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INDEPENDENT TRANSVERSALS OF LONGEST PATHS IN LOCALLY SEMICOMPLETE AND LOCALLY TRANSITIVE DIGRAPHS

Hortensia Galeana-Sánchez, Ricardo Gómez

and

Juan José Montellano-Ballesteros

Instituto de Matemáticas de la Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria
C.P. 04510, México D.F., México

Abstract

We present several results concerning the Laborde-Payan-Xuang conjecture stating that in every digraph there exists an independent set of vertices intersecting every longest path. The digraphs we consider are defined in terms of local semicompleteness and local transitivity. We also look at oriented graphs for which the length of a longest path does not exceed 4.

Keywords: independent set, longest path, locally semicomplete, locally transitive.

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1. Introduction

Given a digraph, does there exist a maximal independent set of vertices that is *transversal* to all longest paths (every longest path has a vertex on the set)? This question was posed by Laborde, Payan and Xuong in 1982 (see [10]). They conjectured that the answer is always yes. The problem is still open in general, but a number of partial results are known even for more general settings. For example, in [9] Galeana-Sánchez and Gómez consider not only longest paths but non-augmentable paths and present results for generalizations of tournaments. Another example is the so called *path* *partition conjecture* (see [7]), proved to be true for several generalizations of tournaments (see [4]). The traceability conjecture (see [8]) is related to a particular case of the path partition conjecture that also generalizes the Laborde, Payan and Xuong conjecture for oriented graphs.

In this paper we restrict ourselves to study the original question for several kinds of digraphs. We divide them into three types according to the hypothesis defining them. The first two types involve local conditions, namely semicompleteness and transitivity. The third type of digraphs that we consider are oriented graphs for which the length of a longest path is at most 4.

Let us fix some definitions and notation. Let D be a digraph with vertex set $V(D)$ and arc set $A(D) \subset V(D) \times V(D)$ with no loops (so $(u,u) \notin A(D)$ for all $u \in V(D)$). We say that D is *oriented* if there exist no vertices $x, y \in V(D)$ such that $(x,y),(y,x) \in A(D)$. A (*directed*) *path* in D is a sequence of distinct vertices $T = (x_0, \ldots, x_n)$ such that $(x_{i-1}, x_i) \in A(D)$ for every $i = 1, \ldots, n$. Let $V(T) = \{x_0, \ldots, x_n\}$ and let n be the *length* of T. Let $u, v \in V(D)$. A *shortest path* from u to v is a path from u to v of minimal length. The (*directed*) *distance* $d_D(u,v)$ is the length of a shortest path in D from u to v (so $d_D(u,v) = 0$ if and only if $u = v$ and $d_D(u,v) = \infty$ if there is no path from u to v). A digraph is (*weakly*) *connected* if the underlying graph is connected. Let $\Gamma^+(u) = \{x \in V(D) \mid d_D(u,x) = 1\}$ and $\Gamma^-(v) = \{y \in V(D) \mid d_D(y,v) = 1\}$; let $\delta^+(u) = |\Gamma^+(u)|$ and $\delta^-(v) = |\Gamma^-(v)|$. Let

$$L^+(D) = \{x \in V(D) \mid \exists \text{ longest path in } D \text{ starting at } x\}$$

and

$$L^-(D) = \{y \in V(D) \mid \exists \text{ longest path in } D \text{ ending at } y\}.$$

A set of vertices $I \subset V(D)$ is *independent* if $d_D(x,y) > 1$ for every $x, y \in I$ with $x \neq y$. An independent set I is *maximal* if for every $z \in V(D) \setminus I$, the set $I \cup \{z\}$ is not independent. An independent set $K \subset V(D)$ is a *kernel* if for every $v \in V(D) \setminus K$ there exists $x \in K$ such that $(v,x) \in A(D)$ (i.e., $d(v,x) = 1$). An independent set $Q \subset V(D)$ is a *quasikernel* if for every $v \in V(D) \setminus K$ there exists $x \in K$ such that $d(v,x) \leq 2$. An independent set $S \subset V(D)$ is a *solution* (*quasisolution*) if for every $v \in V(D) \setminus S$ there exists $x \in S$ such that $d(x,v) = 1$ ($d(x,v) \leq 2$). Given $B \subset V(D)$, let $D[B]$ be the subdigraph induced by B and let $D - B$ be the digraph that results from D by removing the vertices in B. We say that D is *semicomplete* if every pair of vertices in $V(D)$ are adjacent. We say that D is *locally in-semicomplete* (*locally out-semicomplete*) if for every vertex $v \in V(D)$, $D[\Gamma^-(v)]$ $(D[\Gamma^+(v)])$ is semicomplete. A vertex $v \in V(D)$ is a *transitivity point* if for every pair of distinct vertices $x, y \in V(D) \setminus \{v\}$ for which $(x,v),(v,y) \in A(D)$, we have $(x,y) \in A(D)$.

