)| \leqslant k - 2$, and similarly $|C^2_\sigma(v_4)| \leqslant k - 2$. We take

$\varphi(v_1v_2) \in L(v_1v_2)\backslash C_\sigma(v_2)$,
$\varphi(v_3v_4) \in L(v_3v_4)\backslash C_\sigma(v_4)$,
$\varphi(v_2v_3) \in L(v_2v_3)\backslash(C^2_\sigma(v_2) \cup \{\varphi(v_3v_4)\})$ and
$\varphi(v_1v_4) \in L(v_1v_4)\ \backslash(C^2_\sigma(v_4) \cup \{\varphi(v_1v_2)\})$.

Subcase 1.1. $u \neq w$.
If $|C_\sigma(w)| \geqslant k$ then $k + |C^2_\sigma(w)| \leqslant |C^1_\sigma(w)| + 2|C^2_\sigma(w)| = d(w) - 1 \leqslant 2k - 2$, and so $|C^2_\sigma(w)| \leqslant k - 2$. Then we assign v_3w a color
$\varphi(v_3w) \in L(v_3w)\backslash(C^2_\sigma(w) \cup \{\varphi(v_2v_3)\})$ if $|C_\sigma(w)| \geqslant k$, and
$\varphi(v_3w) \in L(v_3w)\backslash C_\sigma(w)$, otherwise. Finally,
$\varphi(v_1u) \in L(v_1u)\backslash(C^2_\sigma(u) \cup \{\varphi(v_1v_4)\})$ if $|C_\sigma(u)| \geqslant k$, and
$\varphi(v_1u) \in L(v_1u)\backslash C_\sigma(u)$, otherwise.
To see that φ is a linear L-coloring of G, we shall check that $|C^i_\varphi(x)| = 0$ for any vertex $x \in \{v_1, v_2, v_3, v_4, u, w\}$ and any $i \geqslant 3$, and there exists no monochromatic cycle containing at least one edge of $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1u, v_3w\}$.

Since $d(v_1) = d(v_3) = 3$, $\varphi(v_1v_4) \neq \varphi(v_1v_2)$ and $\varphi(v_2v_3) \neq \varphi(v_3v_4)$, $|C^i_\varphi(x)| = 0$ for $x \in \{v_1, v_3\}$ and any $i \geqslant 3$. $|C^i_\varphi(v_2)| = 0$ for any $i \geqslant 3$ since $\varphi(v_1v_2) \notin C_\sigma(v_2)$ and $\varphi(v_2v_3) \notin C^2_\sigma(v_2)$. Similarly, $|C^i_\varphi(v_4)| = 0$ for any $i \geqslant 3$. Since $\varphi(v_1u) \notin C^2_\sigma(u)$ and $\varphi(v_3w) \notin C^2_\sigma(w)$, $|C^i_\varphi(u)| = |C^i_\varphi(w)| = 0$ for any $i \geqslant 3$.

By contradiction, suppose C is a monochromatic cycle in G. Since $\varphi(v_4v_1) \neq \varphi(v_1v_2)$ and $\varphi(v_4v_1) \neq \varphi(v_1u)$ or $\varphi(v_1u) \notin C_\sigma(u)$, C cannot contain the edge v_4v_1. Similarly, C cannot contain the edge v_2v_3. Thus C must contain the path uv_1v_2 or the path wv_3v_4. However, since $\varphi(v_1v_2) \notin C_\sigma(v_2)$ and $\varphi(v_3v_4) \notin C_\sigma(v_4)$, C cannot be monochromatic.

Subcase 1.2. $u = w$.
Since $2|C^2_\sigma(u)| \leqslant d(u) - 2 \leqslant 2k - 3$, we have $|C^2_\sigma(u)| \leqslant k - 2$.

Assign v_3u a color $\varphi(v_3u) \in L(v_3u)\backslash(C^2_\sigma(u) \cup \{\varphi(v_2v_3)\})$. A choice for a color for v_1u is somewhat complicated.

If $\varphi(v_3u) = \varphi(v_3v_4) = \varphi(v_1v_4)$ then $\varphi(v_1u) \in L(v_1u)\backslash(C^2_\sigma(u) \cup \{\varphi(v_3u)\})$.

