Discussiones Mathematicae
Graph Theory 29 (2009) 481–498

ON LEE'S CONJECTURE AND SOME RESULTS*

Lixia Fan and Zhihe Liang†

Department of Mathematics, Hebei Normal University
Shijiazhuang 050016, P.R. China

e-mail: zhiheliang@163.com.cn

Abstract

S.M. Lee proposed the conjecture: for any $n > 1$ and any permutation f in $S(n)$, the permutation graph $P(P_n, f)$ is graceful. For any integer $n > 1$ and permutation f in $S(n)$, we discuss the gracefulness of the permutation graph $P(P_n, f)$ if $f = \prod_{k=0}^{l-1}(m + 2k, m + 2k + 1)$, and $\prod_{k=0}^{l-1}(m+4k, m+4k+2)(m+4k+1, m+4k+3)$ for any positive integers m and l.

Keywords: permutation graph; graceful, Lee's conjecture.

2000 Mathematics Subject Classification: 05C78.

1. Introduction

A graph that has order p and size q is called a (p, q)-graph. Let $G = (V, E)$ be a finite simple (p, q)-graph. The graph G is called graceful if there exists an injection g from V to $\{0, 1, 2, \ldots, q\}$ such that the induced edge labels $\{g^*(uv) = |g(u) - g(v)| \mid uv \in E\}$ is equal to $\{1, 2, \ldots, q\}$. The mapping g is said to be a graceful labelling of G, and g^* is said to be an edge labelling of G. The concept of graceful graph is due to Rosa [1]. In 1967, he introduced the notion of β-valuation, which Golomb [2] subsequently called graceful labelling. In general, it is hard to decide whether a given graph is graceful. Even if a graph is known to be graceful, it may still be difficult to find a graceful labelling. Research has focused on specific classes of trees, bipartite

*This work was supported by National Natural Science Foundation of China and NSFHB.

†Corresponding author Z. Liang.

graphs, cycles, cycle with a pendant edge attached to it at each vertex, wheels, cycles with chords, grid graphs, one point union of two cycles, one-point union of complete graphs, unicyclic graphs, etc (see [4]). The theory of graceful graphs has attracted many mathematicians mainly because of its aesthetic aspect, as well as its wide range of applications in such areas as radar pulse codes, x-ray crystallography, circuit design, missile guidance, radio astronomy, sonar ranging, and broadcast frequency assignments (see [4, 5, 6]). Recently, instead of studying special classes of graceful graphs, attention has been turned to their construction. In this paper, we will consider the construction of graceful graphs via permutation graphs.

Chartrand and Harary introduced the concept of permutation graphs. For a graph G with n vertices labelled $1, 2, \ldots, n$, and a permutation f in $S(n)$, the symmetric group on the symbols $\{1, 2, \ldots, n\}$, the f-permutation graph on G, denoted $P(G, f)$, consists of two disjoint copies of G, say G_1 and G_2, along with the n edges obtained by joining $v_{1,i}$ in G_1 with $v_{2,f(i)}$ in G_2, $i = 1, 2, \ldots, n$. It is obvious that the graph $P(G, (1))$ is isomorphic to $G \times K_2$. If G is a cycle C_n, then the (1)-permutation graph of C_n, $P(C_n, (1))$, is a prism graph. Let P_n denote the path with n vertices. In 1983, Lee proposed the graceful conjecture of permutation graphs of paths:

Conjecture 1.1. For any $n > 1$ and any permutation f in $S(n)$, the permutation graph $P(P_n, f)$ is graceful.

Gallian restated this conjecture in [4]. Gracefulness of $P(G, f)$ has been considered in the following literature. In [8], G is hypercube and f is an identity map. In [4, 9], G is a star $K_{1,n}$ and f is an identity map. In [10, 11], G is a cycle and f is an identity map. In [3, 7, 12, 13] G is a path with n vertices and f is a non-identity map.

Lemma 1.2. *Let g be a function and $f = (a_1, a_2, \ldots, a_s) \ldots (a_{n-t+1}, a_{n-t+2}, \ldots, a_{n-1}, a_n)$ be a permutation, define $gf = (g(a_1), g(a_2), \ldots, g(a_s)) \ldots g(a_{n-t+1}), g(a_{n-t+2}), \ldots, g(a_{n-1}), g(a_n))$.*

Let $n > 1$ be a positive integer, f be a permutation in $S(n)$. Define the function g as follows: $g(x) = n + 1 - x$, if $x \in [1, n]$. Then $P(P_n, gf)$ is graceful when permutation graph $P(P_n, f)$ is graceful.

Proof. Since $P(P_n, gf) \cong P(P_n, f)$, then $P(P_n, gf)$ is graceful when permutation graph $P(P_n, f)$ is graceful. ∎

Example 1.3. Taking $n = 4$, $f = (13)(24)$, then $gf = (42)(31)$. Taking $n = 5$, $f = (124)$, then $gf = (542)$.
In this paper, we obtain the following theorem:

Theorem 1.4. *Let $n > 1$ be an integer, and f be a permutation in $S(n)$. Then the permutation graph $P(P_n, f)$ is graceful if $f = \prod_{k=0}^{l-1}(m + 2k, m+2k+1)$, and $\prod_{k=0}^{l-1}(m+4k, m+4k+2)(m+4k+1, m+4k+3)$ for any positive integers m and l.*

Let Z be the ring of integers and let $a, b, k \in Z$ and $k \geq 2$. The following notations are used frequently.

$[a, b] = \{x \mid x \in Z,\ a \leq x \leq b\}$,

$[a, b]_k = \{x \in Z \mid a \leq x \leq b, x \equiv a\ (mod\ k)\}$,

$f(S) = \{f(x)|\ x \in S\}$, where S is a set and f is a function.

Remark 1.5. $[a, b] = [a, b]_k = \emptyset$ if $a > b$, and $f(S) = \emptyset$ if $S = \emptyset$.

Example 1.6. The symbols $[3, 8] = \{3, 4, 5, 6, 7, 8\}$, $[2, 8]_2 = \{2, 4, 6, 8\}$, $[3, 7]_2 = \{3, 5, 7\}$, $[3, 8]_2 = \{3, 5, 7\}$, and for any integer $k \geq 2$, $[3, 3] = [3, 3]_k = \{3\}$, $[3, 1] = [3, 1]_k = \emptyset$.

2. MAIN RESULTS

Let P_n be a path with n vertices, f a permutation in $S(n)$, $V(G_j) = \{v_{j,1}, v_{j,2}, \ldots, v_{j,n}\}$, $j = 1, 2$. Then the vertex set of the permutation graph $P(P_n, f)$ is $V(G_1) \bigcup V(G_2)$ (shortly $V(G)$), and its edge set is $E(G_1) \bigcup E(G_2) \bigcup \{v_{1,j}v_{2,f(j)} \mid j \in [1, n]\}$ (shortly $E(G)$). The permutation graph $P(P_n, f)$ has $2n$ vertices and $3n - 2$ edges.

In the following, the set behind the function g is the label set of corresponding vertices. For instance, "$g(x_{2i+1}) = 2i + 1$ if $i \in [0, m-2]$, $[1, 2m-3]_2$" implies "$g(x_1) = 1$, $g(x_3) = 3, \ldots, g(x_{2m-3}) = 2m-3$, and the set $[1, 2m-3]_2 = \{g(x_{2i+1}) \mid i \in [0, m-2]\}$". (Note: When $m = 1$, the interval $[0, m-2] = [0, -1] = \emptyset$, so the function $g(x_{2i+1})$ has no value in this interval.)

Lemma 2.1. *When $f = \prod_{k=0}^{l-1}(m + 2k, m + 2k + 1)$ for positive integers m and l satisfying $n \geq 2(m-1) + 2l$, the graph $P(P_n, f)$ is graceful.*

Proof. We distinguish between two cases to show this lemma.

Case 1. when m is odd:

(1) When $m = 1$, define the function $g : V(G) \to [0, 3n-2]$ as follows:

$$
\begin{array}{lll}
g(v_{1,2i+1}) = 3n-3i-3 & \text{if } i \in [0, l-1] & [3n-3l, 3n-3]_3, \\
g(v_{2,2i+1}) = 3n-3i-2 & \text{if } i \in [0, l-1] & [3n-3l+1, 3n-2]_3, \\
g(v_{1,2i+2}) = 3i & \text{if } i \in [0, l-1] & [0, 3l-3]_3, \\
g(v_{2,2i+2}) = 3i+2 & \text{if } i \in [0, l-1] & [2, 3l-1]_3, \\
g(v_{1,2l+2i+1}) = 3n-3l-2i-3 & \text{if } i \in [0, \lceil \frac{n-2l}{2} \rceil - 1] & V_1, \\
g(v_{2,2l+2i+1}) = 3l+4i-2 & \text{if } i \in [0, \lceil \frac{n-2l}{2} \rceil - 1] & V_2, \\
g(v_{1,2l+2i+2}) = 3l+4i & \text{if } i \in [0, \lfloor \frac{n-2l}{2} \rfloor - 1] & V_3, \\
g(v_{2,2l+2i+2}) = 3n-3l-2i-4 & \text{if } i \in [0, \lfloor \frac{n-2l}{2} \rfloor - 1] & V_4.
\end{array}
$$

Where

$V_1 = [2n-l-1, 3n-3l-3]_2$ if n is even, and $[2n-l-2, 3n-3l-3]_2$ if n is odd;

$V_2 = [3l-2, 2n-l-6]_4$ if n is even, and $[3l-2, 2n-l-4]_4$ if n is odd;

$V_3 = [3l, 2n-l-4]_4$ if n is even, and $[3l, 2n-l-6]_4$ if n is odd;

$V_4 = [2n-l-2, 3n-3l-4]_2$ if n is even, and $[2n-l-1, 3n-3l-4]_2$ if n is odd.

