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ON TRANSITIVE ORIENTATIONS OF $G - \hat{e}$

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Abstract

A comparability graph is a graph whose edges can be oriented transitively. Given a comparability graph $G = (V, E)$ and an arbitrary edge $\hat{e} \in E$ we explore the question whether the graph $G - \hat{e}$, obtained by removing the undirected edge $\hat{e}$, is a comparability graph as well. We define a new substructure of implication classes and present a complete mathematical characterization of all those edges.

Keywords: (primary): comparability graph, edge deletion; (secondary): transitive orientation, Triangle Lemma, $\dot{\Gamma}$-components, open shop scheduling, irreducibility.

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1. Introduction

A comparability graph is an undirected graph whose edges can be oriented in a transitive way. Properties and structures of comparability graphs and their orientations were investigated by many authors. Basic papers were written by Gilmore and Hoffman [6], Gallai [5] or Golumbic [7], for instance. An alternative interpretation of comparability graphs as representations of partial orders will be of no importance throughout this paper.

Golumbic [8] (or [9]) developed an algorithm for identifying and orienting comparability graphs with running time $\mathcal{O}(\delta m)$, where δ denotes the maximal degree of a vertex and m the number of edges. In Simon [15] an algorithm with running time $\mathcal{O}(n^2)$ is contained, where n is the number of vertices. Both algorithms use the notion of implication classes of the edge set, where the orientation of one edge in an implication class forces the orientation of all other edges in this class.

A closely related problem is the so-called modular decomposition of a graph. Algorithms for modular decomposition of a given graph G can be used to construct an acyclic orientation of G which is transitive, if G is a comparability graph.

Linear time algorithms for modular decomposition were developed by McConnell and Spinrad [10, 11], and Cournier and Habib [4]. In [12] McConnell and Spinrad give an $O(n+m \log n)$ algorithm for modular decomposition of a graph by ordered vertex partitioning. This algorithm was implemented by Moerig [13] by usage of the software LEDA. The author describes in detail that the time complexity $O(n + m \log n)$ is indeed preserved.

We investigate in this paper whether the comparability property of a graph is destroyed by the deletion of a single fixed edge. A similar problem in the literature is the so-called Comparability-Editing Problem. This rises the question whether for a graph $G = (V, E)$ there is a set of edges F of cardinality k such that $G^* = (V, E \nabla F)$ is a comparability graph. With $E \nabla F$ we denote the symmetric difference between E and F. If $F \subseteq E$ is claimed, the question can be restated as follows: Is there a subset F consisting of at most k edges from E that leads to a comparability graph when deleted from G? This variation of the general Comparability-Editing Problem is known as Comparability-Deletion. Both problems were shown to be NP-complete by Natanzon *et al.* Natanzon *et al.* 2001 and Yannakakis [17], respectively.

There are several differences between the Comparability-Deletion problem and our stated question. We only consider the deletion of one single edge and we require the original graph to be a comparability graph. Willenius [16] constructed 1-Deletion sets in a comparability graph. But we investigate the existence of a transitive orientation on a graph obtained from a comparability graph G by removing a *given* edge.

In this paper we present a complete mathematical characterization of all edges of E whose deletion does not destroy the comparability property. We therefore split E in several subsets and show for each the respective result. These subsets are obtained by exploring the properties of the implication class of the given edge e, making use of a new substructure of this implication class, so-called $\dot{\Gamma}$-components.

2. Basic Notation

We consider simple undirected graphs $G = (V, E)$, where an *undirected* edge (or simply, *edge*) $\hat{e} = \widehat{ab}$ consists of the *directed* edge (or *arc*) $e = ab \in E$

together with its *reversal* $e^{-1} = ba \in E$. For simplicity we write $\hat{e} \in E$ instead of $\hat{e} = \{e, e^{-1}\} \subseteq E$. Analogously to single arcs, we denote with A^{-1} the set of the reversed arcs from $A \subseteq E$, and with $\hat{A} = A \cup A^{-1}$ the symmetric closure of A. $V(A)$ denotes the set of vertices induced by an edge set A. For edge sets $A = \{e\}$ consisting of single edges we will omit the braces. The removal of an arc e from $A \subseteq E$ will be denoted by $A - e$ and the addition (union) of an edge set B to A by $A + B$. This commitment is useful when adding and removing edge sets at the same time, and should raise no confusion. Since we only deal with pairwise disjoint sets we can interpret each operator separately having no need for any parentheses.

The graph obtained by removing some edge $\hat{e} \in E$ from $G = (V, E)$ will be denoted by $G - \hat{e}$.

We call a graph $G = (V, E)$ a *comparability graph* if there exists some *transitive orientation* on G, i.e., a set $T \subseteq E$ with $T + T^{-1} = E$, $T \cap T^{-1} = \emptyset$, and the property of transitivity—the existence of ab and bc in T implies the existence of $ac \in T$. An orientation T is transitive if and only if T^{-1} is transitive as well, and we say T_1 *differs* from T_2 if neither $T_1 = T_2$ nor $T_1 = T_2^{-1}$. The set of all transitive orientations of G is denoted by $\mathcal{T}_G = \{T_1, \ldots, T_t, T_1^{-1}, \ldots, T_t^{-1}\}$.

For describing transitive orientations the so-called Γ-relation has been introduced on E,

$$ab \Gamma cd \Leftrightarrow \begin{cases} & a = c \text{ and } \widehat{bd} \notin E \\ \vee & b = d \text{ and } \widehat{ac} \notin E\,, \\ \vee & ab = cd\,. \end{cases}$$

The transitive closure Γ^+ of this relation is an equivalence relation, and the equivalence classes of Γ^+ are called *implication classes*. The orientation of any arc implies the orientation of every other arc from the same implication class. We call two arcs e' and e'' with $e' \Gamma^+ e''$ Γ-*connected*, or *directly* Γ-*connected* for $e' \Gamma e''$, respectively. Then there exists a Γ-*chain* $e' = e_1 \Gamma \ldots \Gamma e_s = e''$ between these two arcs. Consider the graph in Figure 1 (left) on page 427. The (directed) arc $e = ab$ is directly Γ-connected to arcs ax and az (black). The arcs az and ay are Γ-connected, $az \Gamma^+ ay$ ($az \Gamma ab \Gamma ax \Gamma ay$), but not directly Γ-connected. Finally, we denote the Γ-*neighborhood* of e, i.e., the set of all arcs differing from e that are in direct Γ-relation to e, by $\Gamma(e)$.

Furthermore let $\mathcal{I}_G = \{I_1, \ldots, I_k, I_1^{-1}, \ldots, I_k^{-1}\}$ be the set of all implication classes of G, and let $\mathcal{C}_G = \{\hat{I}_1, \ldots, \hat{I}_k\}$ be the set of all *color classes* of G. We will call an implication class $I \in \mathcal{I}_G$ *proper* if $I \cap I^{-1} = \emptyset$. Any graph possesses at least one implication class. And it is a comparability graph if and only if all its implication classes are proper (compare Theorem 3). The graph in Figure 1 (left) consists of two proper implication classes (black and gray).

Every transitive orientation $T = J_1 + \cdots + J_k \in \mathcal{T}_G$ is a combination of transitive orientations of the respective color classes (see Theorem 3), $J_i \in \{I_i, I_i^{-1}\}$ for all $i = 1, \ldots, k$. Moreover, every $T \in \mathcal{T}_G$ is acyclic. But not every arbitrary combination $J_1 + \cdots + J_k$ of transitive orientations of the color classes is acyclic. Consider, for example, triangles with edges from three different color classes. Hence, not every such combination yields a transitive orientation. We will call an arbitrary combination of transitive orientations of the color classes a *potential* transitive orientation, which is either acyclic or not.

Since every proper implication class is a transitive orientation (compare once more Theorem 3), we can derive that every acyclic potential transitive orientation is transitive.

Hence, a potential transitive orientation is a transitive orientation if and only if it is acyclic. Therefore, the number of transitive orientations of G is bounded by 2^k, where k is the number of different color classes.

From the Γ-relation mentioned we now develop a new relation. In the context of this paper we consider some given comparability graph $G = (V, E)$ with some given edge $\hat{e} \in E$. From now on we will regard this edge $\hat{e}$ as being fixed. We therefore may introduce some relations and edge sets referring to $\hat{e}$ without having to index them.