If a digraph D possesses a kernel K, then K is obviously an independent transversal of longest paths in D. Of course, not every digraph possesses a kernel, but we show that if B is a set of vertices in a strongly connected digraph D such that $D - B$ possesses a kernel K, then K is an independent transversal of longest paths in D, provided that the in-neighborhood of every out-neighhor of every vertex in B induces a semicomplete subdigraph in D (Theorem 2.2). A similar result is obtained by switching in and out (Theorem 2.4). It is well known that every strongly connected semi-complete digraph is hamiltonian. More generally, a strongly connected digraph is hamiltonian if it is locally in-semi-complete, i.e., if the in-neighborhood of every vertex induces a semicomplete digraph (see [2], Theorem 5.5.1). We show that if the semicomplete in-neighborhood requirement is restricted to out-neighbors of terminal vertices of longest paths, hamiltonicity is still guaranteed in strongly connected digraphs (Theorem 2.5) and the existence of an independent transversal of longest paths is guaranteed even in digraphs that are not strongly connected (Theorem 2.6). In fact, under these conditions every quasikernel is an independent transversal of longest paths in D (and it is known that every digraph has a quasikernel [6].)

Then we consider local transitivity. We prove that if every vertex in the outer-neighborhood of a vertex which is a terminal vertex of a longest path is a transitivity point, then every quasikernel intersects every longest path (Theorem 3.1), and the dual result is stated (Theorem 3.2).

To finish we proof that if the length of a longest path in an oriented graph D is at most 4, then there exists an independent transversal of longest paths formed by terminal vertices of longest paths (Theorem 4.2).

Remark 1.1. Let D be a connected digraph. If D is hamiltonian, then each subset of vertices of D intersects every longest path in D.

Proof. If D is hamiltonian, then every longest path T has length $|V(D)|-1$ and $V(T) = V(D)$. From here the result follows. ∎

Lemma 1.2. *Let D be a connected digraph. Let T be a longest path in D and suppose that $D[V(T)]$ is hamiltonian. Then D is hamiltonian.*

Proof. Let k be the length of the longest path in D, $(x_0, \ldots, x_k, x_0)$ be a hamiltonian cycle in $D[V(T)]$ and suppose that D is not hamiltonian. Then $V(D) \setminus V(T) \neq \emptyset$. Hence, since D is connected, there exist adjacent vertices $x_i \in V(T)$ and $z \in V(D) \setminus V(T)$. If $(x_i, z) \in A(D)$ then either $(x_{i+1}, \ldots, x_k, x_0, \ldots, x_i, z)$ or $(x_0, \ldots, x_k, z)$ is a path of length $k+1$ which is a contradiction. If $(z, x_i) \in A(D)$ then either $(z, x_i, x_{i+1}, \ldots, x_k, x_0, \ldots, x_{i-1})$ or $(z, x_0, \ldots, x_k)$ is a path of length $k+1$, a contradiction. Therefore, D is hamiltonian. ■

2. Locally Semicomplete Digraphs

Here we look at digraphs having appropriate subsets of vertices inducing semicomplete digraphs. Locally semicomplete digraphs were introduced by Bang-Jensen in [1]. Many classic theorems for tournaments hold for this class of digraphs which are generalizations of tournaments. See [3] for a comprehensive survey on generalizations of tournaments.

Strictly speaking, we are not going to consider locally semicomplete digraphs but a wider class that requires the semicomplete condition to hold on a particular set of vertices. We will use the following result.