Whether n is even or odd, the label-set of vertices of the graph $P(P_n, f)$, $g(V(G))$, is $[0, 3l-3]_3 \cup [2, 3l-1]_3 \cup [3l-2, 2n-l-4]_2 \cup [2n-l-2, 3n-3l-3] \cup [3n-3l, 3n-3]_3 \cup [3n-3l+1, 3n-2]_3$. Since $|g(V(G))| = 2((3l-3)/3+1)+(2n-4l-2)/2+1+(n-2l-1)+1+2((3l-3)/3+1) = 2n$, the g is an injection from $V(G)$ to $[0, 3n-2]$.

In the following, we show that the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. By the definition of the g, we have:

$$
\begin{aligned}
A &= \{g^*(v_{1,2i+1}v_{2,2i+2}), g^*(v_{1,2i+2}v_{2,2i+1}), g^*(v_{1,2i+1}v_{1,2i+2}), g^*(v_{2,2i+1}v_{2,2i+2}) | \; i \in [0, l-1]\} \\
&= \{3n-6i-5, 3n-6i-2, 3n-6i-3, 3n-6i-4 | \; i \in [0, l-1]\} \\
&= [3n-6l+1, 3n-5]_6 \cup [3n-6l+4, 3n-2]_6 \cup [3n-6l+3, 3n-3]_6 \cup [3n-6l+2, 3n-4]_6;
\end{aligned}
$$

$$
\begin{aligned}
B &= \{g^*(v_{1,2i+2}v_{1,2i+3}), g^*(v_{2,2i+2}v_{2,2i+3}) | \; i \in [0, l-2]\} \\
&= \{3n-6i-6, 3n-6i-7 | \; i \in [0, l-2]\} \\
&= [3n-6l+6, 3n-6]_6 \cup [3n-6l+5, 3n-7]_6;
\end{aligned}
$$

$$
\begin{aligned}
C &= \{g^*(v_{1,2l+2i+2}v_{2,2l+2i+2}), g^*(v_{1,2l+2i+1}v_{1,2l+2i+2}), g^*(v_{2,2l+2i+1}v_{2,2l+2i+2}) | \; i \in [0, \lfloor \tfrac{n-2l}{2} \rfloor - 1]\} \\
&= \{3n-6l-6i-4, 3n-6l-6i-3, 3n-6l-6i-2 | \; i \in [0, \lfloor \tfrac{n-2l}{2} \rfloor - 1]\} \\
&= [2, 3n-6l-4]_6 \cup [3, 3n-6l-3]_6 \cup [4, 3n-6l-2]_6 \text{ if } n \text{ is even,}
\end{aligned}
$$

and $[5, 3n-6l-4]_6 \cup [6, 3n-6l-3]_6 \cup [7, 3n-6l-2]_6$ if n is odd;

$D = \{g^*(v_{1,2l+2i+1}v_{2,2l+2i+1})|\ i \in [0, \lceil \frac{n-2l}{2} \rceil - 1]\}$
$\quad = \{3n-6l-6i-1|\ i \in [0, \lceil \frac{n-2l}{2} \rceil - 1]\}$
$\quad = [5, 3n-6l-1]_6$ if n is even, and $[2, 3n-6l-1]_6$ if n is odd;
$E = \{g^*(v_{1,2l+2i+2}v_{1,2l+2i+3}), g^*(v_{2,2l+2i+2}v_{2,2l+2i+3})|\ i \in [0, \lceil \frac{n-2l}{2} \rceil - 2]\}$
$\quad = \{3n-6l-6i-5, 3n-6l-6i-6|\ i \in [0, \lceil \frac{n-2l}{2} \rceil - 2]\}$
$\quad = [7, 3n-6l-5]_6 \cup [6, 3n-6l-6]_6$ if n is even,

and $[4, 3n-6l-5]_6 \cup [3, 3n-6l-6]_6$ if n is odd;

$F = \{g^*(v_{1,2l}v_{1,2l+1}), g^*(v_{2,2l}v_{2,2l+1})\} = \{3n-6l, 1\}$.

Whether n is even or odd, $g^*(E(G))=A \cup B \cup C \cup D \cup E \cup F = [2, 3n-6l-1] \cup [3n-6l+1, 3n-2] \cup \{3n-6l, 1\} = [1, 3n-2]$. Therefore, the g^* is a bijection from $E(G)$ to $[1, 3n-2]$.

(2) When $m \geq 3$, define the function $g : V(G) \to [0, 3n-2]$ as follows:

$g(v_{1,2i+1}) = 2i,$ if $i \in [0, \lceil \frac{m-1}{2} \rceil - 1]$ $[0, m-3]_2,$
$g(v_{2,2i+1}) = 3n-4i-2,$ if $i \in [0, \lceil \frac{m-1}{2} \rceil - 1]$ $[3n-2m+4, 3n-2]_4,$
$g(v_{1,2i+2}) = 3n-4i-4,$ if $i \in [0, \lfloor \frac{m-1}{2} \rfloor - 1]$ $[3n-2m+2, 3n-4]_4,$
$g(v_{2,2i+2}) = 2i+1,$ if $i \in [0, \lfloor \frac{m-1}{2} \rfloor - 1]$ $[1, m-2]_2,$
$g(v_{1,m+2i}) = 3n-2m-3i,$ if $i \in [0, l-1]$ $[3n-2m-3l+3, 3n-2m]_3,$
$g(v_{2,m+2i}) = 3n-2m-3i+1,$ if $i \in [0, l-1]$ $[3n-2m-3l+4, 3n-2m+1]_3,$
$g(v_{1,m+2i+1}) = m+3i-1,$ if $i \in [0, l-1]$ $[m-1, m+3l-4]_3,$
$g(v_{2,m+2i+1}) = m+3i+1,$ if $i \in [0, l-1]$ $[m+1, m+3l-2]_3,$
$g(v_{1,m+2l+2i}) = 3n-2m-3l-2i,$ if $i \in [0, \lceil \frac{n-2l-m+1}{2} \rceil - 1]$ $V_1,$
$g(v_{2,m+2l+2i}) = m+3l+4i-3,$ if $i \in [0, \lceil \frac{n-2l-m+1}{2} \rceil - 1]$ $V_2,$
$g(v_{1,m+2l+2i+1}) = m+3l+4i-1,$ if $i \in [0, \lfloor \frac{n-2l-m+1}{2} \rfloor - 1]$ $V_3,$
$g(v_{2,m+2l+2i+1}) = 3n-2m-3l-2i-1,$ if $i \in [0, \lfloor \frac{n-2l-m+1}{2} \rfloor - 1]$ $V_4.$

Where
$V_1 = [2n-m-l+1, 3n-2m-3l]_2$ if n is even, and $[2n-m-l, 3n-2m-3l]_2$ if n is odd.
$V_2 = [m+3l-3, 2n-m-l-5]_4$ if n is even, and $[m+3l-3, 2n-m-l-3]_4$ if n is odd.
$V_3 = [m+3l-1, 2n-m-l-3]_4$ if n is even, and $[m+3l-1, 2n-m-l-5]_4$ if n is odd.
$V_4 = [2n-m-l, 3n-2m-3l-1]_2$ if n is even, and $[2n-m-l+1, 3n-2m-3l-1]_2$ if n is odd.

Whether n is even or odd, $g(V(G)) = [0, m-2] \cup [m-1, m+3l-4]_3 \cup [m+1, m+3l-2]_3 \cup [m+3l-3, 2n-m-l-3]_2 \cup [2n-m-l, 3n-2m-3l] \cup [3n-2m-3l+3, 3n-2m]_3 \cup [3n-2m-3l+4, 3n-2m+1]_3 \cup [3n-2m+2, 3n-2]_2$.

Since $|g(V(G))| = (m-2)+1+2((3l-3)/3+1)+(2n-2m-4l)/2+1+(n-m-2l)+1+2((3l-3)/3+1)+(2m-4)/2+1 = 2n$, the g is an injection from $V(G)$ to $[0, 3n-2]$.