Definition 1 ($\dot{\Gamma}$-relation)**.** Let $G = (V, E)$ be a comparability graph and let $\hat{e} \in E$ be a firmly given edge. For $e', e'' \in E$ we define

$$e' \dot{\Gamma} e'' \Leftrightarrow e' \Gamma e'' \text{ with } e', e'' \notin \left\{e, e^{-1}\right\}.$$

The transitive closure $\dot{\Gamma}^+$ of this new relation is an equivalence relation as well. We call the emerging equivalence classes $\dot{\Gamma}$-*components*. Two arcs e' and e'' belong to the same $\dot{\Gamma}$-component if they are $\dot{\Gamma}$ *-connected*, i.e., if there exists a $\dot{\Gamma}$-*chain* $e' \dot{\Gamma} \ldots \dot{\Gamma} e''$. The implication class $I(e)$ may thus be split into several $\dot{\Gamma}$-components $\dot{I}_1, \ldots, \dot{I}_p$ and the remaining arc e, $I(e) = \dot{I}_1 + \cdots + \dot{I}_p + e$. All implication classes differing from $I(e)$ are left unchanged. 'Outside' of $\hat{I}(e)$ the terms implication class and $\dot{\Gamma}$-component are equivalent. Provided $I(e) = I_1$, the edge set E can thus be partitioned into $\dot{\Gamma}$-components and both arcs of $\hat{e}$,

$$E = \left[\dot{I}_{1_1} + \cdots \dot{I}_{1_p} + e\right] + \dot{I}_2 + \cdots + \dot{I}_k + \left[\dot{I}_{1_1}^{-1} + \cdots \dot{I}_{1_p}^{-1} + e^{-1}\right] + \dot{I}_2^{-1} + \cdots + \dot{I}_k^{-1},$$

where $\dot{I}_i = I_i \in \mathcal{I}_G$ for $i = 2, \ldots, k$.

Note, that any $\dot{\Gamma}$-component $\dot{I} \subseteq I(e)$ contains at least one edge e^* from the Γ-neighborhood of e, i.e., $\dot{I} \cap \Gamma(e) \neq \emptyset$ for all $\dot{\Gamma}$-components $\dot{I} \subseteq I(e)$. Hence, there is always a $\dot{\Gamma}$-chain from any arc e^{**} of $\dot{I} \subseteq I(e)$ into the Γ-neighborhood of e. We will therefore sometimes speak of a $\dot{\Gamma}$-chain from e^{**} to e, although, formally, it is no $\dot{\Gamma}$-chain, since arc e is involved in the last Γ-connection.

The prospect of this new substructure of the implication class $I(e)$ is that $\dot{\Gamma}$-components remain connected when the edge $\hat{e}$ is removed from G. The implication class $I(e)$ (black) in Figure 1 (left) splits into two $\dot{\Gamma}$-components. The arc az is not $\dot{\Gamma}$-connected to any other arc in $I(e)$—every Γ-chain to the remaining arcs ax and ay contains $e = ab$. Therefore we have $I(e) = \dot{I}_1 + \dot{I}_2 + e$ with $\dot{I}_1 = \{az\}$ and $\dot{I}_2 = \{ax, ay\}$. Note, that in $G - \hat{e}$ (right) the $\dot{\Gamma}$-component $\dot{I}_2$ merges with one orientation of the second color class (gray) of G.

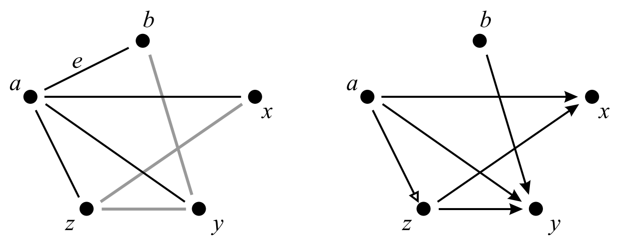


Figure 1. The relations Γ,Γ^+ and $\dot{\Gamma}$.

Different $\dot{\Gamma}$-components $\dot{I}'$ and $\dot{I}''$ of $I(e)$ almost behave like different implication classes. Two arcs $e' \in \dot{I}'$ and $e'' \in \dot{I}''$ are Γ-connected only through the arc e. Therefore two arcs $e' = ab$ and $e'' = ac$ from different $\dot{\Gamma}$-components sharing a common vertex a force the existence of the connecting edge $\widehat{bc} \in E$ finishing the triangle. Otherwise they would be directly Γ-connected, and could thus not belong to different $\dot{\Gamma}$-components of $I(e)$.

We have defined the $\dot{\Gamma}$-relation for the whole edge set E, although nothing 'happens' outside of $\hat{I}(e)$. By making this convention we do not have to distinguish between different $\dot{\Gamma}$-components within $I(e)$ on the one hand and different implication classes $I \neq I(e)$ on the other hand. Thus we may make use of statements like $A \neq B$ for $\dot{\Gamma}$-components A and B without having to know, whether A and B are two different implication classes, two different $\dot{\Gamma}$-components of $I(e)$, or a mixture of both possibilities.

3. Motivation

The answer to the question whether $G - \hat{e}$ is still a comparability graph for some comparability graph $G = (V, E)$ and a given edge $\hat{e} \in E$ is of some importance for the search for so-called irreducible sequences for the open-shop scheduling problem.

In this section we will try to give some explanations for this context. The remaining part of this paper and the result itself, however, are by no means limited to applications in the realm of scheduling theory. Merely the examples presented there are chosen as a reference to this background.

A scheduling problem generally consists of some given set of *jobs* $J_1, \ldots, J_n$ which have to be processed on some set of *machines* $M_1, \ldots, M_m$ minimizing some sort of target function as, for example, the completion time. A solution to such a problem is called a *sequence* and consists of a feasible combination of the order of machines for every job A_i (*machine order*) and the order of jobs to be processed on every machine M_j (*job order*). The corresponding *schedule* contains the information on the completion time for every *operation* (A_i, M_j).

If neither the machine order nor the job order are limited by some preset settings, we speak of an *open-shop* scheduling problem. Then it does not matter on which machine we start processing job A_i, and to which machine we hand this job over, and so on. All that matters is that *somehow* every job is being processed by every machine.

Such an open-shop problem can easily be translated into a problem on graphs. Following Bräsel [1, 2] each operation (A_i, M_j) can be identified with a vertex (ij), where different operations are connected by an edge if they cannot be processed at the same time. The resulting graph G is thus isomorphic to the so-called Hamming graph $K_m \times K_n$. A feasible combination of machine order and job order—a sequence—then translates into an acyclic orientation of G (see Figure 2). If each vertex is now weighted with the given processing times for the respective operation, the problem of minimizing the completion time $C_{\max}$ for all operations can be stated as the problem of finding an acyclic orientation on G minimizing the maximal weighted path.

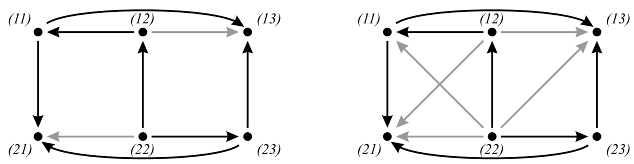


Figure 2. A sequence graph $G(B)$ representing the sequence $B = \binom{3\ 2\ 4}{4\ 1\ 2}$ and its transitive closure.

Furthermore we can associate each sequence A with a uniquely determined comparability graph by computing the transitive closure of the directed graph $G(A)$.

We say, a sequence B is *reduced* by some other sequence A, $A \preceq B$ if the processing times cannot be chosen such that the completion time for sequence B is less than that for sequence A, $C_{\max}(A) \leq C_{\max}(B)$. Two sequences with $C_{\max}(A) = C_{\max}(B)$ are called *similar*. If a sequence B is reduced by a sequence A not similar to B, it is reduced *strongly*, $A \prec B$. Finally, a sequence A is called *irreducible* if it is not strongly reduced by some other sequence B.

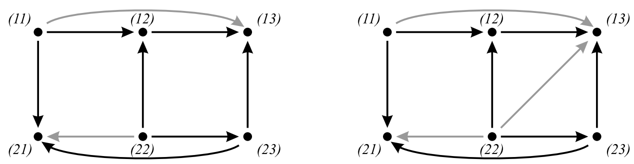


Figure 3. A sequence graph $G(A)$ representing an irreducible sequence $A = \binom{1\ 2\ 3}{3\ 1\ 2}$ and its transitive closure.

The set of all irreducible sequences of an open-shop problem is of interest, because it is a set containing an optimal solution for any choice of processing times, a so-called *universally optimal set*, considerably smaller than the set of all sequences.

Up to now it is an open problem whether a given sequence can be detected as irreducible in polynomial time. For two given sequences A and B, on the other hand, Bräsel *et al.* [3] have presented a simple polynomial test for deciding whether one is the reduction of the other.

Theorem 1 (reducibility of a sequence) [3]. *Let A and B be two sequences on the same operation set SIJ. Then A reduces B, $A \preceq B$, if and only if the comparability graph belonging to A is a subgraph of the comparability graph belonging to B.*

Consider, for example, the sequences A and B in Figures 3 and 2. The comparability graph belonging to A (symmetric closure of the transitive closure of $G(A)$) is a subgraph of the comparability graph belonging to B which contains two additional edges. Hence, A reduces B. Furthermore, A is irreducible, since there is only one edge left not belonging to the Hamming graph $K_2 \times K_3$ which is no comparability graph itself.