Lemma 2.1. *Let D be a digraph. Suppose that there exists $B \subset V(D)$ such that $D - B$ possesses a kernel. Let K be a kernel in $D - B$ and let $T = (x_0, \ldots, x_n)$ be a longest path in D. If $V(T) \cap K = \emptyset$, then $x_n \in B$.*

Proof. If $x_n \notin B$, then there exists $z \in K$ such that $(x_n, z) \in A(D)$ and hence $(x_0, \ldots, x_n, z)$ is a path in D contradicting that T is a longest path. ■

Theorem 2.2. *Let D be a strongly connected digraph. Suppose that there exists $B \subset V(D)$ such that*

1. *$D - B$ possesses a kernel.*
2. *For every $x \in B$ and $v \in \Gamma^+(x)$, $D[\Gamma^-(v)]$ is semicomplete.*

Then every kernel of $D - B$ intersects every longest path in D.

Proof. Let $K \subset V(D) \setminus B$ be a kernel in $D - B$. Suppose that there exists a longest path $T = (x_0, \ldots, x_n)$ in D such that $V(T) \cap K = \emptyset$. Then, by Remark 1.1, D is not hamiltonian. Lemma 2.1 implies that $x_n \in B$. Since D is strongly connected, $\delta^+(x_n) > 0$. Moreover, $\Gamma^+(x_n) \subset \{x_0, \ldots, x_{n-1}\}$

because T is a longest path in D. Choose T so that $i_0 = \min\{i \mid (x_n, x_i) \in A(D)\}$ is minimal among all longest paths in D not intersecting K. If $i_0 = 0$, then $D[V(T)]$ is hamiltonian and hence Lemma 1.2 implies that D is hamiltonian, a contradiction. Henceforth we assume that $i_0 > 0$.

By 2, $D[\Gamma^-(x_{i_0})]$ is semicomplete since $x_n \in B$ and $x_{i_0} \in \Gamma^+(x_n)$. Now, $x_n, x_{i_0-1} \in \Gamma^-(x_{i_0})$ and thus x_n and x_{i_0-1} are adjacent, but $(x_n, x_{i_0-1}) \notin A(D)$ because we chose i_0 minimal. It follows that $(x_{i_0-1}, x_n) \in A(D)$ and hence $T_1 = (x_0, \ldots, x_{i_0-1}, x_n, x_{i_0}, \ldots, x_{n-1})$ is a longest path in D with $V(T_1) = V(T)$. Now we proceed by induction. Suppose that $j \geq 1$ is such that $j < n - i_0 - 1$ and $x_{n-j-1} \in B$, $(x_{i_0-1}, x_{n-j}) \in A(D)$ and also that we have the longest path $T_{j+1} = (x_0, \ldots, x_{i_0-1}, x_{n-j}, \ldots, x_n, x_{i_0}, \ldots, x_{n-j-1})$. Since $V(T_{j+1}) = V(T)$, Lemma 2.1 implies $x_{n-j-1} \in B$. Since $x_{n-j} \in \Gamma^+(n-j-1)$, $D[\Gamma^-(x_{n-j})]$ is semicomplete. We have $x_{n-j-1}, x_{i_0-1} \in \Gamma^-(x_{n-j})$ and therefore x_{n-j-1} and x_{i_0-1} are adjacent but $(x_{n-j-1}, x_{i_0-1}) \notin A(D)$ because we chose i_0 minimal. It follows that $(x_{i_0-1}, x_{n-j-1}) \in A(D)$ and $T_{j+2} = (x_0, \ldots, x_{i_0-1}, x_{n-j}, \ldots, x_n, x_{i_0}, \ldots, x_{n-j-1})$ is a longest path in D (see Figure 1).

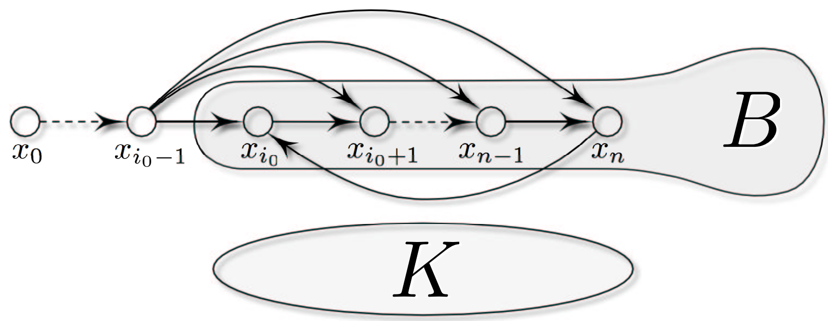

Figure 1. Vertices of T in B and existence of arcs.