In the following, we show that the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. By the definition of the g, we have:

$$A = \{g^*(v_{1,2i+1}v_{2,2i+1}) \mid i \in [0, \lceil \tfrac{m-1}{2} \rceil - 1]\} = \{3n-6i-2 \mid i \in [0, \lceil \tfrac{m-1}{2} \rceil - 1]\}$$
$$= [3n-3m+7, 3n-2]_6;$$

$$B = \{g^*(v_{1,2i+2}v_{2,2i+2}), g^*(v_{1,2i+1}v_{1,2i+2}), g^*(v_{2,2i+1}v_{2,2i+2}) \mid i \in [0, \lfloor \tfrac{m-1}{2} \rfloor - 1]\}$$
$$= \{3n-6i-5, 3n-6i-4, 3n-6i-3 \mid i \in [0, \lfloor \tfrac{m-1}{2} \rfloor - 1]\}$$
$$= [3n-3m+4, 3n-5]_6 \cup [3n-3m+5, 3n-4]_6 \cup [3n-3m+6, 3n-3]_6;$$

$$C = \{g^*(v_{1,2i+2}v_{1,2i+3}), g^*(v_{2,2i+2}v_{2,2i+3}) \mid i \in [0, \lceil \tfrac{m-1}{2} \rceil - 2]\}$$
$$= \{3n-6i-6, 3n-6i-7 \mid i \in [0, \lceil \tfrac{m-1}{2} \rceil - 2]\}$$
$$= [3n-3m+9, 3n-6]_6 \cup [3n-3m+8, 3n-7]_6;$$

$$D = \{g^*(v_{1,m+2i}v_{2,m+2i+1}), g^*(v_{1,m+2i+1}v_{2,m+2i}), g^*(v_{1,m+2i}v_{1,m+2i+1}), g^*(v_{2,m+2i}v_{2,m+2i+1}) \mid i \in [0, l-1]\}$$
$$= \{3n-3m-6i-1, 3n-3m-6i+2, 3n-3m-6i+1, 3n-3m-6i \mid i \in [0, l-1]\}$$
$$= [3n-3m-6l+5, 3n-3m-1]_6 \cup [3n-3m-6l+8, 3n-3m+2]_6 \cup [3n-3m-6l+7, 3n-3m+1]_6 \cup [3n-3m-6l+6, 3n-3m]_6;$$

$$E = \{g^*(v_{1,m+2i+1}v_{1,m+2i+2}), g^*(v_{2,m+2i+1}v_{2,m+2i+2}) \mid i \in [0, l-2]\}$$
$$= \{3n-3m-6i-2, 3n-3m-6i-3 \mid i \in [0, l-2]\}$$
$$= [3n-3m-6l+10, 3n-3m-2]_6 \cup [3n-3m-6l+9, 3n-3m-3]_6;$$

$$F = \{g^*(v_{1,m+2l+2i+1}v_{2,m+2l+2i+1}), g^*(v_{1,m+2l+2i}v_{1,m+2l+2i+1}), g^*(v_{2,m+2l+2i}v_{2,m+2l+2i+1}) \mid i \in [0, \lfloor \tfrac{n-2l-m+1}{2} \rfloor - 1]\}$$
$$= \{3n-3m-6l-6i, 3n-3m-6l-6i+1, 3n-3m-6l-6i+2 \mid i \in [0, \lfloor \tfrac{n-2l-m+1}{2} \rfloor - 1]\}$$
$= [3, 3n-3m-6l]_6 \cup [4, 3n-3m-6l+1]_6 \cup [5, 3n-3m-6l+2]_6$ if n is even, and $[6, 3n-3m-6l]_6 \cup [7, 3n-3m-6l+1]_6 \cup [8, 3n-3m-6l+2]_6$ if n is odd;

$$G = \{g^*(v_{1,m+2l+2i}v_{2,m+2l+2i}) \mid i \in [0, \lceil \tfrac{n-2l-m+1}{2} \rceil - 1]\}$$
$$= \{3n-3m-6l-6i+3 \mid i \in [0, \lceil \tfrac{n-2l-m+1}{2} \rceil - 1]\}$$
$= [6, 3n-3m-6l+3]_6$ if n is even, and $[3, 3n-3m-6l+3]_6$ if n is odd;

$$H = \{g^*(v_{1,m+2l+2i+1}v_{1,m+2l+2i+2}), g^*(v_{2,m+2l+2i+1}v_{2,m+2l+2i+2}) \mid i \in [0, \lceil \tfrac{n-2l-m+1}{2} \rceil - 2]\}$$
$$= \{3n-3m-6l-6i-1, 3n-3m-6l-6i-2 \mid i \in [0, \lceil \tfrac{n-2l-m+1}{2} \rceil - 2]\}$$

$= [8, 3n - 3m - 6l - 1]_6 \cup [7, 3n - 3m - 6l - 2]_6$ if n is even,
and $[5, 3n - 3m - 6l - 1]_6 \cup [4, 3n - 3m - 6l - 2]_6$ if n is odd;
$I = \{g^*(v_{1,m-1}v_{1,m}), g^*(v_{2,m-1}v_{2,m}), g^*(v_{1,2l+m-1}v_{1,2l+m}), g^*(v_{2,2l+m-1}v_{2,2l+m})\}$
$= \{2, 3n - 3m + 3, 3n - 3m - 6l + 4, 1\}$.

Whether n is even or odd, $g^*(E(G))=A \cup B \cup C \cup D \cup E \cup F \cup G \cup H \cup I = [1, 3n - 2]$. Therefore, the g^* is a bijection from $E(G)$ to $[1, 3n - 2]$.

Case 2. when m is even:
We define the function $g : V(G) \to [0, 3n - 2]$ as follows:

$g(v_{1,2i+1}) = 2i$, if $i \in [0, \lceil \frac{m-1}{2} \rceil - 1]$ $[0, m - 2]_2$,
$g(v_{2,2i+1}) = 3n - 4i - 2$, if $i \in [0, \lceil \frac{m-1}{2} \rceil - 1]$ $[3n - 2m + 2, 3n - 2]_4$,
$g(v_{1,2i+2}) = 3n - 4i - 4$, if $i \in [0, \lfloor \frac{m-1}{2} \rfloor - 1]$ $[3n - 2m + 4, 3n - 4]_4$,
$g(v_{2,2i+2}) = 2i + 1$, if $i \in [0, \lfloor \frac{m-1}{2} \rfloor - 1]$ $[1, m - 3]_2$,
$g(v_{1,m+2i}) = m + 3i$, if $i \in [0, l-1]$ $[m, m + 3l - 3]_3$,
$g(v_{2,m+2i}) = m + 3i - 1$, if $i \in [0, l-1]$ $[m - 1, m + 3l - 4]_3$,
$g(v_{1,m+2i+1}) = 3n - 2m - 3i + 1$, if $i \in [0, l-1]$ $[3n - 2m - 3l + 4, 3n - 2m + 1]_3$,
$g(v_{2,m+2i+1}) = 3n - 2m - 3i - 1$, if $i \in [0, l-1]$ $[3n - 2m - 3l + 2, 3n - 2m - 1]_3$,
$g(v_{1,m+2l+2i}) = m + 3l + 2i$, if $i \in [0, \lceil \frac{n-2l-m+1}{2} \rceil - 1]$ V_1,
$g(v_{2,m+2l+2i}) = 3n - 2m - 3l - 4i + 3$, if $i \in [0, \lceil \frac{n-2l-m+1}{2} \rceil - 1]$ V_2,
$g(v_{1,m+2l+2i+1}) = 3n - 2m - 3l - 4i + 1$, if $i \in [0, \lfloor \frac{n-2l-m+1}{2} \rfloor - 1]$ V_3,
$g(v_{2,m+2l+2i+1}) = m + 3l + 2i + 1$, if $i \in [0, \lfloor \frac{n-2l-m+1}{2} \rfloor - 1]$ V_4.

Where
$V_1 = [m + 3l, n + l]_2$ if n is even, and $[m + 3l, n + l - 1]_2$ if n is odd;
$V_2 = [n+l+3, 3n-2m-3l+3]_4$ if n is even,and $[n+l+5, 3n-2m-3l+3]_4$ if n is odd;
$V_3 = [n+l+5, 3n-2m-3l+1]_4$ if n is even, and $[n+l+3, 3n-2m-3l+1]_4$ if n is odd;
$V_4 = [m + 3l + 1, n + l - 1]_2$ if n is even, and $[m + 3l + 1, n + l]_2$ if n is odd.

Whether n is even or odd, $g(V(G)) = [0, m - 2] \cup [m - 1, m + 3l - 4]_3 \cup [m, m + 3l - 3]_3 \cup [m + 3l, n + l] \cup [n + l + 3, 3n - 2m - 3l + 3]_2 \cup [3n - 2m - 3l + 2, 3n - 2m - 1]_3 \cup [3n - 2m - 3l + 4, 3n - 2m + 1]_3 \cup [3n - 2m + 2, 3n - 2]_2$. Since $|g(V(G))| = (m - 2) + 1 + 2((3l - 3)/3 + 1) + (n - m - 2l) + 1 + (2n - 2m - 4l)/2 + 1 + 2((3l - 3)/3 + 1) + (2m - 4)/2 + 1 = 2n$, the g is an injection from $V(G)$ to $[0, 3n - 2]$.

In the following, we show that the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. By the definition of the g, we can get the sets $A \sim I$ the same as Case 1 (2), except the following six sets:
$A = [3n-3m+4, 3n-2]_6$;
$B = [3n-3m+7, 3n-5]_6 \cup [3n-3m+8, 3n-4]_6 \cup [3n-3m+9, 3n-3]_6$;
$C = [3n-3m+6, 3n-6]_6 \cup [3n-3m+5, 3n-7]_6$;
$F = [6, 3n-3m-6l]_6 \cup [7, 3n-3m-6l+1]_6 \cup [8, 3n-3m-6l+2]_6$ if n is even, and $[3, 3n-3m-6l]_6 \cup [4, 3n-3m-6l+1]_6 \cup [5, 3n-3m-6l+2]_6$ if n is odd;
$G = [3, 3n-3m-6l+3]_6$ if n is even, and $[6, 3n-3m-6l+3]_6$ if n is odd;
$H = [5, 3n-3m-6l-1]_6 \cup [4, 3n-3m-6l-2]_6$ if n is even, and $[8, 3n-3m-6l-1]_6 \cup [7, 3n-3m-6l-2]_6$ if n is odd.