For finding these irreducible sequences it may thus be an appropriate strategy to start with the complete graph K_{mn} containing the Hamming graph $K_m \times K_n$ as well as all possible additional '*irregular*' diagonal edges, and one by one removing these irregular edges until we find some graph where no set of irregular edges can be removed without leading to a graph that is no comparability graph any more. Then this graph is the comparability graph belonging to an irreducible sequence—which then can be obtained in polynomial time.

For this strategy the answer to the question whether the graph obtained by the deletion of a given edge is still a comparability graph is of obvious importance.*

4. Preliminaries

For the main result of this work the so-called Triangle Lemma by Golumbic [9] (with origins from Gilmore and Hoffmann [6]) will play an important role. We cite this theorem and prove it in detail, since the proof given by Golumbic [9] contains an error. Moreover, we present an extension of the Triangle Lemma for the new defined $\dot{\Gamma}$-components. The chapter closes with some important applications of both Triangle Lemmas and further theorems which are relevant for the theoretical investigations throughout this paper.

The Triangle Lemma can be interpreted as follows: Consider a graph $G = (V, E)$ and a triangle $\triangle\, abc$ in G, with arcs $ab \in B$, $ac \in C$, and $bc \in A$ belonging to different color classes, basically. Now, consider a further arc

*Unfortunately, it may happen that for some comparability graphs belonging to irreducible sequences there exists no chain of comparability graphs with each containing exactly one edge more than the previous up to the complete K_{mn}.

$b'c'$ anywhere in G, having color A. Then the Triangle Lemma states that the arcs from a to the vertexes b' and c' not only exist, but also have the same colors as the arcs from a to b and a to c, respectively. Moreover, is arc $b'c' \in A$ part of a triangle with $a'b'$ having color B, then arc $a'c'$ exists and has color C, making $\triangle\ a'b'c'$ congruent to the original one. Finally, there is no edge with color A in G touching vertex a.

For the purpose of keeping the following proofs as simple as possible, we will first introduce the notion of a canonical Γ-chain (see Golumbic [9]). Let $ab = a_0b_0\Gamma a_1b_1\Gamma \dots \Gamma a_kb_k = a'b'$ be a Γ-chain. For every pair of Γ-related arcs $a_ib_i\Gamma a_{i+1}b_{i+1}$ ($i = 1, \dots, k$) we can now insert w.l.o.g. the arc $a_{i+1}b_i$ into the chain, yielding $a_ib_i\Gamma a_{i+1}b_i\Gamma a_{i+1}b_{i+1}$. There, the new arc equals one of the other two. Thus we may assume any Γ-chain from ab to $a'b'$ to be a *canonical* Γ*-chain*, i.e., a chain of the form

$$ab = a_0b_0\Gamma a_1b_0\Gamma a_1b_1\Gamma a_2b_1\Gamma \dots \Gamma a_kb_k = a'b'.$$

Theorem 2 (Triangle Lemma) [9]**.** *Let $A, B, C \in \mathcal{I}_G$ be implication classes of a graph $G = (V, E)$ with $A \neq B^{-1}$ and $A \neq C$ and having the triangle $ab \in B$, $ac \in C$ and $bc \in A$.*

(i) *If there exists $b'c' \in A$ then there exist $ab' \in B$ and $ac' \in C$ as well:*
$\exists\, b'c' \in A \Rightarrow ab' \in B$ *and* $ac' \in C$.

(ii) *If there exist $b'c' \in A$ and $a'b' \in B$ then there exists $a'c' \in C$ as well:*
$\exists\, b'c' \in A$ *and* $a'b' \in B \Rightarrow a'c' \in C$.

(iii) *No arc in A touches vertex a:*
$a \notin V(A)$.

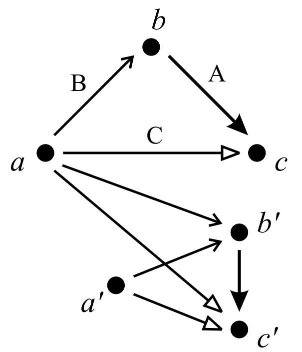


Figure 4. The Triangle Lemma.

Proof. Compare Golumbic [9] and correctness for (ii).

(i) Since $b'c' \in A$, there exists a Γ-chain in A which we write down as canonical Γ-chain: $bc = b_0c_0\Gamma b_1c_0\Gamma b_1c_1\Gamma b_2c_1\Gamma \dots \Gamma b_kc_k = b'c'$. Now, we prove by induction the existence of $ab_l \in B$ and $ac_l \in C$ for all $0 \le l \le k$. For $l = 0$ there is nothing to do, since there exist $ab_0 = ab \in B$ and $ac_0 = ac \in C$. Suppose now $l \ge 1$. Then the existence of $ab_l \in B$ and $ac_l \in C$ has to follow from the existence of $ab_{l-1} \in B$ and $ac_{l-1} \in C$. From $ac_{l-1} \in C$, $b_lc_{l-1} \in A$ and $A \neq C$, we can conclude the existence of edge $\widehat{ab_l}$. Since b_lc_{l-1} is directly Γ-connected to $b_{l-1}c_{l-1}$, we have $\widehat{b_lb_{l-1}} \notin E$. This implies $ab_l\Gamma ab_{l-1}$, i.e., ab_l and ab_{l-1} belong to the same implication class, which is B by assumption.

Since $b_lc_l \in A$ and $A \neq B^{-1}$, b_lc_l is not directly Γ-connected to b_la ($ab_l \in B$). Therefore, edge $\widehat{ac_l}$ must exist. Now, since c_l and c_{l-1} are not adjacent ($b_lc_l\Gamma b_lc_{l-1}$), arc ac_l is directly Γ-connected to $ac_{l-1} \in C$. Hence, $ac_l \in C$.

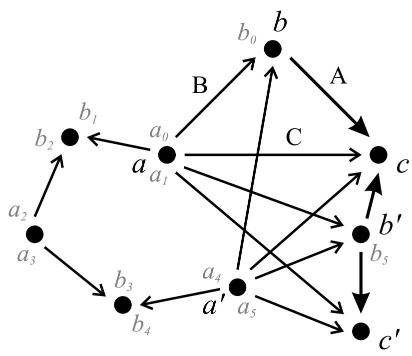


Figure 5. A Counterexample to the proof of (ii) by Golumbic [9].

(ii) Golumbic reasons by induction from property (i) and the existence of a canonical Γ-chain from ab to $a'b'$ in B to the existence of the chain $ac' = a_0c'\Gamma a_1c'\Gamma a_2c' \dots \Gamma a_lc' = a'c'$. We present in Figure 5 a counterexample for this conclusion. By assumption there have to exist two canonical Γ-chains: one in B from ab to $a'b' = a_lb_l$ and another one in A connecting bc and $b'c'$. In our counterexample we have chosen the latter very short. The first one, nevertheless, contains six arcs $e_i = a_ib_i$, $i = 0, \dots, l = 5$. We write this canonical Γ-chain the following way, $ab = a_0b_0 = a_1b_0\Gamma a_1b_1\Gamma a_2b_1 = a_2b_2 = a_3b_2\Gamma a_3b_3\Gamma a_4b_3 = a_4b_4 = a_5b_4\Gamma a_5b_5 = a'b'$. As already mentioned, Golumbic [9] concludes from this chain in B the existence of some chain $ac' = a_0c'\Gamma \ldots \Gamma a_lc' = a'c'$.

In our counterexample in Figure 5 this chain obviously does not exist. Vertex $a_2 = a_3$ is not adjacent to c. Therefore, the induction given by Golumbic is incorrect. The result, however, remains true: $ac' \in C$ (by (i)) and $a'c'$ *are* Γ-connected, indeed. But not necessarily through the proposed Γ-chain.

Note, that in our example we have made use of the feasible claim $B = C$. In fact, when claimed $B \neq C$, the proposed Γ-chain exists, indeed.

So suppose $B \neq C$. We show by induction that in each step i, $i = 0, \ldots, l$, a triangle $\triangle\ a_ib_ic$ isomorphic to triangle $\triangle\ abc$ exists. Then, by means of (i) all arcs $a_ib' \in B$ and $a_ic' \in C$, $i = 1, \ldots, l$, exist. In particular, $a_lc' = a'c'$. Consider now the canonical Γ-chain $ab = a_0b_0\Gamma a_1b_0\Gamma a_1b_1\Gamma a_2b_1\Gamma \ldots \Gamma a_lb_l = a'b'$ in B. Clearly, by assumption, $\triangle\ a_0b_0c$ exists. Let $i \geq 1$ and suppose that for all $i = 1, \ldots, r-1$ the required triangle exists. For $i = r$ then the existence of $\widehat{a_rc}$ and $\widehat{b_rc}$ follow from $A \neq B^{-1}$ and $B \neq C$, respectively. In the former case we then have a_rc being directly Γ-connected to $a_{r-1}c$, i.e., $a_rc \in C$. In the latter, we find b_rc being directly Γ-connected to $b_{r-1}c$, i.e., $b_rc \in A$. Thus, we have generated a new triangle $a_rb_r \in B$, $b_rc \in A$ and $a_rc \in C$, which is isomorphic to Δabc. To this new triangle we can apply part (i), gaining the desired existence of arcs a_ib' and a_ic'.