Clearly, $\Gamma^+(x_k) \cap (V(D) \setminus V(T)) = \emptyset$ for all $k = i_0, \ldots, n$ because otherwise D would have a path of length $n+1$. Moreover, $\Gamma^+(x_k) \cap \{x_0, \ldots, x_{i_0-1}\} = \emptyset$ because we chose i_0 to be minimal. But then there exists no path from any vertex in $\{x_{i_0}, \ldots, x_n\}$ to any vertex in $\{x_0, \ldots, x_{i_0-1}\}$, contradicting that D is strongly connected. ∎

Remark 2.3. Theorem 2.5 is a generalization of the fact that every strongly connected and locally semicomplete digraph is hamiltonian (see Theorem 5.5.1 in [2]).

Now we look at what happens when switching between inner and outer neighborhoods in Theorem 2.2.

Theorem 2.4. *Let D be a strongly connected digraph. Suppose that there exists $B \subset V(D)$ such that*

1. *$D - B$ possesses a kernel.*
2. *For every $x \in B$ and $v \in \Gamma^-(x)$, $D[\Gamma^+(v)]$ is semicomplete.*

Then every kernel of $D - B$ intersects every longest path in D.

Proof. Let $K \subset V(D) \setminus B$ be a kernel in $D - B$. Suppose that there exists a longest path $T = (x_0, \ldots, x_n)$ in D such that $V(T) \cap K = \emptyset$. If there exists $y \in B$ and $z \in K$ such that $(y, z) \in A(D)$, then we can let $B' = B - \{y\}$ so that K is now a kernel in $D - B'$ and for every $x \in B'$ and $v \in \Gamma^-(x)$, $D[\Gamma^+(v)]$ is semicomplete and $V(T) \cap K = \emptyset$. Thus we can assume that B is minimal in the sense that there exist no edges starting in B and ending in K.

Since $V(T) \cap K = \emptyset$ and T is a longest path, $x_n \in B$ by Lemma 2.1. Suppose that $V(T) \cap (V(D) \setminus B) \neq \emptyset$ and let $j = \max\{i \mid x_i \notin B\}$. Then there exists $z \in K$ such that $(x_j, z) \in A(D)$ because K is a kernel of $D - B$. By 2.4, z and x_{j+1} are adjacent because both belong to $\Gamma^+(x_j)$ and $x_j \in \Gamma^-(x_{j+1})$. Since $x_{j+1} \in B$ and $z \in K$, $(x_{j+1}, z) \notin A(D)$ and thus $(z, x_{j+1}) \in A(D)$, but then $(x_0, \ldots, x_j, z, x_{j+1}, \ldots x_n)$ is a path in D of length $n + 1$, contradicting that T is a longest path.

Now suppose that $V(T) \cap (V(D) \setminus B) = \emptyset$, that is, suppose that $V(T) \subset B$. Since D is strongly connected, there exists $0 \leq j \leq n$ such that $\Gamma^+(x_j) \nsubseteq V(T)$. Let j maximal with the property that $\Gamma^+(x_j) \nsubseteq V(T)$ and let $z \in \Gamma^+(x_j) - V(T)$. If $j = n$, then $(x_0, \ldots, x_n, z)$ is a path in D of length $n + 1$, contradicting that T is a longest path. Therefore $j < n$. Since $x_j \in \Gamma^-(x_{j+1})$, 2.4 implies that $\Gamma^+(x_j)$ is semicomplete. Hence z and x_{j+1} are adjacent. We have $(x_{j+1}, z) \notin A(D)$ because we chose j maximal. If $(z, x_{j+1}) \in A(D)$, then $(x_0, \ldots, x_j, z, x_{j+1}, \ldots, x_n)$ is path in D of length $n + 1$, contradicting that T is a longest path. ■

Theorem 2.5. *Let D be a strongly connected digraph. Suppose that for every vertex $x \in L^-(D)$ and $v \in \Gamma^+(x)$, $D[\Gamma^-(v)]$ is semicomplete. Then D is hamiltonian.*

Proof. Let $T = (x_0, \ldots, x_n)$ be a longest path in D. We have that $\Gamma^+(x_n) \neq \emptyset$ because D is strongly connected and $\Gamma^+(x_n) \subset V(T)$ because