If it is not, $\varphi(v_1u) \in L(v_1u)\backslash C_\sigma(u)$ when $|C_\sigma(u)| < k$. For the case $|C_\sigma(u)| \geqslant k$, we have $k + |C^2_\sigma(u)| \leqslant |C^1_\sigma(u)| + 2|C^2_\sigma(u)| = d(u) - 2 \leqslant 2k - 3$, and thus $|C^2_\sigma(u)| \leqslant k - 3$. Then assign a color $\varphi(v_1u) \in L(v_1u)\backslash(C^2_\sigma(u) \cup \{\varphi(v_1v_4), \varphi(v_3u)\})$ for v_1u.

To see φ is a linear L-coloring of G, we verify that $|C^i_\varphi(x)| = 0$ for any vertex $x \in \{v_1, v_2, v_3, v_4, u\}$ and any $i \geqslant 3$, and and show that there exists no monochromatic cycle containing at least one edge of $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1u, v_3u\}$. We can check that $|C^i_\varphi(x)| = 0$ for any vertex $x \in \{v_1, v_2, v_3, v_4\}$ and any $i \geqslant 3$ by a similar argument as Subcase 1.1. The selection of colors for v_1u and v_3u ensure that $|C^i_\varphi(u)| = 0$ for any $i \geqslant 3$. By contradiction, suppose G contains a monochromatic cycle C. One can see that C cannot contain the edge v_2v_3 since $\varphi(v_2v_3) \neq \varphi(v_3u)$ and $\varphi(v_2v_3) \neq \varphi(v_3v_4)$. Clearly, $C \neq v_1uv_3v_4v_1$ by the choice of the color of v_1u. Since $\varphi(v_1v_2) \notin C_\sigma(v_2)$ and $\varphi(v_3v_4) \notin C_\sigma(v_4)$, C cannot contain the edges v_1v_2 and v_3v_4. Thus C must contain the path v_4v_1u, but $\varphi(v_1u) \notin C_\sigma(u)$ or $\varphi(v_1u) \neq \varphi(v_1v_4)$, C is not monochromatic.

Case 2. $|C_\sigma(v_i)| < k$ and $|C_\sigma(v_j)| \geqslant k$ for $\{i, j\} = \{2, 4\}$.
By the symmetry of the roles of v_2 and v_4, assume $|C_\sigma(v_2)| < k$ and $|C_\sigma(v_4)| \geqslant k$. By the similar argument as in proof of Case 1, we have $|C^2_\sigma(v_2)| \leqslant k-2$ and $|C^2_\sigma(v_4)| \leqslant k-3$. We take

$\varphi(v_1v_2) \in L(v_1v_2)\backslash C_\sigma(v_2)$,
$\varphi(v_2v_3) \in L(v_2v_3)\backslash (C^2_\sigma(v_2) \cup \{\varphi(v_1v_2)\})$,
$\varphi(v_3w) \in L(v_3w)\backslash C_\sigma(w)$ if $|C_\sigma(w)| < k$, and $\varphi(v_3w) \in L(v_3w)\backslash (C^2_\sigma(w) \cup \{\varphi(v_2v_3)\})$, otherwise. Then we successively take
$\varphi(v_3v_4) \in L(v_3v_4)\backslash (C^2_\sigma(v_4) \cup \{\varphi(v_2v_3), \varphi(v_3w)\})$ and
$\varphi(v_1v_4) \in L(v_1v_4)\backslash (C^2_\sigma(v_4) \cup \{\varphi(v_3v_4), \varphi(v_1v_2)\})$.

Finally we assign a color for v_1u as follows. If $|C_\sigma(u)| < k$, $\varphi(v_1u) \in L(v_1u)\backslash C_\sigma(u)$. If $|C_\sigma(u)| \geqslant k$, $\varphi(v_1u) \in L(v_1u)\backslash\ (C^2_\sigma(u) \cup \{\varphi(v_1v_4)\})$ if $u \neq w$; $\varphi(v_1u) \in L(v_1u)\backslash (C^2_\sigma(u) \cup \{\varphi(v_1v_4), \varphi(v_3w)\})$, otherwise.

It is easy to check that φ is a linear L-coloring of G by a similar argument as in proof of Case 1.