Suppose now $B = C$: From $A \neq B^{-1}$ follows the existence of edge $\widehat{a'c'}$. By part (i) we have $ac' \in C$. Suppose $a'c' \in D \neq C$. Now, we can apply part (i) to the reversed triangle $\triangle\ a'b'c'$ (with $c'b' \in A^{-1}$, $b'a' \in B^{-1}$ and $c'a' \in D^{-1}$), with respect to $ba \in B^{-1}$. (This application is feasible, since $B^{-1} = C^{-1} \neq D^{-1}$ and $B^{-1} \neq (A^{-1})^{-1}$.) This, particularly, yields $ac' \in D$, contradicting $ac' \in C \neq D$.

(iii) $a \notin V(A)$ directly follows from (i). ∎

A direct application of this Triangle Lemma is the following theorem. The important part of this statement from our point of view is that proper implication classes are transitive orientations. Hence, a potential transitive orientation is indeed transitive if it is acyclic.

Theorem 3 (implication classes are transitive) [9]**.** *Let $I \in \mathcal{I}_G$ be an implication class of a graph $G = (V, E)$. Then either $I = \hat{I} = I^{-1}$, or $I \cap I^{-1} = \emptyset$ and I and I^{-1} are (the only) transitive orientations of $\hat{I}$.*

As mentioned above, the Triangle Lemma can be extended on the $\dot{\Gamma}$-components introduced earlier. The understanding of the following proposition is crucial for understanding the proofs to come. Since in most cases of our applications either two or all three arcs of each triangle will belong to the same implication class, the original Triangle Lemma is of little help. But because of the similar behavior of different $\dot{\Gamma}$-components compared to different implication classes its main result (part (i)) can be taken over almost one to one. Just the special role of the firmly given edge $\hat{e}$ requires some special attention.

Lemma 1 (extended Triangle Lemma)**.** *Let $G = (V, E)$ be a graph with the triangle ab, bc, ac, and with $e \in E$ fixed below. Let $\dot{A}$, $\dot{B}$ and $\dot{C}$ be $\dot{\Gamma}$-components referring to $\hat{e}$, with $\dot{A} \neq \dot{B}^{-1}$ and $\dot{A} \neq \dot{C}$. Consider the following cases:*

(1) *$ab \in \dot{B}$, $ac \in \dot{C}$, and $bc \in \dot{A}$,*
with $a \notin V(e)$ or $\{\dot{C} \nsubseteq I(e)$ and $[B \nsubseteq I(e)$ or $A \subseteq I(e)]\}$;

(2) *$ab \in \dot{B}$, $ac \in \dot{C}$, and $e = bc$, with $\dot{A} \subseteq I(e)$;*

(3) *$ab \in \dot{B}$, $ac \in \dot{C}$, and $bc \in \dot{A}$, with $e = b'c'$, $\dot{A} \subseteq I(e)$,*
and $\{\dot{C} \nsubseteq I(e)$ or $[a \notin V(e)$ and $\widehat{ab'} \in E]\}$.

Then $b'c' \in \dot{A}$ or $e = b'c'$, respectively, implies $ab' \in \dot{B}$ and $ac' \in \dot{C}$ $(1, 2, 3)$, and $a \notin V(\dot{A})$ $(1, 2)$ or $a \notin V(e)$ (3) (for $\dot{C} \nsubseteq I(e)$), respectively.

Statement (2) shall be understood in the following way. If there is some arc $b'c'$ belonging to an *arbitrary* $\dot{\Gamma}$-component $\dot{A} \subseteq I(e)$ in G with $\dot{A} \neq \dot{B}^{-1}$ and $\dot{A} \neq \dot{C}$, then $ab' \in \dot{B}$, $ac' \in \dot{C}$ and $a \notin V(\dot{A})$.

Proof. The proof is similar to the proof of the Triangle Lemma (Theorem 2). As in the case for implication classes, two adjacent arcs from different $\dot{\Gamma}$-components $\dot{A}$, $\dot{B}$ or $\dot{C}$ are not in Γ-relation to each other (definition of $\dot{\Gamma}$-components).

Suppose $b'c' \in \dot{A}$ $(1, 2)$, or $e = b'c'$ (3). There exists a canonical $\dot{\Gamma}$-chain $bc = b_0c_0 \dot{\Gamma} b_1c_0 \dot{\Gamma} b_1c_1 \dot{\Gamma} \ldots \dot{\Gamma} b_kc_k = b'c'$ in $\dot{A}$. (Note, $\dot{A} \cap \Gamma(e) \neq \emptyset$ $(2, 3)$.) Alternating, the following holds. From each $\dot{\Gamma}$-relation $b_{i-1}c_{i-1} \dot{\Gamma} b_ic_{i-1}$ follows $\widehat{b_{i-1}b_i} \notin E$. $\dot{A} \neq \dot{C}$ (together with $ac_{i-1} \neq e$, and $b_ic_{i-1} \neq e$ for $b_i \neq b_0$) implies the existence of $\widehat{ab_i} \in E$ with $ab_i \in \dot{B}$ $(ab_i \dot{\Gamma} ab_{i-1})$ $(1, 2, 3)$. For case (1), $ab_i \neq e$ is ensured by the given condition. For $a \in V(e)$ either $B \nsubseteq I(e)$ or $A \subseteq I(e)$ (with $C \nsubseteq I(e)$) suffices to exclude $ab_i = e$. In the latter case $(A \subseteq I(e))$ $ab_i = e$ would lead to $B \subseteq I(e)$, and hence, by transitivity (Theorem 3) $C \subseteq I(e)$ as well (contradiction). For case (3) in the first place only follows $ab_i \Gamma ab_{i-1}$, where $ab_i = e = b'c'$ is feasible. But then the remaining part of the induction may be conducted with some $\dot{\Gamma}$-component $\dot{B}' \subseteq I(e)$ instead of $\dot{B} \subseteq I(e)$, yielding $ac' = ab_i = e \in C$ as well (contradiction to $C \nsubseteq I(e)$).

On the other hand, each $\dot{\Gamma}$-relation $b_ic_{i-1}\dot{\Gamma}b_ic_i$ implies $\widehat{c_{i-1}c_i} \notin E$. From $\dot{A} \neq \dot{B}^{-1}$ and $ab_i \in \dot{B}$ as well as $b_ic_i \in \dot{A}$ then follows $\widehat{ac_i} \in E$ with $ac_i \in \dot{C}$ $(ac_i\dot{\Gamma}ac_{i-1})$ $(1, 2, 3)$.

By induction on i there exist $ab_i \in \dot{B}$ and $ac_i \in \dot{C}$ for all $i = 0, \dots, k-1$ (with $ac_i \neq e$ for all i, and e not inner part of the $\dot{\Gamma}$-chain) $(1, 2, 3)$. For the cases $(1, 2)$ the induction yields $ab_k \in \dot{B}$ and $ac_k \in \dot{C}$ as well. For the case $e = b'c'$ (3) either claim, $\dot{C} \nsubseteq I(e)$ or $\widehat{ab'} \in E$, may be necessary to force the *existence* of $ab_k = ab' \in \dot{B}$. (This necessity only arises for $b_{k-1}c_{k-1}$ with $c_{k-1} = c_k$ being the last arc before $b_kc_k = b'c' = e$ in the $\dot{\Gamma}$-chain. Then edge $\widehat{ab'}$ need not exist for $\dot{C} \subseteq I(e)$.) Hence, $ab' \in \dot{B}$ and $ac' \in \dot{C}$ $(1, 2, 3)$. This immediately implies $a \notin V(\dot{A})$ or $a \notin V(e)$, respectively. ∎

Part (ii) of the Triangle Lemma cannot be taken over for $\dot{\Gamma}$-components as easily as part (i) and part (iii). For obtaining this as well, further (restricting) assumptions have to be made. As we have no need for this application we do not undertake this challenge.

For $\dot{\Gamma}$-components we can show as well as for implication classes that, in principal, they are transitive orientations.