T is a longest path. Choose T so that $i_0 = \min\{i \mid (x_n, x_i) \in A(D)\}$ is minimal among all longest paths in D. Now, suppose that D is not hamiltonian. It follows from Lemma 1.2 that $D[V(T)]$ is not hamiltonian and hence $i_0 > 0$. Then as in the proof of Theorem 2.2, we can show that $\Gamma^+(x_k) \cap (V(D) \setminus V(T)) = \emptyset$ for all $k = i_0, \ldots, n$ and $\Gamma^+(x_k) \cap \{x_0, \ldots, x_{i_0-1}\} = \emptyset$, contradicting that D is strongly connected. ■

Now we look at quasikernels. It is a well known fact that every digraph D possesses a quasikernel (see [6]). Given a quasikernel Q of D, the set of vertices $V(D)$ are partitioned into three groups, namely,

$$V(D) = Q \cup \Gamma^-(Q) \cup \big(\Gamma^-(\Gamma^-(Q)) - \big(Q \cup \Gamma^-(Q)\big)\big)$$

where

$$\Gamma^-(A) = \bigcup_{z \in A} \Gamma^-(z)$$

for every $A \subset V(D)$ (see Figure 2).

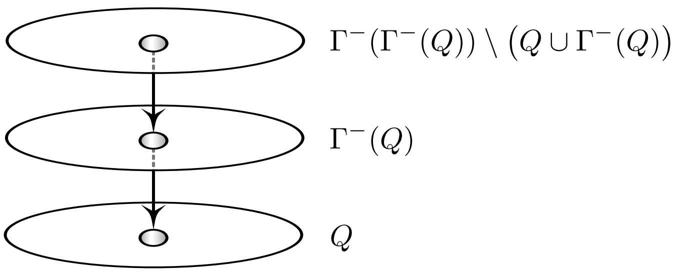

Figure 2. Partition of the set of vertices of a digraph D for a given quasi-kernel Q.

We now show that a digraph D that satisfies the local semi-complete condition of Theorem 2.6 has an independent transversal of longest paths, even if D is not strongly connected.

Theorem 2.6. *Let D be a digraph. Suppose that for every $x \in L^-(D)$ and $z \in \Gamma^+(x)$, $D[\Gamma^-(z)]$ is semicomplete. Then every longest path intersects every quasikernel Q.*

Proof. Let $Q \subset V(D)$ be a quasikernel.Suppose that $T = (x_0, \ldots, x_n)$ is a longest path in D with $V(T) \cap Q = \emptyset$. There exists $u \in Q$ such that either $(x_n, u) \in A(D)$ or there exists $w \in \Gamma^-(Q)$ such that (x_n, w, u) is a path in

D. If $(x_n, u) \in A(D)$, then $(x_0, \ldots, x_n, u)$ is a path in D of length $n+1$, contradicting that T is a longest path. So we have $\Gamma^+(x_n) \subset V(T)$. Choose T so that $i_0 = \min\{i \mid (x_n, x_i) \in A(D)\}$ is minimal among all longest paths in D not intersecting Q. Proceeding as in the proof of Theorem 2.2, the result follows. ■

We can state the corresponding dual results for solutions, the dual definition of kernel, and quasisolutions, the dual definition of quasikernel. Again, every digraph possesses a quasisolution; and given one, there is a partition of the set of vertices similar to the one described in Figure 2, with the arrows pointing up and with all occurrences of '$-$' replaced by '$+$'. We let the reader verify the details.

3. Locally Transitive Digraphs

Theorem 3.1. *Let D be a digraph. Suppose that for every $x \in L^-(D)$, if $v \in \Gamma^+(x)$, then v is a transitivity point. Then every quasikernel of D intersects every longest path in D.*

Proof. Suppose that there exists a quasikernel $Q \subset V(D)$ and a longest path $T = (x_0, \ldots, x_n)$ such that $V(T) \cap Q = \emptyset$. Since Q is a quasikernel and $x_n \notin Q$, there exists $u \in Q$ such that either $(x_n, u) \in A(D)$ or there exists $w \in \Gamma^-(Q)$ such that (x_n, w, u) is a path in D. If $(x_n, u) \in A(D)$, then $(x_0, \ldots, x_n, u)$ is a path in D of length $n+1$, contradicting that T is a longest path. Hence we must have $w \in V(T)$, say $w = x_i$ for some $i \in \{0, \ldots, n-1\}$ (see Figure 3). By hypothesis x_i is a transitivity point and therefore $(x_n, u) \in A(D)$, a contradiction. ■