Whether n is even or odd, $g^*(E(G)) = A \cup B \cup C \cup D \cup E \cup F \cup G \cup H \cup I = [1, 3n-2]$. Therefore, the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. ■

Lemma 2.2. *When* $f = \prod_{k=0}^{l-1}(m+4k, m+4k+2)(m+4k+1, m+4k+3)$ *for positive integers* m *and* l *satisfying* $n \geq 2(m-1)+4l$*, the graph* $P(P_n, f)$ *is graceful.*

Proof. We distinguish between two cases to show this lemma.

Case 1. When m is odd,
(1) $m = 1$,
(i) $n = 4l$, we define the function $g: V(G) \to [0, 3n-2]$ as follows:

$g(v_{1,4i+1}) = 3n-6i-3$	if $i \in [0, l-1]$	$[6l+3, 12l-3]_6$,
$g(v_{2,4i+1}) = 6i+5$	if $i \in [0, l-1]$	$[5, 6l-1]_6$,
$g(v_{1,4i+2}) = 6i$	if $i \in [0, l-1]$	$[0, 6l-6]_6$,
$g(v_{2,4i+2}) = 3n-6i-5$	if $i \in [0, l-1]$	$[6l+1, 12l-5]_6$,
$g(v_{1,4i+3}) = 3n-6i-6$	if $i \in [0, l-1]$	$[6l, 12l-6]_6$,
$g(v_{2,4i+3}) = 6i+2$	if $i \in [0, l-1]$	$[2, 6l-4]_6$,
$g(v_{1,4i+4}) = 6i+3$	if $i \in [0, l-1]$	$[3, 6l-3]_6$,
$g(v_{2,4i+4}) = 3n-6i-2$	if $i \in [0, l-1]$	$[6l+4, 12l-2]_6$.

The label-set of vertices of the graph $P(P_n, f)$ is $g(V(G)) = [0, 6l-3]_3 \cup [2, 6l-1]_3 \cup [6l, 12l-3]_3 \cup [6l+1, 12l-2]_3$. Since $|g(V(G))| = 4((6l-3)/3+1) = 8l = 2n$, the g is an injection from $V(G)$ to $[0, 3n-2]$.

In the following, we show that the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. By the definition of the g, we have:

$$\begin{aligned} A &= \{g^*(v_{1,4i+1}v_{2,4i+3}), g^*(v_{1,4i+2}v_{2,4i+4}), g^*(v_{1,4i+3}v_{2,4i+1}),\\ &\quad g^*(v_{1,4i+4}v_{2,4i+2}) |\, i \in [0, l-1]\}\\ &= \{3n-12i-5, 3n-12i-2, 3n-12i-11, 3n-12i-8 |\ i \in [0, l-1]\}\\ &= [7, 3n-5]_{12} \cup [10, 3n-2]_{12} \cup [1, 3n-11]_{12} \cup [4, 3n-8]_{12};\\ B &= \{g^*(v_{1,4i+1}v_{1,4i+2}), g^*(v_{1,4i+2}v_{1,4i+3}), g^*(v_{1,4i+3}v_{1,4i+4}),\\ &\quad g^*(v_{2,4i+3}v_{2,4i+4}), g^*(v_{2,4i+2}v_{2,4i+3}), g^*(v_{2,4i+1}v_{2,4i+2}) |\ i \in [0, l-1]\}\\ &= \{3n-12i-3, 3n-12i-6, 3n-12i-9, 3n-12i-4, 3n-12i-7,\\ &\quad 3n-12i-10 |\ i \in [0, l-1]\}\\ &= [9, 3n-3]_{12} \cup [6, 3n-6]_{12} \cup [3, 3n-9]_{12} \cup [8, 3n-4]_{12} \cup [5, 3n-7]_{12}\\ &\quad \cup [2, 3n-10]_{12};\\ C &= \{g^*(v_{1,4i+4}v_{1,4i+5}), g^*(v_{2,4i+4}v_{2,4i+5}) |\ i \in [0, l-2]\}\\ &= \{3n-12i-12, 3n-12i-13 |\ i \in [0, l-2]\}\\ &= [12, 3n-12]_{12} \cup [11, 3n-13]_{12}. \end{aligned}$$

We can get $g^*(E(G)) = A \cup B \cup C = [1, 3n-2]$. Therefore, the g^* is a bijection from $E(G)$ to $[1, 3n-2]$.

(ii) $n = 4l+1$,

(a) $l = 1$, we define the function $g : V(G) \to [0, 3n-2]$ as follows:

$$\begin{array}{lllll} g(v_{1,1}) = 12, & g(v_{1,2}) = 0, & g(v_{1,3}) = 8, & g(v_{1,4}) = 5, & g(v_{1,5}) = 6,\\ g(v_{2,1}) = 3, & g(v_{2,2}) = 9, & g(v_{2,3}) = 2, & g(v_{2,4}) = 13, & g(v_{2,5}) = 4. \end{array}$$

Clearly, the g is an injection from $V(G)$ to $[0, 3n-2]$ and the g^* is a bijection from $E(G)$ to $[1, 3n-2]$.

(b) $l \geq 2$, we define the function $g :\ V(G) \to [0, 3n-2]$ as follows:

$$\begin{array}{lll} g(v_{1,4i+1}) = 3n-6i-3 & \text{if } i \in [0, l-2] & [6l+12, 12l]_6,\\ g(v_{2,4i+1}) = 6i+5 & \text{if } i \in [0, l-2] & [5, 6l-7]_6,\\ g(v_{1,4i+2}) = 6i & \text{if } i \in [0, l-1] & [0, 6l-6]_6,\\ g(v_{2,4i+2}) = 3n-6i-5 & \text{if } i \in [0, l-2] & [6l+10, 12l-2]_6,\\ g(v_{1,4i+3}) = 3n-6i-6 & \text{if } i \in [0, l-2] & [6l+9, 12l-3]_6,\\ g(v_{2,4i+3}) = 6i+2 & \text{if } i \in [0, l-2] & [2, 6l-10]_6,\\ g(v_{1,4i+4}) = 6i+3 & \text{if } i \in [0, l-1] & [3, 6l-3]_6,\\ g(v_{2,4i+4}) = 3n-6i-2 & \text{if } i \in [0, l-1] & [6l+7, 12l+1]_6,\\ g(v_{1,n-4}) = 6l+5, & g(v_{2,n-4}) = 6l-2, & g(v_{2,n-3}) = 6l+3,\\ g(v_{1,n-2}) = 6l+1, & g(v_{2,n-2}) = 6l-5, & g(v_{1,n}) = 6l+6,\\ g(v_{2,n}) = 6l+8. & & \end{array}$$

The label-set of vertices of the graph $P(P_n, f)$ is $g(V(G)) = [0, 6l-3]_3 \cup [2, 6l-7]_3 \cup [6l+7, 12l+1]_3 \cup [6l+9, 12l]_3 \cup \{6l-5, 6l-2, 6l+1,$

$6l+3, 6l+5, 6l+6, 6l+8\}$. Since $|g(V(G))| = (6l-3)/3+1+2((6l-9)/3+1)+(6l-6)/3+1+7 = 8l+2 = 2n$, the g is an injection from $V(G)$ to $[0, 3n-2]$.

In the following, we show that the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. By the definition of the g, we have:

$$\begin{aligned} A &= \{g^*(v_{1,4i+1}v_{2,4i+3}), g^*(v_{1,4i+2}v_{2,4i+4}), g^*(v_{1,4i+3}v_{2,4i+1}), g^*(v_{1,4i+4}v_{2,4i+2})| \\ &\quad i \in [0, l-2]\} \\ &= \{3n-12i-5, 3n-12i-2, 3n-12i-11, 3n-12i-8|\ i \in [0, l-2]\} \\ &= [22, 3n-5]_{12} \cup [25, 3n-2]_{12} \cup [16, 3n-11]_{12} \cup [19, 3n-8]_{12}; \end{aligned}$$

$$\begin{aligned} B &= \{g^*(v_{1,4i+1}v_{1,4i+2}), g^*(v_{1,4i+2}v_{1,4i+3}), g^*(v_{1,4i+3}v_{1,4i+4}), \\ &\quad g^*(v_{1,4i+4}v_{1,4i+5}), g^*(v_{2,4i+3}v_{2,4i+4}), g^*(v_{2,4i+2}v_{2,4i+3}), \\ &\quad g^*(v_{2,4i+1}v_{2,4i+2}), g^*(v_{2,4i+4}v_{2,4i+5})|\ i \in [0, l-2]\} \\ &= \{3n-12i-3, 3n-12i-6, 3n-12i-9, 3n-12i-12, 3n-12i-4, \\ &\quad 3n-12i-7, 3n-12i-10, 3n-12i-13|\ i \in [0, l-2]\} \\ &= [24, 3n-3]_{12} \cup [21, 3n-6]_{12} \cup [18, 3n-9]_{12} \cup [15, 3n-12]_{12} \cup \\ &\quad [23, 3n-4]_{12} \cup [20, 3n-7]_{12} \cup [17, 3n-10]_{12} \cup [14, 3n-13]_{12}; \end{aligned}$$

$$\begin{aligned} C &= \{g^*(v_{1,n-4}v_{2,n-2}), g^*(v_{1,n-3}v_{2,n-1}), g^*(v_{1,n-2}v_{2,n-4}), \\ &\quad g^*(v_{1,n-1}v_{2,n-3}), g^*(v_{1,n}v_{2,n}), g^*(v_{1,n-4}v_{1,n-3}), g^*(v_{1,n-3}v_{1,n-2}), \\ &\quad g^*(v_{1,n-2}v_{1,n-1}), g^*(v_{1,n-1}v_{1,n}), g^*(v_{2,n-4}v_{2,n-3}), g^*(v_{2,n-3}v_{2,n-2}), \\ &\quad g^*(v_{2,n-2}v_{2,n-1}), g^*(v_{2,n-1}v_{2,n})\} \\ &= \{10, 13, 3, 6, 2, 11, 7, 4, 9, 5, 8, 12, 1\} = [1, 13]. \end{aligned}$$

We can get $g^*(E(G)) = A \cup B \cup C = [1, 3n-2]$. Therefore, the g^* is a bijection from $E(G)$ to $[1, 3n-2]$.