Lemma 2 ($\dot{\Gamma}$-components are transitive)**.** *Let $G = (V, E)$ be a comparability graph, $e \in E$ an arbitrary arc and $\dot{I}$ a $\dot{\Gamma}$-component of $I(e) \in \mathcal{I}_G$ referring to $\hat{e}$. Then $\dot{I} + e$ and $\dot{I}^{-1} + e^{-1}$ are transitive orientations of $\hat{I} + \hat{e}$.*

Proof. From G being a comparability graph follows $I(e) \cap I(e)^{-1} = \emptyset$ (each implication class is proper) and therefore $\dot{I} \cap \dot{I}^{-1} = \emptyset$ as well. Let ab und bc be two arcs in $\dot{I}$. With Theorem 3 the transitive arc ac belongs to $I(e)$ as well. The statement is trivial for $ac = e$. So suppose $ac \in \dot{J} \neq \dot{I}$, and $a \notin V(e)$. Then the triangle $\triangle\ abc$ (with edges ab, bc and ac) holds the assumptions of case (1) in Lemma 1, and it follows $a \notin V(\dot{I})$—contradicting $ab \in \dot{I}$. So suppose now $a \in V(e)$ with $e \neq ac$. Then Lemma 1 (case (1)) may be applied to arcs ab^{-1}, bc^{-1} and ac^{-1} ($\dot{I}^{-1} \neq (\dot{I}^{-1})^{-1}$, $\dot{I}^{-1} \neq \dot{J}^{-1}$) yielding $c \notin V(\dot{I}^{-1})$—in contradiction to $cb \in \dot{I}^{-1}$. ∎

In addition to the Triangle Lemma and its applications the following considerations will be of some importance. As already mentioned above each transitive orientation $T = J_1 + \cdots + J_k \in \mathcal{T}_G$ with $J_i \in \{I_i, I_i^{-1}\}$ $(i = 1, \dots, k)$ is an acyclic combination of transitive orientations of the color classes of G.

Let us now consider the consequences of a removal of an arbitrary edge $\hat{e} \in E$ from $G = (V, E)$. First of all, no direct Γ-relation $e'\Gamma e''$ will be destroyed—beside those involving e or e^{-1} directly, of course. Two arcs from an implication class $J \neq I(e), I(e)^{-1}$ are thus Γ-connected in $G - \hat{e}$ as well. Hence, the deletion of $\hat{e}$ has no consequences on the connectivity of color classes differing from $\hat{I}(e)$.

On the other hand it may happen that there emerge new Γ-relations in $G - \hat{e}$. If we remove $\hat{e} = \widehat{ab}$ from a triangle $\triangle\ abc$, then the remaining arcs ac and bc, as well as ca and cb suddenly are directly Γ-related. Thus it may happen that different implication classes merge (if ac and bc belong to different implication classes in G). The merger of an implication class I with its reversal I^{-1} is by Theorem 3 only for $I = I(e)$ or $I = I(e)^{-1}$ possible.

We gather these considerations in the following proposition.

Lemma 3 (Γ-connections in $G - e$). *Let $G = (V, E)$ be a comparability graph and let $\mathcal{I}_G = \{I_1, \ldots, I_k, I_1^{-1}, \ldots, I_k^{-1}\}$ be the set of its implication classes. If $\hat{e} = \widehat{ab} \in E$ is an edge of G, then for $G - \hat{e}$ holds*

(i) *$e'\Gamma e''$ in $G \Rightarrow e'\Gamma e''$ in $G - \hat{e}$ for all $e', e'' \neq e, e^{-1}$.*
(*Implication classes $I_i \neq I(e)$ do not split up in $G - \hat{e}$.*)

(ii) *$e'\Gamma e''$ in $G - \hat{e}$ for all $e' = ac, e'' = bc \in E$.*
(*There may arise new Γ-connections in $G - \hat{e}$ which may connect different implication classes of G.*)

(iii) *Let $J \in \mathcal{I}_{G-\hat{e}}$ be an implication class of $G - \hat{e}$ which contains no arc from $I(e)$ or $I(e^{-1})$. Then J is an union of implication classes from $\mathcal{I}_G \setminus \{I(e), I(e)^{-1}\}$ with $J \cap J^{-1} = \emptyset$.*

Thus, the question, whether $G - \hat{e}$ is a comparability graph, only depends on the implication class $I(e)$ containing e (and its reversal $I(e^{-1})$, of course). The deletion of $\hat{e}$ splits $I(e)$ into its $\dot{\Gamma}$-components (if there is more than one). These $\dot{\Gamma}$-components may merge with other implication classes or with reversals of some $\dot{\Gamma}$-components, but not with other $\dot{\Gamma}$-components of $I(e)$ (because of the 'transitivity' of each $\dot{\Gamma}$-component). If some $\dot{\Gamma}$-component of $I(e)$ merges with some implication class J and at the same time with its reversal J^{-1} as well the resulting implication class is improper, and $G - \hat{e}$ is no comparability graph. If, on the other hand, every $\dot{\Gamma}$-component of $I(e)$ only merges with either some J *or* J^{-1}, the resulting implication classes of $G - \hat{e}$ are proper, and hence $G - \hat{e}$ is a comparability graph.

Corollary 1 (Γ-connections in $G - e$)**.** *Let $G = (V, E)$ be a comparability graph and let $\hat{e} \in E$ be an arbitrary edge. Then $G - \hat{e}$ is a comparability graph if and only if each implication class $J \in \mathcal{I}_{G-\hat{e}}$ that contains any arcs from $I(e) \in \mathcal{I}_G$ is a proper implication class.*

The general aim of this paper is to explain the circumstances under which $G - \hat{e}$ is still a comparability graph. This problem is almost completely solved by the following theorem by Willenius [16]. Every edge $\hat{e}$ for which there exists a transitive orientation $T \in \mathcal{T}_G$ containing neither e nor e^{-1} as transitive edge, may be removed without causing any harm.

Theorem 4 (edge from transitive reduction) [16]**.** *Let $T \in \mathcal{T}_G$ be a transitive orientation of $G = (V, E)$ with $e \in T$. $T - e$ is a transitive orientation of $G - \hat{e}$ if and only if e is not transitive in T, i.e., e belongs to the transitive reduction of T.*

This leaves our stated question unanswered only for those edges $\hat{e}$ that are transitive in *every* transitive orientation, i.e., for arcs, where either e or e^{-1} is transitive in T for every $T \in \mathcal{T}_G$. We will call such edges *always transitive.* Theorem 4 only states that none of *these* orientations T is transitive in $G - \hat{e}$ any more. But this does not necessarily mean that there are no transitive orientations on $G - \hat{e}$. Indeed, it may happen that there exist transitive orientations on $G - \hat{e}$ having no correspondents in G.

Consider, for example, the graph displayed in Figure 6 on page 440. Both edged $\hat{e}_1 = \widehat{15}$ and $\hat{e}_2 = \widehat{28}$ are transitive in each of the 4 transitive orientations. Nevertheless, both $G - \hat{e}_1$ and $G - \hat{e}_2$ are comparability graphs as well.

Finally, we will need some aspect of the following considerations on the number of transitive orientations by Golumbic [7].

A complete subgraph of $G = (V, E)$ on $r + 1$ vertices with all edges belonging to different color classes is called a *simplex* (V_S, S) of rank r. Adding all other edges of these color classes to S as well will lead us to a so-called *multiplex* (V_M, M) of rank r, i.e., $M = \bigcup_{\hat{I} \cap S \neq \emptyset} \hat{I}$. Such a simplex (multiplex) is called *maximal* if it is not part of a larger one. It can easily be shown that a multiplex M is maximal if and only if each simplex S inducing M is maximal. Golumbic shows in [7] and [9] that the edge set E of each graph $G = (V, E)$ has a unique partition into maximal multiplices. Furthermore each multiplex is a comparability graph, and each transitive orientation of G has a partition into transitive orientations of the respective multiplices. Thus the number of transitive orientations of $G = (V, E)$ can be computed from the partition of E into multiplices.

Theorem 5 (number of transitive orientations) [7]. *Let $G = (V, E)$ be a graph and let $E = M_1 + \cdots + M_k$, where each M_i is a maximal multiplex of E.*

(i) *If T is a transitive orientation of G, then $T \cap M_i$ is a transitive orientation of M_i.*

(ii) *Are $T_1, \ldots, T_k$ transitive orientations of $M_1, \ldots, M_k$, respectively, then $T = T_1 + \cdots + T_k$ is a transitive orientation of G.*

(iii) $t(G) = t(M_1) \cdot \ldots \cdot t(M_k)$.

(iv) *If G is a comparability graph and r_i is the rank of M_i, respectively, then $t(G) = \prod_{i=1}^{k}(r+1)!$.*

Although this result gives a remarkable insight into the structure of comparability graphs we only cite it because of a small application. This theorem states that transitive orientations of different multiplices can be combined independently. Therefore we may conclude that for each tricolored triangle (simplex of rank 2) in a comparability graph G there is a combination of transitive orientations of the remaining color classes leading to a transitive orientation of G for *any* acyclic orientation of the tricolored triangle. This is obviously if the triangle is a maximal simplex, but works as well, if the triangle is only a part of a larger simplex. (Consider, for example, orientations of the maximal simplex where all edges touching the vertices of the triangle are directed either to, or away from these).