Case 3. $|C_\sigma(v_i)| \geqslant k$ for each $i \in \{2, 4\}$.
Then $|C^2_\sigma(v_2)| \leqslant k-3$ and $|C^2_\sigma(v_4)| \leqslant k-3$. We take $\varphi(v_1u) \in L(v_1u)\backslash C_\sigma(u)$ if $|C_\sigma(u)| < k$, and $\varphi(v_3w) \in L(v_3w)\backslash C_\sigma(w)$ if $|C_\sigma(w)| < k$. Next we suppose that $|C_\sigma(u)| \geqslant k$ and $|C_\sigma(w)| \geqslant k$.

If $L(v_1v_2)\backslash C^2_\sigma(v_2) \nsubseteq C^1_\sigma(v_2) \cap C^1_\sigma(v_4)$, we take
$\varphi(v_1v_2) \in L(v_1v_2)\backslash (C^2_\sigma(v_2) \cup (C^1_\sigma(v_2) \cap C^1_\sigma(v_4)))$, and then
$\varphi(v_1u) \in L(v_1u)\backslash (C^2_\sigma(u) \cup \{\varphi(v_1v_2)\})$,
$\varphi(v_3w) \in L(v_3w)\backslash (C^2_\sigma(w) \cup \{\varphi(v_1u)\})$,
$\varphi(v_2v_3) \in L(v_2v_3)\backslash (C^2_\sigma(v_2) \cup \{\varphi(v_1v_2), \varphi(v_3w)\})$,
$\varphi(v_3v_4) \in L(v_3v_4)\backslash (C^2_\sigma(v_4) \cup \{\varphi(v_2v_3), \varphi(v_3w)\})$,
$\varphi(v_1v_4) \in L(v_1v_4)\backslash (C^2_\sigma(v_4) \cup \{\varphi(v_3v_4), \varphi(v_1u)\})$.

By the similar argument as in the proof of Case 1, one can show that φ is a linear L-coloring of G.

By symmetry, we consider that $L(v_1v_2)\backslash C^2_\sigma(v_2)$, $L(v_2v_3)\backslash C^2_\sigma(v_2)$, $L(v_3v_4)\backslash C^2_\sigma(v_4)$ and $L(v_4v_1)\backslash C^2_\sigma(v_4)$ are all contained in $C^1_\sigma(v_2)\cap C^1_\sigma(v_4)$.

We claim that $(L(v_1v_2)\backslash C^2_\sigma(v_2))\cap(L(v_3v_4)\backslash C^2_\sigma(v_4))=\emptyset$. Suppose it is false, and $|C^2_\sigma(v_2)|\geqslant|C^2_\sigma(v_4)|$, without loss of generality. Therefore,

$$\begin{aligned}
2k-2|C^2_\sigma(v_2)| &\leqslant k-|C^2_\sigma(v_2)|+k-|C^2_\sigma(v_4)|\\
&\leqslant |L(v_1v_2)\backslash C^2_\sigma(v_2)|+|L(v_3v_4)\backslash C^2_\sigma(v_4)|\\
&\leqslant |C^1_\sigma(v_2)\cap C^1_\sigma(v_4)|\\
&\leqslant |C^1_\sigma(v_2)|\\
&\leqslant d(v_2)-2|C^2_\sigma(v_2)|\\
&\leqslant 2k-1-2|C^2_\sigma(v_2)|.
\end{aligned}$$

It follows that $2k\leqslant 2k-1$, a contradiction.
Thus we take

$\varphi(v_1v_2)=\varphi(v_3v_4)\in(L(v_1v_2)\backslash C^2_\sigma(v_2))\cap(L(v_3v_4)\backslash C^2_\sigma(v_4))$ and
$\varphi(v_1u)\in L(v_1u)\backslash(C^2_\sigma(u)\cup\{\varphi(v_1v_2)\})$. And then
$\varphi(v_3w)\in L(v_3w)\backslash\ (C^2_\sigma(w)\cup\{\varphi(v_3v_4)\})$ if $u\neq w$;
$\varphi(v_3w)\in L(v_3w)\backslash\ (C^2_\sigma(w)\cup\{\varphi(v_3v_4),\varphi(v_1u)\})$, otherwise. Finally,
$\varphi(v_2v_3)\in L(v_2v_3)\backslash(C^2_\sigma(v_2)\cup\{\varphi(v_3v_4),\varphi(v_3w)\})$ and
$\varphi(v_1v_4)\in L(v_1v_4)\backslash(C^2_\sigma(v_4)\cup\{\varphi(v_3v_4),\varphi(v_1u)\})$.

One can verify that φ is a linear L-coloring of G.
The proof is complete. ■

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