(iii) $n \geq 4l+2$, we define the function $g:\ V(G) \to [0, 3n-2]$ as follows:

$$\begin{array}{lll} g(v_{1,4i+1}) = 3n-6i-3 & \text{if } i \in [0, l-1] & [3n-6l+3, 3n-3]_6, \\ g(v_{2,4i+1}) = 6i+5 & \text{if } i \in [0, l-1] & [5, 6l-1]_6, \\ g(v_{1,4i+2}) = 6i & \text{if } i \in [0, l-1] & [0, 6l-6]_6, \\ g(v_{2,4i+2}) = 3n-6i-5 & \text{if } i \in [0, l-2] & [3n-6l+7, 3n-5]_6, \\ g(v_{1,4i+3}) = 3n-6i-6 & \text{if } i \in [0, l-2] & [3n-6l+6, 3n-6]_6, \\ g(v_{2,4i+3}) = 6i+2 & \text{if } i \in [0, l-1] & [2, 6l-4]_6, \\ g(v_{1,4i+4}) = 6i+3 & \text{if } i \in [0, l-1] & [3, 6l-3]_6, \\ g(v_{2,4i+4}) = 3n-6i-2 & \text{if } i \in [0, l-1] & [3n-6l+4, 3n-2]_6, \end{array}$$

$$\begin{array}{lll} g(v_{1,4l+2i+1}) = 3n-6l-2i-4 & \text{if } i \in [0, \lceil \frac{n-4l}{2} \rceil - 1] & V_1, \\ g(v_{2,4l+2i+1}) = 6l+4i-2 & \text{if } i \in [0, \lceil \frac{n-4l}{2} \rceil - 1] & V_2, \\ g(v_{1,4l+2i+2}) = 6l+4i & \text{if } i \in [0, \lfloor \frac{n-4l}{2} \rfloor - 1] & V_3, \\ g(v_{2,4l+2i+2}) = 3n-6l-2i-5 & \text{if } i \in [0, \lfloor \frac{n-4l}{2} \rfloor - 1] & V_4, \\ g(v_{1,4l-1}) = 3n-6l-1, & g(v_{2,4l-2}) = 3n-6l. & \end{array}$$

Where
$V_1 = [2n-2l-2, 3n-6l-4]_2$ if n is even, and $[2n-2l-3, 3n-6l-4]_2$ if n is odd;
$V_2 = [6l-2, 2n-2l-6]_4$ if n is even, and $[6l-2, 2n-2l-4]_4$ if n is odd;
$V_3 = [6l, 2n-2l-4]_4$ if n is even, and $[6l, 2n-2l-6]_4$ if n is odd;
$V_4 = [2n-2l-3, 3n-6l-5]_2$ if n is even, and $[2n-2l-2, 3n-6l-5]_2$ if n is odd.

Whether n is even or odd, the label-set of vertices of the graph $P(P_n, f)$ is $g(V(G)) = [0, 6l-3]_3 \cup [2, 6l-1]_3 \cup [6l-2, 2n-2l-4]_2 \cup [2n-2l-3, 3n-6l-4]_1 \cup [3n-6l+3, 3n-3]_3 \cup [3n-6l+4, 3n-2]_3 \cup \{3n-6l-1, 3n-6l\}$. Since $|g(V(G))| = 2((6l-3)/3+1) + (2n-8l-2)/2 + 1 + (n-4l-1) + 1 + 2((6l-6)/3+1) + 2 = 2n$, the g is an injection from $V(G)$ to $[0, 3n-2]$.

In the following, we show that the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. By the definition of the g, we have:

$$\begin{aligned} A &= \{g^*(v_{1,4i+1}v_{2,4i+3}), g^*(v_{1,4i+2}v_{2,4i+4}), g^*(v_{1,4i+1}v_{1,4i+2}), \\ &\quad g^*(v_{2,4i+3}v_{2,4i+4}) |\ i \in [0, l-1]\} \\ &= \{3n-12i-5, 3n-12i-2, 3n-12i-3, 3n-12i-4 |\ i \in [0, l-1]\} \\ &= [3n-12l+7, 3n-5]_{12} \cup [3n-12l+10, 3n-2]_{12}, \cup \\ &\quad [3n-12l+9, 3n-3]_{12} \cup [3n-12l+8, 3n-4]_{12}; \end{aligned}$$

$$\begin{aligned} B &= \{g^*(v_{1,4i+3}v_{2,4i+1}), g^*(v_{1,4i+4}v_{2,4i+2}), g^*(v_{1,4i+2}v_{1,4i+3}), \\ &\quad g^*(v_{1,4i+3}v_{1,4i+4}), g^*(v_{1,4i+4}v_{1,4i+5}), g^*(v_{2,4i+2}v_{2,4i+3}), \\ &\quad g^*(v_{2,4i+1}v_{2,4i+2}), g^*(v_{2,4i+4}v_{2,4i+5}) |\ i \in [0, l-2]\} \\ &= \{3n-12i-11, 3n-12i-8, 3n-12i-6, 3n-12i-9, \\ &\quad 3n-12i-12, 3n-12i-7, 3n-12i-10, 3n-12i-13 |\ i \in [0, l-2]\} \\ &= [3n-12l+13, 3n-11]_{12} \cup [3n-12l+16, 3n-8]_{12} \cup [3n-12l+18, \\ &\quad 3n-6]_{12} \cup [3n-12l+15, 3n-9]_{12} \cup [3n-12l+12, 3n-12]_{12} \cup \\ &\quad [3n-12l+17, 3n-7]_{12} \cup [3n-12l+14, 3n-10]_{12} \cup [3n-12l+11, 3n-13]_{12}; \end{aligned}$$

$$\begin{aligned} C &= \{g^*(v_{1,4l+2i+1}v_{2,4l+2i+1}) |\ i \in [0, \lceil \tfrac{n-4l}{2} \rceil - 1]\} \\ &= \{3n-12l-6i-2 |\ i \in [0, \lceil \tfrac{n-4l}{2} \rceil - 1]\} \\ &= [4, 3n-12l-2]_6 \text{ if } n \text{ is even, and } [1, 3n-12l-2]_6 \text{ if } n \text{ is odd;} \end{aligned}$$

$$\begin{aligned} D &= \{g^*(v_{1,4l+2i+2}v_{2,4l+2i+2}), g^*(v_{1,4l+2i+1}v_{1,4l+2i+2}), \\ &\quad g^*(v_{2,4l+2i+1}v_{2,4l+2i+2}) |\ i \in [0, \lfloor \tfrac{n-4l}{2} \rfloor - 1]\} \end{aligned}$$

$= \{3n-12l-6i-5, 3n-12l-6i-4, 3n-12l-6i-3|\ i \in [0, \lfloor\frac{n-4l}{2}\rfloor-1]\}$
$= [1, 3n-12l-5]_6 \cup [2, 3n-12l-4]_6 \cup [3, 3n-12l-3]_6$ if n is even,
and $[4, 3n-12l-5]_6 \cup [5, 3n-12l-4]_6 \cup [6, 3n-12l-3]_6$ if n is odd;

$E = \{g^*(v_{1,4l+2i+2}v_{1,4l+2i+3}), g^*(v_{2,4l+2i+2}v_{2,4l+2i+3})|\ i \in [0, \lceil\frac{n-4l}{2}\rceil-2]\}$
$= \{3n-12l-6i-6, 3n-12l-6i-7|\ i \in [0, \lceil\frac{n-4l}{2}\rceil-2]\}$
$= [6, 3n-12l-6]_6 \cup [5, 3n-12l-7]_6$ if n is even,
and $[3, 3n-12l-6]_6 \cup [2, 3n-12l-7]_6$ if n is odd;

$F = \{g^*(v_{1,4l}v_{1,4l+1}), g^*(v_{1,4l-1}v_{2,4l-3}), g^*(v_{2,4l-3}v_{2,4l-2}), g^*(v_{1,4l-1}v_{1,4l}),$
$g^*(v_{1,4l}v_{2,4l-2}), g^*(v_{2,4l-2}v_{2,4l-1}), g^*(v_{1,4l-2}v_{1,4l-1}), g^*(v_{2,4l}v_{2,4l+1})\}$
$= \{3n-12l-1, 3n-12l, 3n-12l+1, 3n-12l+2, 3n-12l+3, 3n-$
$12l+4, 3n-12l+5, 3n-12l+6\}$
$= [3n-12l-1, 3n-12l+6]$.