The background of this application is given by our need to make sure that for a given transitive orientation $T \in \mathcal{T}_G$ of G containing some tricolored triangle it is always possible to choose the orientations of the remaining color classes such that each implication class belonging to that triangle can be reversed independently, leading always to a transitive orientation T^* of G.

5. Always Transitive Edges

As mentioned above we will characterize those edges whose removal from G leads to a graph that is still a comparability graph, by a suitable partition of the edge set E.

5.1. The sets E_N, E_T and E_R

First, we split the edges of $G = (V, E)$ into the subsets of all *never transitive* edges E_N, all *always transitive* edges E_T, and all remaining edges E_R. We already have mentioned always transitive edges in the previous section.

Definition 2 (E_N, E_T and E_R)**.** Let $G = (V, E)$ be a comparability graph and let $\mathcal{T}_G$ be the set of transitive orientations of G. We define E_N, E_T and E_R with $E = E_N + E_T + E_R$ the following way,

$$
\begin{aligned}
E_N &= \{e \in E \colon \hat{e} \text{ not transitive in } T, \text{ for all } T \in \mathcal{T}_G\}, \\
E_T &= \{e \in E \colon \hat{e} \text{ transitive in } T, \text{ for all } T \in \mathcal{T}_G\}, \\
E_R &= \{e \in E \colon \exists\, T_1, T_2 \in \mathcal{T}_G \colon \hat{e} \text{ transitive in } T_1, \hat{e} \text{ not transitive in } T_2\}.
\end{aligned}
$$

By an undirected edge $\hat{e}$ being transitive in some transitive orientation $T \in \mathcal{T}_G$ we mean that either e or e^{-1} is transitive in T, or, analogously, that e is transitive in either T or T^{-1}.

From Theorem 4 directly follows that we only have to cope with the subset E_T.

Lemma 4 ($e \in E_N$, $e \in E_R$)**.** *Let $G = (V, E)$ be a comparability graph and let $\hat{e} \in E_N + E_R$. Then $G - \hat{e}$ is a comparability graph as well.*

Proof. For every edge $\hat{e} \in E_N + E_R$ there exists a transitive orientation, in which $\hat{e}$ is not transitive. Any such orientation is transitive on $G - \hat{e}$ by Theorem 4. ∎

5.2. The sets E_{T_0} and E_{T_1}

We will now partition the set of all always transitive edges E_T further into two subsets E_{T_0} and E_{T_1}. We therefore consider the consequences of a removal of some edge $\hat{e}$ from G to the implication class $I(e) \in \mathcal{I}_G$. This splits $I(e)$ into its $\dot{\Gamma}$-components. If $I(e)$ consists of only one $\dot{\Gamma}$-component, e belongs to E_{T_0}. If, on the other hand, $I(e)$ splits into several $\dot{\Gamma}$-components by the removal of e, the edge e belongs to E_{T_1}.

Definition 3 (E_{T_0} and E_{T_1})**.** Let $G = (V, E)$ be a comparability graph and let E_T be the set of all always transitive edges. We define E_{T_0} and E_{T_1} with

$E_T = E_{T_0} + E_{T_1}$ the following way,

$$\begin{aligned} E_{T_0} &= \Big\{ e \in E_T \colon e_1 \dot{\Gamma}^+ e_2 \text{ for all } e_1, e_2 \in I(e) - e \Big\}, \\ E_{T_1} &= \Big\{ e \in E_T \colon \exists\, e_1, e_2 \in I(e) - e \text{ with } e_1 \not{\dot{\Gamma}}^+ e_2 \Big\} \\ &= \Big\{ e \in E_T \colon \exists\, e_1, e_2 \in \Gamma(e) \text{ with } e_1 \not{\dot{\Gamma}}^+ e_2 \Big\}. \end{aligned}$$

Particularly, $|\Gamma(e)| \geq 2$ for $e \in E_{T_1}$.

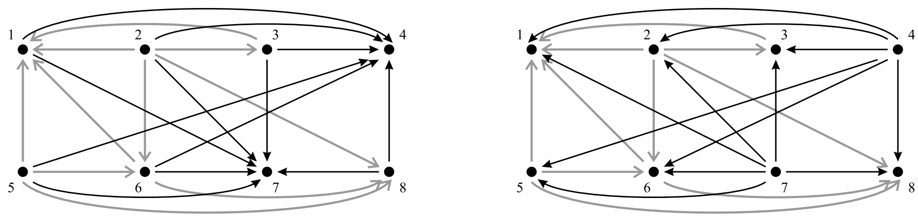


Figure 6. Example 1 ($e \in E_{T_{11}}$). Both transitive orientations of G (two different implication classes). For $\hat{e}_1 = \widehat{15} \in E_{T_1}$ and $\hat{e}_2 = \widehat{28} \in E_{T_1}$ we find that both $G - \hat{e}_1$ and $G - \hat{e}_2$ are comparability graphs, indeed.

For reasons of symmetry e^{-1} belongs to the same set as e. To illustrate these definitions we will now consider an example.

Example 1 ($e \in E_{T_{11}}$)**.** Consider the graph $G = (V, E)$ from Figure 6 which has two different color classes and hence exactly two 'different' transitive orientations. There are eight always transitive edges, $E_T = \{\widehat{12}, \widehat{15}, \widehat{28}, \widehat{34}, \widehat{37}, \widehat{46}, \widehat{58}, \widehat{67}\}$, where two belong to E_{T_1}, $E_{T_1} = \{\widehat{15}, \widehat{28}\}$, while the remaining belong to E_{T_0}. While $G - \hat{e}$ is indeed a comparability graph for each of the edges from E_{T_1}, it is not for each edge from E_{T_0}. Later we will find $E_{T_1} = E_{T_{11}}$ for this example.

As we will see it is not very difficult to show that there is no possibility for $G - \hat{e}$ to be a comparability graph if e belongs to E_{T_0}. By Theorem 4 neither a transitive orientation of G can survive in $G - \hat{e}$, nor can any new potential transitive orientation emerge by definition of E_{T_0} ($I(e)$ contains only one $\dot{\Gamma}$-component) and Lemma 3. For E_{T_1}, on the other hand, such a general statement is not possible.

Lemma 5 ($e \in E_{T_0}$). *Let $G = (V, E)$ be a comparability graph and let $\hat{e} \in E_{T_0}$. Then $G - \hat{e}$ is no comparability graph.*

Proof. Let $\mathcal{C}_G = \{\hat{I}_1, \ldots, \hat{I}_k\}$ be the set of color classes of G. We associate every potential transitive orientation, i.e., every combination of transitive orientations of the color classes, with a vector $v \in \mathcal{V}_G = \{0,1\}^k$, where v_i represents the orientation of $\hat{I}_i$ (I_i or I_i^{-1}). Any chosen v represents a *transitive* orientation $T_v \in \mathcal{T}_G$ of G if and only if it is *acyclic* (with Theorem 3). Let $\mathcal{V}_G^{tr}$ be the set of points in $\mathcal{V}_G$ that correspond to orientations of G that are transitive.

By our claim (together with Lemma 3) no implication class splits up. Therefore we may identify any potential transitive orientations of $G - \hat{e}$ with one of G—new combinations of implication classes cannot arise—so $\mathcal{V}_{G-\hat{e}} = \mathcal{V}_G = \mathcal{V}$.

Let us now assume, there exists a transitive orientation $T \in \mathcal{T}_{G-\hat{e}}$ on $G - \hat{e}$. Then T is represented by some $w \in \mathcal{V}$, i.e., $T = T_w - e$ for some potential transitive orientation T_w of G. The given arc e is always transitive ($E_{T_0} \subseteq E_T$). Therefore w cannot correspond to a transitive orientation of G, i.e., $w \notin \mathcal{V}_G^{tr}$, since otherwise, T would not be transitive. Hence, T_w is not acyclic. Let C be a (smallest) cycle in T_w. W.l.o.g. (with Theorem 3), C has length 3 and contains e. Otherwise, $T_w - e$ would not be acyclic. If $e = ab \in C$, then $bc, ca \in C$ as well. But then $bc, ca \in T$ and $ba \notin T$. This contradicts the transitivity of $T \in \mathcal{T}_{G-\hat{e}}$. ∎

5.3. The sets $E_{T_{10}}$ and $E_{T_{11}}$ (1)

As we have mentioned it is not possible to deduce a general statement for always transitive edges $\hat{e}$ whose implication class $I(e)$ splits into several $\dot{\Gamma}$-components ($\hat{e} \in E_{T_1}$). While $G - \hat{e}$ is indeed a comparability graph for all edges $e \in E_{T_1}$ for the graph in Example 1 (Figure 6), there exists an edge $\hat{e} = \widehat{46} \in E_{T_1}$ in the graph in Example 2 (Figure 7), such that $G - \hat{e}$ is no comparability graph.