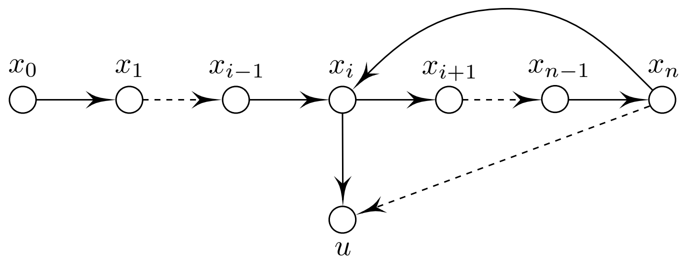

Figure 3. Existence of a vertex $u \in Q$ such that $(x_n, u) \in A(D)$.

Theorem 3.1 admits a dual version. We state it and let the reader verify the details.

Theorem 3.2. *Let D be a digraph. Suppose that for every $x \in L^+(D)$, if $v \in \Gamma^-(x)$, then v is a transitivity point. Then every quasisolution of D intersects every longest path in D.*

4. DIGRAPHS WITH "SHORT" LONGEST PATHS

In this section we will prove that if the length of a longest path of an oriented graph is at most 4, then there exists an independent set of vertices in $L^-(D)$ that intersects every longest path.

Lemma 4.1. *Let D be an oriented graph and let $k \geq 1$ be the length of a longest path in D. Let $B \subseteq L^-(D)$ be an independent set that intersects the maximum number of longest paths in D. Suppose P is a longest path in D such that $V(P) \cap B = \emptyset$. Then there exist a vertex $x \in B$ and longest paths Q and T in D such that the following holds. Write $P = (p_0, \ldots, p_k)$, $Q = (q_0, \ldots, q_k)$ and $T = (t_0, \ldots, t_k)$.*

1. $(x, p_k) \in A(D)$.
2. $q_k = x$.
3. $t_s = x$ *for some* $s \in \{0, \ldots, k-1\}$ *and* $p_k \notin V(T)$.
4. $p_k = q_r$ *for some* $r \in \{1, \ldots, k-2\}$.
5. $q_{r+1} = p_n$ *for some* $n \in \{1, \ldots, k-2\}$.
6. $t_{s+1} = q_m$ *for some* $m \in \{1, \ldots, k-2\}$.

Proof. If $B \cup \{p_k\}$ is independent, then it intersects more longest paths in D than B, a contradiction. Then it is not independent and hence p_k is adjacent to at least one vertex in B. If $(p_k, z) \in A(D)$ for some $z \in B$, then $(p_0, \ldots, p_k, z)$ is a path of length $k+1$, a contradiction. Then $B_{p_k} = \{z \in B \mid (z, p_k) \in A(D)\} \neq \emptyset$. Suppose that for each $z \in B_{p_k}$ and every longest path T in D with $z \in V(T)$, $p_k \in V(T)$. Then $(B \setminus B_{p_k}) \cup \{p_k\}$ is independent, intersects every path intersected by B but also intersects P, a contradiction. Then there exist $x \in B_{p_k}$ and a longest path T in D such that $x \in V(T)$ and $p_k \notin V(T)$. Now, T does not end at x because otherwise $(t_0, \ldots, t_k = x, p_k)$ is a path in D of length $k+1$, a contradiction. Then there exists $s \in \{0, \ldots, k-1\}$ such that $x = t_s$. Now, since $x \in B \subseteq L^-(D)$, there exists a longest path Q in D with $q_k = x$. From here 1, 2 and 3 follow. Now, observe that for every longest path $Q = (q_0, \ldots, q_k)$ in D, if for some $z \in V(D)$, $(q_k, z) \in A(D)$, then we have $z \in V(Q)$ (otherwise $(q_0, \ldots, q_k = x, z)$ is a path in D of length $k+1$, a contradiction) and therefore $z = q_s$ for some $s \in \{0, \ldots, k\}$. Since D has no loops, $z \neq q_k$; and since $(q_{k-1}, q_k) \in A(D)$ and D is oriented, $z \neq q_{k-1}$. Also, if $z = q_0$, then $(z = q_0, q_1, \ldots, q_k)$ is a hamiltonian cycle in $V(Q)$, which by Lemma 1.2 is a contradiction. Thus, $z = q_s$ with $s \in \{1, \ldots, k-2\}$. From here, since $\{(x = q_k, p_k), (p_k = q_r, q_{r+1}), (q_k = t_s, t_{s+1})\} \subseteq A(D)$, 4, 5 and 6 follow. ■