Whether n is even or odd, $g^*(E(G))$=$A \cup B \cup C \cup D \cup E \cup F = [1, 3n-2]$. Therefore, the g^* is a bijection from $E(G)$ to $[1, 3n-2]$.

(2) $m \geq 3$, we define the function $g : V(G) \to [0, 3n-2]$ as follows:

$g(v_{1,2i+1}) = 3n-4i-2$, if $i \in [0, \lceil\frac{m-1}{2}\rceil-1]$ $[3n-2m+4, 3n-2]_4$,
$g(v_{2,2i+1}) = 2i$, if $i \in [0, \lceil\frac{m-1}{2}\rceil-1]$ $[0, m-3]_2$,
$g(v_{1,2i+2}) = 2i+1$, if $i \in [0, \lfloor\frac{m-1}{2}\rfloor-1]$ $[1, m-2]_2$,
$g(v_{2,2i+2}) = 3n-4i-4$, if $i \in [0, \lfloor\frac{m-1}{2}\rfloor-1]$ $[3n-2m+2, 3n-4]_4$,
$g(v_{1,m+4i}) = 3n-2m-6i-1$, if $i \in [1, l-1]$ $[3n-2m-6l+5, 3n-2m-7]_6$,
$g(v_{2,m+4i}) = m+6i+4$, if $i \in [0, l-1]$ $[m+4, m+6l-2]_6$,
$g(v_{1,m+4i+1}) = m+6i-1$, if $i \in [0, l-1]$ $[m-1, m+6l-7]_6$,
$g(v_{2,m+4i+1}) = 3n-2m-6i-3$, if $i \in [0, l-2]$ $[3n-2m-6l+9, 3n-2m-3]_6$,
$g(v_{1,m+4i+2}) = 3n-2m-6i-4$, if $i \in [0, l-2]$ $[3n-2m-6l+8, 3n-2m-4]_6$,
$g(v_{2,m+4i+2}) = m+6i+1$, if $i \in [0, l-1]$ $[m+1, m+6l-5]_6$,
$g(v_{1,m+4i+3}) = m+6i+2$, if $i \in [0, l-1]$ $[m+2, m+6l-4]_6$,
$g(v_{2,m+4i+3}) = 3n-2m-6i$, if $i \in [0, l-1]$ $[3n-2m-6l+6, 3n-2m]_6$,
$g(v_{1,m+4l+2i}) = 3n-2m-6l-2i-2$, if $i \in [0, \lceil\frac{n-4l-m+1}{2}\rceil-1]$ V_1,
$g(v_{2,m+4l+2i}) = m+6l+4i-3$, if $i \in [0, \lceil\frac{n-4l-m+1}{2}\rceil-1]$ V_2,
$g(v_{1,m+4l+2i+1}) = m+6l+4i-1$, if $i \in [0, \lfloor\frac{n-4l-m+1}{2}\rfloor-1]$ V_3,
$g(v_{2,m+4l+2i+1}) = 3n-2m-6l-2i-3$, if $i \in [0, \lfloor\frac{n-4l-m+1}{2}\rfloor-1]$ V_4,
$g(v_{1,m}) = 3n-2m+1$,
$g(v_{2,m+4l-3}) = 3n-2m-6l+2$,
$g(v_{1,m+4l-2}) = 3n-2m-6l+1$.

Where
$V_1 = [2n-m-2l-1, 3n-2m-6l-2]_2$ if n is even,
and $[2n-m-2l-2, 3n-2m-6l-2]_2$ if n is odd;

$V_2 = [m+6l-3, 2n-m-2l-5]_4$ if n is even, and $[m+6l-3, 2n-m-2l-3]_4$ if n is odd;
$V_3 = [m+6l-1, 2n-m-2l-3]_4$ if n is even, and $[m+6l-1, 2n-m-2l-5]_4$ if n is odd;
$V_4 = [2n-m-2l-2, 3n-2m-6l-3]_2$ if n is even, and $[2n-m-2l-1, 3n-2m-6l-3]_2$ if n is odd.

Whether n is even or odd, the label-set of vertices of the graph $P(P_n, f)$ is $g(V(G)) = [0, m-2]_1 \cup [m-1, m+6l-4]_3 \cup [m+1, m+6l-2]_3 \cup [m+6l-3, 2n-m-2l-3]_2 \cup [2n-m-2l-2, 3n-2m-6l-2]_1 \cup [3n-2m-6l+5, 3n-2m-4]_3 \cup [3n-2m-6l+6, 3n-2m]_3 \cup [3n-2m+2, 3n-2]_2 \cup \{3n-2m+1, 3n-2m-6l+2, 3n-2m-6l+1\}$. Since $|g(V(G))| = (m-2)+1+2((6l-3)/3+1)+(2n-2m-8l)/2+1+(n-m-4l)+1+(6l-9)/3+1+(6l-6)/3+1+(2m-4)/2+1+3 = 2n$, the g is an injection from $V(G)$ to $[0, 3n-2]$.

In the following, we show that the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. By the definition of the g, we have:

$$\begin{aligned} A &= \{g^*(v_{1,2i+1}v_{2,2i+1}) \mid i \in [0, \lceil \tfrac{m-1}{2} \rceil - 1]\} \\ &= \{3n-6i-2 \mid i \in [0, \lceil \tfrac{m-1}{2} \rceil - 1]\} \\ &= [3n-3m+7, 3n-2]_6; \end{aligned}$$

$$\begin{aligned} B &= \{g^*(v_{1,2i+2}v_{2,2i+2}), g^*(v_{1,2i+1}v_{1,2i+2}), g^*(v_{2,2i+1}v_{2,2i+2}) \mid \\ &\quad i \in [0, \lfloor \tfrac{m-1}{2} \rfloor - 1]\} \\ &= \{3n-6i-5, 3n-6i-3, 3n-6i-4 \mid i \in [0, \lfloor \tfrac{m-1}{2} \rfloor - 1]\} \\ &= [3n-3m+4, 3n-5]_6 \cup [3n-3m+6, 3n-3]_6 \cup [3n-3m+5, 3n-4]_6; \end{aligned}$$

$$\begin{aligned} C &= \{g^*(v_{1,2i+2}v_{1,2i+3}), g^*(v_{2,2i+2}v_{2,2i+3}) \mid i \in [0, \lceil \tfrac{m-1}{2} \rceil - 2]\} \\ &= \{3n-6i-7, 3n-6i-6 \mid i \in [0, \lceil \tfrac{m-1}{2} \rceil - 2]\} \\ &= [3n-3m+8, 3n-7]_6 \cup [3n-3m+9, 3n-6]_6; \end{aligned}$$

$$\begin{aligned} D &= \{g^*(v_{1,m+4i}v_{2,m+4i+2}), g^*(v_{1,m+4i+1}v_{2,m+4i+3}), g^*(v_{1,m+4i}v_{1,m+4i+1}), \\ &\quad g^*(v_{2,m+4i+2}v_{2,m+4i+3}) \mid i \in [1, l-1]\} \\ &= \{3n-3m-12i-2, 3n-3m-12i+1, 3n-3m-12i, 3n-3m-12i-1 \mid \\ &\quad i \in [1, l-1]\} \\ &= [3n-3m-12l+10, 3n-3m-14]_{12} \cup [3n-3m-12l+13, 3n-3m-11]_{12} \cup \\ &\quad [3n-3m-12l+12, 3n-3m-12]_{12} \cup [3n-3m-12l+11, 3n-3m-13]_{12}; \end{aligned}$$

$$\begin{aligned} E &= \{g^*(v_{1,m+4i+2}v_{2,m+4i}), g^*(v_{1,m+4i+3}v_{2,m+4i+1}), g^*(v_{1,m+4i+1}v_{1,m+4i+2}), \\ &\quad g^*(v_{2,m+4i+1}v_{2,m+4i+2}), g^*(v_{1,m+4i+2}v_{1,m+4i+3}), g^*(v_{2,m+4i}v_{2,m+4i+1}), \\ &\quad g^*(v_{1,m+4i+3}v_{1,m+4i+4}), g^*(v_{2,m+4i+3}v_{2,m+4i+4}) \mid i \in [0, l-2]\} \\ &= \{3n-3m-12i-8, 3n-3m-12i-5, 3n-3m-12i-3, 3n-3m-12i-4, \\ &\quad 3n-3m-12i-6, 3n-3m-12i-7, 3n-3m-12i-9, 3n-3m-12i-10 \mid \end{aligned}$$

$i \in [0, l-2]\}$
$= [3n-3m-12l+16, 3n-3m-8]_{12} \cup [3n-3m-12l+19, 3n-3m-5]_{12} \cup$
$[3n-3m-12l+21, 3n-3m-3]_{12} \cup [3n-3m-12l+20, 3n-3m-4]_{12} \cup$
$[3n-3m-12l+18, 3n-3m-6]_{12} \cup [3n-3m-12l+17, 3n-3m-7]_{12} \cup$
$[3n-3m-12l+15, 3n-3m-9]_{12} \cup [3n-3m-12l+14, 3n-3m-10]_{12}$;

$F = \{g^*(v_{1,m+4l+2i}v_{2,m+4l+2i}) | \ i \in [0, \lceil \frac{n-4l-m+1}{2} \rceil - 1]\}$
$= \{3n-3m-12l-6i+1 | \ i \in [0, \lceil \frac{n-4l-m+1}{2} \rceil - 1]\}$
$= [4, 3n-3m-12l+1]_6$ if n is even, and $[1, 3n-3m-12l+1]_6$ if n is odd;