If $\hat{e} \in E$ is an always transitive edge, then there exists at least one pair of edges $\{\hat{e}_1, \hat{e}_2\}$ in every orientation $T \in \mathcal{T}_G$ that makes $\hat{e}$ transitive in T, i.e., $e = ac$ is transitive in T through $e_1 = ab$ and $e_2 = bc$, or e^{-1} is transitive through e_1^{-1} and e_2^{-1}, respectively. Let us now consider the set of all such pairs of *transitiving edges* throughout all transitive orientations. It becomes clear immediately that there is at least one *minimal* set (by inclusion) $E_{\hat{e}} = \{\hat{P}_1, \ldots, \hat{P}_s\}$ of such pairs $\hat{P} = \{\hat{e}_1, \hat{e}_2\}$, such that $\hat{e}$ is transitive in *each* orientation $T \in \mathcal{T}_G$ through the edges of *some* pair $\hat{P}_i$ ($i \in \{1, \ldots, s\}$) in $E_{\hat{e}}$.

There may exist several such minimal sets $E_{\hat{e}}$. So let $E_{\hat{e}}^{(1)}, \ldots, E_{\hat{e}}^{(r)}$ each be a minimal set of pairs of transitiving edges,

$$\begin{aligned} E_{\hat{e}}^{(1)} &= \left\{\hat{P}_1^{(1)}, \ldots, \hat{P}_{s_1}^{(1)}\right\}, \\ &\vdots \\ E_{\hat{e}}^{(r)} &= \left\{\hat{P}_1^{(r)}, \ldots, \hat{P}_{s_r}^{(r)}\right\}. \end{aligned}$$

Thus, $\hat{e}$ is transitive in every transitive orientation through the edges of some $\hat{P}_i^{(j)}$ for at least one $i \in \{1, \ldots, s_j\}$ for every $j = 1, \ldots, r$.

In Example 1. For $\hat{e} = \widehat{12}$ we have, for example, $E_{\hat{e}}^{(1)} = \hat{P}_1^{(1)} = \{\widehat{13}, \widehat{32}\}$ and $E_{\hat{e}}^{(2)} = \hat{P}_1^{(2)} = \{\widehat{16}, \widehat{62}\}$, while for $\hat{e} = \widehat{34}$ there is $E_{\hat{e}}^{(1)} = \{\hat{P}_1^{(1)}, \hat{P}_2^{(1)}\}$ with $\hat{P}_1^{(1)} = \{\widehat{31}, \widehat{14}\}$ and $\hat{P}_2^{(1)} = \{\widehat{32}, \widehat{24}\}$.

The small number of such pairs in each minimal set in our example is no coincidence. We will show in the next section that every minimal set of pairs of transitiving edges consists either of only one pair or of exactly two pairs, i.e., $s_j \leq 2$ for all $j = 1, \ldots, r$. Furthermore we will understand that at least one edge of each pair belongs to $\hat{I}(e) = \hat{I}_1$, while all other edges in case of $s_j = 2$ belong to the same color class $\hat{I}_2 \neq \hat{I}_1$. Thus there are exactly two possibilities for making an edge $\hat{e}$ a transitive one; e is transitive within its own implication class $I(e)$ (at least one minimal set consists of only one pair of transitiving edges), or e is transitive by two pairs of transitiving edges from two different color classes, one of which being $\hat{I}(e)$.

We finally denote with $E_{\hat{e}}^*$ the union of all these minimal sets, $E_{\hat{e}}^* = \bigcup_{j=1}^r E_{\hat{e}}^{(j)} = \hat{Q}_1 + \cdots + \hat{Q}_t$. Since some of these pairs $\hat{Q}_i = \{\hat{e}_1, \hat{e}_2\}$ may belong to several minimal sets $E_{\hat{e}}^{(j)}$, we have $s_{\max} \leq t \leq \sum_{j=1}^r s_j$.

With these notations in mind we can now present our last partition. Since our main question is only unanswered for edges from E_{T_1} we will split this edge set further into the subsets $E_{T_{10}}$ ($G - \hat{e}$ is no comparability graph) and $E_{T_{11}}$ ($G - \hat{e}$ is a comparability graph). Remember that an always transitive arc e belongs to E_{T_1} if its implication class is split into several $\dot{\Gamma}$-components by the removal of e.

Definition 4 ($E_{T_{10}}$ and $E_{T_{11}}$)**.** Let $G = (V, E)$ be a comparability graph and let E_{T_1} be the set of all always transitive edges whose implication classes contain more than one $\dot{\Gamma}$-component. We define $E_{T_{10}}$ and $E_{T_{11}}$ with $E_{T_1} = E_{T_{10}} + E_{T_{11}}$ the following way,

$$E_{T_{10}} = \left\{ e \in E_{T_1} : \begin{array}{l} \exists\, Q_i = \{e', e''\} \subseteq E^*_{\hat{e}} \text{ with } e'\dot{\Gamma}^+ e'', \text{ or} \\ \exists\, e' \in P^{(j)}_l \cap I(e), e'' \in P^{(j)}_k \cap I(e) \ \ (l \neq k) \\ \text{with } e'\dot{\Gamma}^+ e'' \text{ for some } j \in \{1, \ldots, r\} \end{array} \right\},$$

$$E_{T_{11}} = \left\{ e \in E_{T_1} : \begin{array}{l} e' \not{\dot{\Gamma}}^+ e'' \text{ for all } \{e', e''\} = Q_i \subseteq E^*_{\hat{e}}, \text{ and} \\ e' \not{\dot{\Gamma}}^+ e'' \text{ for all } e' \in P^{(j)}_l \cap I(e), \\ e'' \in P^{(j)}_k \cap I(e) \ (l \neq k) \text{ for any } j = 1, \ldots, r \end{array} \right\}.$$

An always transitive arc $e \in E_{T_1}$ belongs to $E_{T_{11}}$ if and only if the Γ-connections between all arcs in $I(e)$ that play some role for the transitivity of $\hat{e}$ are destroyed. This requirement has to be met by all arcs from pairs $Q = \{e', e''\}$, where both e' and e'' belong to $I(e)$, as well as for all arcs $e', e'' \in I(e)$ from different pairs $P^{(j)}_l$ and $P^{(j)}_k$ from the same minimal set $E^{(j)}_{\hat{e}}$.

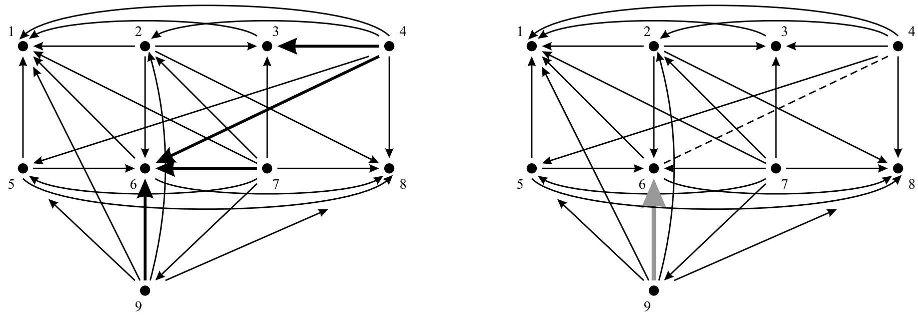


Figure 7. Example 2 ($e \in E_{T_{10}}$). For $\hat{e} = \widehat{46} \in E_{T_1}$ we find that $G - \hat{e}$ is no comparability graph.

On the other hand, an always transitive arc $e \in E_{T_1}$ belongs to $E_{T_{10}}$ if and only if the removal of $\hat{e}$ does not sufficently destroy *every* Γ-chain in $I(e)$ between arcs relevant for e being always transitive.

As for all other edge sets we have the symmetric closure for these sets as well—both directions of each edge $\hat{e} \in E_{T_1}$ belong to the same subset. Now, consider an example for illustration again.

Example 2 ($e \in E_{T_{10}}$)**.** Consider the graph $G = (V, E)$ displayed in Figure 7. G has an unique transitive orientation (left). Consider further the (always) transitive arc $e = 46$ which splits $I(e)$ into two $\dot{\Gamma}$-components $\dot{I}'$ and $\dot{I}''$, where $\dot{I}' = \{e, 96\}$ and $\dot{I}'' = I(e) - 96$. Hence, $e \in E_{T_1}$. But there are two pairs of transitiving edges, $E_{\hat{e}} = \hat{P}^{(1)} = \{\widehat{42}, \widehat{26}\}$ and $E_{\hat{e}} = \hat{P}^{(2)} = \{\widehat{45}, \widehat{56}\}$, where both edges belong to the same $\dot{\Gamma}$-component $\dot{I}''$. Hence, $e \in E_{T_{10}}$. $G - \hat{e}$ is no comparability graph (right), since the transitive edge from vertex 4 to vertex 6 is missing (the $\dot{\Gamma}$-component $\dot{I}''$ merges with its reversal, thus forming an improper implication class of $G - \hat{e}$).