Theorem 4.2 (Laborde-Payan-Xuong for $k \leq 4$)**.** *Let D be an oriented graph. If the length of a longest path in D is at most 4, then there exists an independent set $B \subset L^-(D)$ that intersects every longest path.*

Proof. Here we present the proof for the case $k = 4$. By analogous arguments as in this proof, the reader can verify the cases $k = 1, 2, 3$. Henceforth, let $k = 4$.

Let $B \subset L^-(D)$ be an independent set intersecting the maximum number of longest paths in D. Suppose that there exists a longest path P in D such that $V(P) \cap B = \emptyset$. Let x, Q and T be as in Lemma 4.1 so that

1. $(x, p_4) \in A(D)$.
2. $q_4 = x$.
3. $t_s = x$ for some $s \in \{0, 1, 2, 3\}$ and $p_4 \notin V(T)$.
4. $p_4 = q_r$ for some $r \in \{1, 2\}$.
5. $q_{r+1} = p_n$ for some $n \in \{1, 2\}$.
6. $t_{s+1} = q_m$ for some $m \in \{1, 2\}$.

Case 1. $p_4 = q_1$.
By 6, $t_{s+1} = q_m$ for some $m \in \{1, 2\}$. By 3, $q_1 = p_4 \notin V(T)$ and therefore $t_{s+1} = q_2$. By 1 and 3 we see that $(t_s, p_4) = (x, p_4) \in A(D)$ and $p_4 \notin V(T)$; and since $(p_4, t_{s+1}) = (q_1, q_2) \in A(D)$, we see that $(t_0, \ldots, t_s, p_4, t_{s+1}, \ldots, t_4)$ is a path of lenght 5 in D, which is a contradiction.

Case 2. $p_4 = q_2$.
By 5, $q_3 = p_n$ for some $n \in \{1, 2\}$, and by 6, $t_{s+1} = q_m$ for some $m \in \{1, 2\}$. Since $q_2 = p_4 \notin V(T)$, $t_{s+1} = q_1$. We claim that $t_{s+1} \in V(P)$. Suppose

this is not the case. Hence $(p_0, \ldots, p_n = q_3, q_4 = t_s, t_{s+1} = q_1, q_2 = p_4)$ is a path in D of lenght at most 4 which implies that $n = 1$. Thus $(p_2, \ldots, p_4 = q_2, q_3 = p_1, q_4 = t_s, t_{s+1})$ is path of lenght 5 which is a contradiction. Thus the claim is proved.

Therefore $q_1 = t_{s+1} = p_r$ for some $r \in \{0, \ldots, 4\}$. By 3, $p_4 \notin V(T)$, hence $t_{s+1} \neq p_4$. If $t_{s+1} = p_0$, then $(q_4 = t_s, t_{s+1} = p_0, \ldots, p_4)$ is a path of length 5 which is a contradiction. So, $t_{s+1} \neq p_0$. $t_{s+1} \neq p_n$ because D is oriented and $(p_n, t_s) = (q_3, q_4) \in A(D)$. If $t_{s+1} = p_{n+1}$, then $(p_0, \ldots, p_n = q_3, q_4 = t_s, t_{s+1} = p_{n+1}, \ldots, p_4)$ is a path of lenght 5, which is a contradiction. Hence $t_{s+1} \neq p_{n+1}$. Thus, $t_{s+1} \in \{p_1, p_2, p_3\} \setminus \{p_n, p_{n+1}\}$ where $q_3 = p_n$. This allows only two possible subcases, either $n = 1$ and therefore $q_3 = p_1$ and $t_{s+1} = p_3$; or $n = 2$ and then $q_3 = p_2$ and $t_{s+1} = p_1$.