$G = \{g^*(v_{1,m+4l+2i+1}v_{2,m+4l+2i+1}), g^*(v_{1,m+4l+2i}v_{1,m+4l+2i+1}),$
$g^*(v_{2,m+4l+2i}v_{2,m+4l+2i+1}) | \ i \in [0, \lfloor \frac{n-4l-m+1}{2} \rfloor - 1]\}$
$= \{3n-3m-12l-6i-2, 3n-3m-12l-6i-1, 3n-3m-12l-6i |$
$i \in [0, \lfloor \frac{n-4l-m+1}{2} \rfloor - 1]\}$
$= [1, 3n-3m-12l-2]_6 \cup [2, 3n-3m-12l-1]_6 \cup [3, 3n-3m-12l]_6$ if n is even,
and $[4, 3n-3m-12l-2]_6 \cup [5, 3n-3m-12l-1]_6 \cup [6, 3n-3m-12l]_6$ if n is odd;

$H = \{g^*(v_{1,m+4l+2i+1}v_{1,m+4l+2i+2}), g^*(v_{2,m+4l+2i+1}v_{2,m+4l+2i+2}) |$
$i \in [0, \lceil \frac{n-4l-m+1}{2} \rceil - 2]\}$
$= \{3n-3m-12l-6i-3, 3n-3m-12l-6i-4 | \ i \in [0, \lceil \frac{n-4l-m+1}{2} \rceil - 2]\}$
$= [6, 3n-3m-12l-3]_6 \cup [5, 3n-3m-12l-4]_6$ if n is even,
and $[3, 3n-3m-12l-3]_6 \cup [2, 3n-3m-12l-4]_6$ if n is odd;

$I = \{g^*(v_{1,m}v_{2,m+2}), g^*(v_{1,m+1}v_{2,m+3}), g^*(v_{1,m+4l-2}v_{2,m+4l-4}),$
$g^*(v_{1,m+4l-1}v_{2,m+4l-3}), g^*(v_{1,m-1}v_{1,m}), g^*(v_{1,m}v_{1,m+1}),$
$g^*(v_{1,m+4l-3}v_{1,m+4l-2}), g^*(v_{1,m+4l-2}v_{1,m+4l-1}), g^*(v_{1,m+4l-1}v_{1,m+4l}),$
$g^*(v_{2,m-1}v_{2,m}), g^*(v_{2,m+2}v_{2,m+3}), g^*(v_{2,m+4l-4}v_{2,m+4l-3}),$
$g^*(v_{2,m+4l-3}v_{2,m+4l-2}), g^*(v_{2,m+4l-1}v_{2,m+4l})\}$
$= \{3n-3m, 3n-3m+1, 3n-3m-12l+3, 3n-3m-12l+6, 3n-3m+3, 3n-3m+2, 3n-3m-12l+8, 3n-3m-12l+5, 3n-3m-12l+2, 3n-3m-2, 3n-3m-1, 3n-3m-12l+4, 3n-3m-12l+7, 3n-3m-12l+9\} = [3n-3m-12l+2, 3n-3m-12l+9] \cup [3n-3m-2, 3n-3m+3]$.

Whether n is even or odd, $g^*(E(G)) = A \cup B \cup C \cup D \cup E \cup F \cup G \cup H \cup I = [1, 3n-2]$. Therefore, the g^* is a bijection from $E(G)$ to $[1, 3n-2]$.

Case 2. When m is even,

(1) except the case $m = 2$ and $n = 4l+2$, we define the function $g : V(G) \rightarrow [0, 3n-2]$ as follows:

$g(v_{1,2i+1}) = 3n-4i-2,$ if $i \in [0, \lceil \frac{m-1}{2} \rceil - 1]$ $[3n-2m+2, 3n-2]_4,$
$g(v_{2,2i+1}) = 2i,$ if $i \in [0, \lceil \frac{m-1}{2} \rceil - 1]$ $[0, m-2]_2,$
$g(v_{1,2i+2}) = 2i+1,$ if $i \in [0, \lfloor \frac{m-1}{2} \rfloor - 1]$ $[1, m-3]_2,$
$g(v_{2,2i+2}) = 3n-4i-4,$ if $i \in [0, \lfloor \frac{m-1}{2} \rfloor - 1]$ $[3n-2m+4, 3n-4]_4,$
$g(v_{1,m+4i}) = m+6i+1,$ if $i \in [1, l-1]$ $[m+7, m+6l-5]_6,$
$g(v_{2,m+4i}) = 3n-2m-6i-4,$ if $i \in [0, l-1]$ $[3n-2m-6l+2, 3n-2m-4]_6,$
$g(v_{1,m+4i+1}) = 3n-2m-6i+1,$ if $i \in [0, l-1]$ $[3n-2m-6l+7, 3n-2m+1]_6,$
$g(v_{2,m+4i+1}) = m+6i+3,$ if $i \in [0, l-2]$ $[m+3, m+6l-9]_6,$
$g(v_{1,m+4i+2}) = m+6i+4,$ if $i \in [0, l-2]$ $[m+4, m+6l-8]_6,$
$g(v_{2,m+4i+2}) = 3n-2m-6i-1,$ if $i \in [0, l-1]$ $[3n-2m-6l+5, 3n-2m-1]_6,$
$g(v_{1,m+4i+3}) = 3n-2m-6i-2,$ if $i \in [0, l-1]$ $[3n-2m-6l+4, 3n-2m-2]_6,$
$g(v_{2,m+4i+3}) = m+6i,$ if $i \in [0, l-1]$ $[m, m+6l-6]_6,$
$g(v_{1,m+4l+2i}) = m+6l+2i+2$ if $i \in [0, \lceil \frac{n-4l-m+1}{2} \rceil - 1]$ $V_1,$
$g(v_{2,m+4l+2i}) = 3n-2m-6l-4i+3$ if $i \in [0, \lceil \frac{n-4l-m+1}{2} \rceil - 1]$ $V_2,$
$g(v_{1,m+4l+2i+1}) = 3n-2m-6l-4i+1$ if $i \in [0, \lfloor \frac{n-4l-m+1}{2} \rfloor - 1]$ $V_3,$
$g(v_{2,m+4l+2i+1}) = m+6l+2i+3$ if $i \in [0, \lfloor \frac{n-4l-m+1}{2} \rfloor - 1]$ $V_4,$
$g(v_{1,m}) = m-1,$ $g(v_{2,m+4l-3}) = m+6l-2,$ $g(v_{1,m+4l-2}) = m+6l-1.$

Where
$V_1 = [m+6l+2, n+2l+2]_2$ if n is even, and $[m+6l+2, n+2l+1]_2$ if n is odd;
$V_2 = [n+2l+3, 3n-2m-6l+3]_4$ if n is even, and $[n+2l+5, 3n-2m-6l+3]_4$ if n is odd;
$V_3 = [n+2l+5, 3n-2m-6l+1]_4$ if n is even, and $[n+2l+3, 3n-2m-6l+1]_4$ if n is odd;
$V_4 = [m+6l+3, n+2l+1]_2$ if n is even, and $[m+6l+3, n+2l+2]_2$ if n is odd.
Whether n is even or odd, the label-set of vertices of the graph $P(P_n, f)$ is $g(V(G)) =[0, m-2]_1 \cup [m, m+6l-6]_3 \cup [m+4, m+6l-5]_3 \cup [m+6l+2, n+2l+2]_1 \cup [n+2l+3, 3n-2m-6l+3]_2 \cup [3n-2m-6l+2, 3n-2m-1]_3 \cup [3n-2m-6l+4, 3n-2m+1]_3 \cup [3n-2m+2, 3n-2]_2 \cup \{m-1, m+6l-2, m+6l-1\}$. Since $|g(V(G))| = (m-2)+1+(6l-6)/3+1+(6l-9)/3+1+(n-m-4l)+1+(2n-2m-8l)/2+1+2((6l-3)/3+1)+(2m-4)/2+1+3 = 2n$, the g is an injection from $V(G)$ to $[0, 3n-2]$.

In the following, we show that the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. By the definition of the g, we can get the sets $A \sim I$ the same as Case 1 (2), except the following six sets:
$A = [3n-3m+4, 3n-2]_6;$
$B = [3n-3m+7, 3n-5]_6 \cup [3n-3m+9, 3n-3]_6 \cup [3n-3m+8, 3n-4]_6;$
$C = [3n-3m+5, 3n-7]_6 \cup [3n-3m+6, 3n-6]_6;$

$F = [1, 3n - 3m - 12l + 1]_6$ if n is even , and $[4, 3n - 3m - 12l + 1]_6$ if n is odd;
$G = [4, 3n - 3m - 12l - 2]_6 \cup [5, 3n - 3m - 12l - 1]_6 \cup [6, 3n - 3m - 12l]_6$ if n is even,
and $[1, 3n - 3m - 12l - 2]_6 \cup [2, 3n - 3m - 12l - 1]_6 \cup [3, 3n - 3m - 12l]_6$ if n is odd;
$H = [3, 3n - 3m - 12l - 3]_6 \cup [2, 3n - 3m - 12l - 4]_6$ if n is even, and $[6, 3n - 3m - 12l - 3]_6 \cup [5, 3n - 3m - 12l - 4]_6$ if n is odd.
Whether n is even or odd, $g^*(E(G)) = A \cup B \cup C \cup D \cup E \cup F \cup G \cup H \cup I = [1, 3n - 2]$. Therefore, the g^* is a bijection from $E(G)$ to $[1, 3n - 2]$.

(2) $m = 2$ and $n = 4l + 2$,

(a) $l = 1$, we define the function $g : V(G) \to [0, 3n - 2]$ as follows:
$g(v_{1,1}) = 16$, $g(v_{1,2}) = 1$, $g(v_{1,3}) = 15$, $g(v_{1,4}) = 7$, $g(v_{1,5}) = 12$, $g(v_{1,6}) = 3$, $g(v_{2,1}) = 0$, $g(v_{2,2}) = 10$, $g(v_{2,3}) = 6$, $g(v_{2,4}) = 13$, $g(v_{2,5}) = 2$, $g(v_{2,6}) = 4$.

Clearly, the g is an injection from $V(G)$ to $[0, 3n - 2]$ and the g^* is a bijection from $E(G)$ to $[1, 3n - 2]$.

(b) $l \geq 2$, we define the function $g : V(G) \to [0, 3n - 2]$ as follows:

$g(v_{1,4i+2}) = 6i + 3$	if $i \in [1, l - 2]$	$[9, 6l - 9]_6$,
$g(v_{2,4i+2}) = 3n - 6i - 8$	if $i \in [0, l - 2]$	$[6l + 10, 12l - 2]_6$,
$g(v_{1,4i+3}) = 3n - 6i - 3$	if $i \in [0, l - 1]$	$[6l + 9, 12l + 3]_6$,
$g(v_{2,4i+3}) = 6i + 5$	if $i \in [0, l - 2]$	$[5, 6l - 7]_6$,
$g(v_{1,4i+4}) = 6i + 6$	if $i \in [0, l - 2]$	$[6, 6l - 6]_6$,
$g(v_{2,4i+4}) = 3n - 6i - 5$	if $i \in [0, l - 2]$	$[6l + 13, 12l + 1]_6$,
$g(v_{1,4i+5}) = 3n - 6i - 6$	if $i \in [0, l - 1]$	$[6l + 6, 12l]_6$,
$g(v_{2,4i+5}) = 6i + 2$	if $i \in [1, l - 1]$	$[8, 6l - 4]_6$,
$g(v_{1,1}) = 3n - 2$	$g(v_{2,1}) = 0$	$g(v_{1,2}) = 1$
$g(v_{2,5}) = 2$	$g(v_{1,n-4}) = 6l - 2$	$g(v_{2,n-4}) = 6l + 5$
$g(v_{2,n-3}) = 6l$	$g(v_{1,n-2}) = 6l + 2$	$g(v_{2,n-2}) = 6l + 8$
$g(v_{1,n}) = 6l - 3$	$g(v_{2,n}) = 6l - 5$.	

The label-set of vertices of the graph $P(P_n, f)$ is $g(V(G)) = [5, 6l - 4]_3 \cup [6, 6l - 6]_3 \cup [6l + 6, 12l + 3]_3 \cup [6l + 10, 12l + 1]_3 \cup \{0, 1, 2, 6l - 5, 6l - 3, 6l - 2, 6l, 6l + 2, 6l + 5, 6l + 8, 12l + 4\}$. Since $|g(V(G))| = 2((6l - 9)/3 + 1) + (6l - 12)/3 + 1 + (6l - 3)/3 + 1 + 11 = 8l + 4 = 2(4l + 2) = 2n$, the g is an injection from $V(G)$ to $[0, 3n - 2]$.

In the following, we show that the g^* is a bijection from $E(G)$ to $[1, 3n - 2]$. By the definition of the g, we have:

$$\begin{aligned}
A &= \{g^*(v_{1,4i+2}v_{2,4i+4}), g^*(v_{1,4i+3}v_{2,4i+5}), g^*(v_{1,4i+2}v_{1,4i+3}),\\
&\quad g^*(v_{2,4i+4}v_{2,4i+5}) |\ i \in [1, l-2]\}\\
&= \{3n-12i-8, 3n-12i-5, 3n-12i-6, 3n-12i-7 |\ i \in [1, l-2]\}\\
&= [22, 3n-20]_{12} \cup [25, 3n-17]_{12} \cup [24, 3n-18]_{12} \cup [23, 3n-19]_{12};
\end{aligned}$$

$$\begin{aligned}
B &= \{g^*(v_{1,4i+4}v_{2,4i+2}), g^*(v_{1,4i+5}v_{2,4i+3}), g^*(v_{1,4i+3}v_{1,4i+4}),\\
&\quad g^*(v_{1,4i+4}v_{1,4i+5}), g^*(v_{1,4i+5}v_{1,4i+6}), g^*(v_{2,4i+3}v_{2,4i+4}),\\
&\quad g^*(v_{2,4i+2}v_{2,4i+3}), g^*(v_{2,4i+5}v_{2,4i+6}) |\ i \in [0, l-2]\}\\
&= \{3n-12i-14, 3n-12i-11, 3n-12i-9, 3n-12i-12, 3n-12i-15,\\
&\quad 3n-12i-10, 3n-12i-13, 3n-12i-16 |\ i \in [0, l-2]\}\\
&= [16, 3n-14]_{12} \cup [19, 3n-11]_{12} \cup [21, 3n-9]_{12} \cup [18, 3n-12]_{12} \cup\\
&\quad [15, 3n-15]_{12} \cup [20, 3n-10]_{12} \cup [17, 3n-13]_{12} \cup [14, 3n-16]_{12};
\end{aligned}$$

$$\begin{aligned}
C &= \{g^*(v_{1,1}v_{2,1}), g^*(v_{1,2}v_{2,4}), g^*(v_{1,3}v_{2,5}), g^*(v_{1,1}v_{1,2}), g^*(v_{1,2}v_{1,3}),\\
&\quad g^*(v_{2,1}v_{2,2}), g^*(v_{2,4}v_{2,5}), g^*(v_{1,n-4}v_{2,n-2}), g^*(v_{1,n-3}v_{2,n-1}),\\
&\quad g^*(v_{1,n-2}v_{2,n-4}), g^*(v_{1,n-1}v_{2,n-3}), g^*(v_{1,n}v_{2,n}), g^*(v_{1,n-4}v_{1,n-3}),\\
&\quad g^*(v_{1,n-3}v_{1,n-2}), g^*(v_{1,n-2}v_{1,n-1}), g^*(v_{1,n-1}v_{1,n}), g^*(v_{2,n-4}v_{2,n-3}),\\
&\quad g^*(v_{2,n-3}v_{2,n-2}), g^*(v_{2,n-2}v_{2,n-1}), g^*(v_{2,n-1}v_{2,n})\}\\
&= \{3n-2, 3n-6, 3n-5, 3n-3, 3n-4, 3n-8, 3n-7, 10, 13, 3, 6, 2, 11, 7, 4,\\
&\quad 9, 5, 8, 12, 1\}\\
&= [1, 13] \cup [3n-8, 3n-2].
\end{aligned}$$

We can get $g^*(E(G)) = A \cup B \cup C = [1, 3n-2]$. Therefore, the g^* is a bijection from $E(G)$ to $[1, 3n-2]$. ∎

***Proof of Theorem* 1.4.** The conclusion comes from Lemmas 1.2, 2.1 and 2.2. ∎

References

[1] A. Rosa, *On certain valuations of the vertices of a graph*, in: Theory of Graphs, Proc. of Intemational Symposium, Rome 1966 (Gordon Breach, New York 1967), 349–355.

[2] S.W. Golomb, *How to number a graph*? in: R.C. Read, ed., Graph Theory and Computing (Academic Press, New York, 1972) 23–27.

[3] Z. Liang, *On the graceful conjecture of permutation graphs of paths*, Ars Combin. **91** (2009) 65–82.

[4] J.A. Gallian, *A dynamic survey of graph labeling*, Electronic J. Combin. **14** (2007) DS#6, 1–180.

[5] J.C. Bermond, *Graceful graphs, radio antennae and Fench windmills*, in: R.J. Wilson, ed., Graph Theory and Combinatorics (Pitman, London, 1979), 13–37.

[6] A.K. Dewdney, *The search for an invisible ruler that will help radio astronomers to measure the earth*, Scientific American, Dec. (1986) 16–19.

[7] S.M. Lee, K.Y. Lai, Y.S. Wang and M.K. Kiang, *On the graceful permutation graphs conjecture*, Congr. Numer. **103** (1994) 193–201.

[8] M. Maheo, *Strongly graceful graphs*, Discrete Math. **29** (1980) 39–46.

[9] C. Delorme, *Two sets of graceful graphs*, J. Graph Theory **4** (1980) 247–250.

[10] R.W. Frucht and J.A. Gallian, *Labelling prisms*, Ars Combin. **26** (1988) 69–82.

[11] J.A. Gallian, *Labelling prisms and prism related graphs*, Congress. Numer. **59** (1987) 89–100.

[12] Z. Liang, H. Zhang, N. Xu, S. Ye, Y. Fan and H. Ge, *Gracefulness of five permutation graphs of paths*, Utilitas Mathematica **72** (2007) 241–249.

[13] N. Han and Z. Liang, *On the graceful permutation graphs conjecture*, J. Discrete Math. Sci. & Cryptography **11** (2008) 501–526.

Received 13 November 2007
Revised 21 April 2009
Accepted 4 June 2009