Subcase 2.1. $q_3 = p_1$ and $t_{s+1} = p_3$.
Since, by 3, $p_4 \notin V(T)$, we have that $(t_0, \ldots, t_s, t_{s+1} = p_3, p_4)$ is a path in D and therefore $s \leq 2$. Since T is a path of length 4, there is $z \in V(D)$ such that $z = t_{s+2}$. The path $(q_4 = x, p_4 = q_2, q_3 = p_1, p_2, p_3 = t_{s+1})$ has length 4, and thus $t_{s+2} \in \{q_4, q_2, q_3, p_2, t_{s+1}\}$. Clearly $t_{s+2} \neq t_{s+1}$ and also $t_{s+2} \neq q_4 = t_s$. Since $q_2 = p_4 \notin V(T)$, $t_{s+2} \neq q_2$. Finally, $t_{s+2} \neq p_2$ because $(p_2, p_3) = (p_2, t_{s+1}) \in A(D)$ and D is oriented. Then $t_{s+2} = q_3 = p_1$. Since $p_4 \notin V(T)$, it follows that $(t_0, \ldots, t_s, t_{s+1} = p_3, p_4 = q_2, q_3 = t_{s+2}, \ldots, t_4)$ is a path of length 5, a contradiction.

Subcase 2.2. $q_3 = p_2$ and $t_{s+1} = p_1$.
Recall that $t_{s+1} = q_1$. Now, $(t_0, \ldots, t_s, t_{s+1} = q_1, q_2)$ is a path in D because $q_2 = p_4 \notin V(T)$ and therefore $s \leq 2$. As before, there is $z \in V(D)$ such that $z = t_{s+2}$. Since $(p_3, p_4 = q_2, q_3, q_4 = t_s, t_{s+1})$ is a path in D of length 4, $t_{s+2} \in \{p_3, p_4, q_3, q_4, t_{s+1}\}$. Clearly $t_{s+2} \neq t_{s+1}$ because D is oriented. Since $p_4 \notin V(T)$, $t_{s+2} \neq p_4$. Next, $t_{s+2} \neq q_4$ because $q_4 = t_s$ and finally $t_{s+2} \neq q_3$ because otherwise $(t_0, \ldots, t_s, t_{s+1} = q_1, q_2 = p_4, q_3 = t_{s+2}, \ldots, t_4)$ is a path of length 5, a contradiction. Then $t_{s+2} = p_3$ and the length of the path $(p_0, p_1 = t_{s+1}, t_{s+2} = p_3, p_4 = q_2, q_3 = p_2, q_4 = t_s)$ is 5, a contradiction. ∎

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References

[1] J. Bang-Jensen, *Locally semicomplete digraphs: A generalization of tournaments*, J. Graph Theory **14** 1990) 371–390.

[2] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications (Springer Monographs in Mathematics, 2001).

[3] J. Bang-Jensen and G. Gutin, *Generalization of tournaments: A survey*, J. Graph Theory **14** (1998) 371–390.

[4] J. Bang-Jensen, M.H. Nielsen and A. Yeo, *Longest path partitions in generalizations of tournaments*, Discrete Math. **306** (2006) 1830–1839.

[5] E. Boros and V. Gurvich, *Perfect graphs, kernels, and cores of cooperative games*, Discrete Math. **306** (2006) 2336–2354.

[6] V. Chvátal and L. Lovász, *Every directed graph has a semi-kernel*, Lecture Notes in Math. Vol. 411 (Springer, Berlin, 1974).

[7] M. Frick, S. Van Aardt, G. Dlamini, J. Dunbar and O. Oellermann, *The directed path partition conjecture*, Discuss. Math. Graph Theory **25** (2005) 331–343.

[8] M. Frick, S. Van Aardt, J. Dunbar, M. Nielsen and O. Oellermann, *A traceability conjecture for oriented graphs*, The Electronic Journal of Combinatorics **15** (2008) #R150.

[9] H. Galeana-Sánchez and R. Gómez, *Independent sets and non-augmentable paths in generalization of tournaments*, Discrete Math. **308** (2008) 2460–2472.

[10] J.M. Laborde, C. Payan and N.H. Xuong, *Independent sets and longest paths in digraphs*, in: Graphs and other combinatorial topics, Proceedings of the Third Czechoslovak Symposium of Graph Theory (1982) 173–177.

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