As we have already mentioned, these subsets are defined such that $G - \hat{e}$ is always a comparability graph for $e \in E_{T_{11}}$ (Lemma 10), but never for $e \in E_{T_{10}}$ (Lemma 9). But although these characterizations are somewhat intuitive—even in spite of the formalities of their definitions—the actual proofs are quite lengthy, especially for the case $e \in E_{T_{11}}$. We will explore these results in the subsection next to the following. In the next subsection, however, we need to lay some further ground for finishing our consideration.

5.4. Properties of minimal sets of pairs of transitiving edges

For proving our claim regarding the remaining sets $E_{T_{10}}$ and $E_{T_{11}}$ in the next section (Lemmata 9 and 10) we will need some knowledge about the structure of those minimal sets of pairs of transitiving edges (Lemmata 6, 7 and 8).

Lemma 6 (transitiving edges)**.** *Let $G = (V, E)$ be a comparability graph, $\hat{e} \in E_T$, and $E_{\hat{e}} = \{\hat{P}_1, \ldots, \hat{P}_s\}$ a minimal set of pairs of transitiving edges. Then each of this pairs $\hat{P}_i$ $(i = 1, \ldots, s)$ contains at least one edge from $\hat{I}(e)$.*

Proof. Suppose, there is a graph G that does not hold this claim. Then there exists an always transitive arc $e = ac \in I_3$ in G and a pair $\hat{P} \in E_{\hat{e}}$ that contains two transitiving arcs $e_1 = ab \in I_1$ and $e_2 = bc \in I_2$, where $\hat{I}_1$, $\hat{I}_2$ and $\hat{I}_3$ are pairwise different color classes ($\hat{I}_1 \neq \hat{I}_3$ and $\hat{I}_2 \neq \hat{I}_3$ by assumption, and $\hat{I}_1 \neq \hat{I}_2$ by Theorem 3 and assumption).

Let $T = I_1 + I_2 + I_3 + J \in \mathcal{T}_G$ be a transitive orientation of G, such that e is transitive through the arcs e_1 and e_2. By Theorem 5 the proper combination J of transitive orientations on the remaining color classes may be chosen such, that not only T is acyclic but $T_1 = I_1^{-1} + I_2 + I_3 + J \in \mathcal{T}_G$ and $T_2 = I_1 + I_2^{-1} + I_3 + J \in \mathcal{T}_G$ as well.

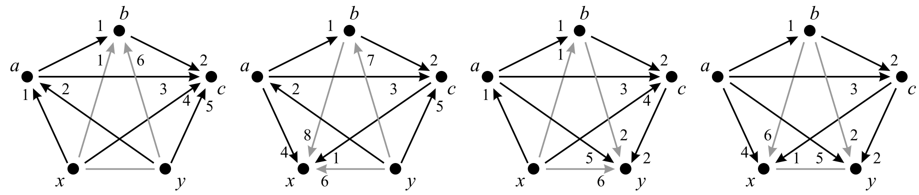


Figure 8. Cases a,b,c,d (presented in orientation $T = I_1 + I_2 + I_3 + J$).

By assumption, e is transitive in T_1 and T_2 as well. So there are transitiving arcs $l_1 = ax$ and $l_2 = xc$ in T_1, and $k_1 = ay$ and $k_2 = yc$ in T_2. W.l.o.g. all these arcs belong to $E_{\hat{e}}$ as well. Let L_1, L_2, K_1 and K_2 be the implication classes of l_1, l_2, k_1 and k_2, respectively.

From the minimality of $E_{\hat{e}}$ follows that neither l_1 and l_2, nor k_1 and k_2 belong to T at the same time. On the other side, at least one arc of each pair has to belong to T—otherwise both would belong to I_1^{-1} or I_2^{-1}, respectively, leading to a cycle in T. So either $l_1 \notin T$ or $l_2 \notin T$, and either $k_1 \notin T$ or $k_2 \notin T$. By construction of T_1 and T_2 then either $L_1 = I_1^{-1}$ or $L_2 = I_1^{-1}$, and $K_1 = I_2^{-1}$ or $K_2 = I_2^{-1}$, respectively. This gives rise to 4 different cases that have to be considered (see Figure 8). We show that none of these cases can occur.

The Triangle Lemma (part (iii)), applied to $\triangle\ abc$, implies $a \notin V(I_2)$. Therefore, the cases with $K_1 = I_2^{-1}$ (cases a and b, with $L_1 = I_1^{-1}$ or $L_2 = I_1^{-1}$, respectively) cannot occur.

Hence, $K_2 = I_2^{-1}$ (cases c and d, with $L_1 = I_1^{-1}$ or $L_2 = I_1^{-1}$, respectively). Then the Triangle Lemma may be applied to $bc \in I_2$ in $\triangle\ abc$ and $cy \in I_2$. By part (i) ac (in $\triangle\ acy$) has to be in the same implication class as ab (in $\triangle\ abc$). This contradicts $I_3 \neq I_1$. ■

Lemma 7 (number of pairs of transitiving edges)**.** *Let $G = (V, E)$ be a comparability graph, $\hat{e} \in E_T$, and $E_{\hat{e}} = \{\hat{P}_1, \ldots, \hat{P}_s\}$ a minimal set of pairs of transitiving edges. Then $s \leq 2$, and all edges in $E_{\hat{e}}$ not belonging to $\hat{I}(e)$ belong to the same color class.*

Proof. Let $T_1 \in \mathcal{T}_G$ be a transitive orientation of G and let $P_1 = \{e_1', e_1''\}$ with $e_1' = ab$ and $e_1'' = bc$ be an arbitrary pair of transitiving arcs for $e = ac \in E_T$ in T_1. Suppose $s > 1$. Then, by Lemma 6, exactly one of these two arcs belongs to $I(e) = I_1$. We may assume w.l.o.g. $e_1' \in I_1$ and $e_1'' \in I_2$ with $\hat{I}_2 \neq \hat{I}_1$ (otherwise consider the reversal orientation).

Let $T_2 = T_1 - I_2 + I_2^{-1} \in \mathcal{T}_G$ be the transitive orientation arising from T_1 by reversing I_2 (by Theorem 5 T_2 may be assumed to be transitive as well, whether or not $\hat{I}_1$ and $\hat{I}_2$ belong to the same maximal multiplex). By construction we have e, e_1' and $e_1''^{-1}$ belonging to T_2. In T_2 we again have a pair $P_2 = \{e_2', e_2''\} \in E_{\hat{e}}$. Again, exactly one of these two edges has to belong to I_1. Let this be the case for e_2'. From the minimality of $E_{\hat{e}}$ (together with $s > 1$) then follows $e_2'' \notin T_1$—otherwise e would be transitive in T_1 and T_2 through the edges of P_2, and P_1 could be removed from $E_{\hat{e}}$. Thus $e_2'' \in I_2^{-1}$.

Hence, $\hat{e}$ is transitive by either $\hat{P}_1$ or $\hat{P}_2$ for any combination of transitive orientations of $\hat{I}_1$ and $\hat{I}_2$—so $E_{\hat{e}} = \{\hat{P}_1, \hat{P}_2\}$, i.e., $s = 2$. In addition, for each pair $\hat{P}_i$ exactly one edge belongs to $\hat{I}_1$ and the other to $\hat{I}_2$ ($i = 1, \ldots, s$). ■

Lemma 8 (configuration $(*)$). *Let $G = (V, E)$ be a comparability graph, $e = ab \in E_T$, and $E_{\hat{e}} = \{\hat{P}_1, \ldots, \hat{P}_s\}$ a minimal set of pairs of transitiving edges. If $s = 2$, then $\hat{P}_1 = \{\widehat{ax}, \widehat{xb}\}$ and $\hat{P}_2 = \{\widehat{ay}, \widehat{yb}\}$ with $ab, xb, yb \in I_1$ and $ya, ax, yx \in I_2$, where $\hat{I}_1 \neq \hat{I}_2$.*

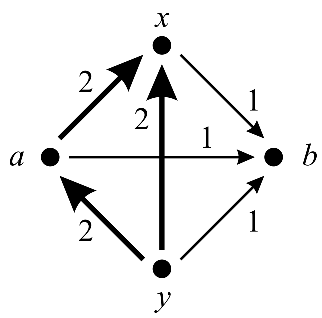


Figure 9. Configuration $(*)$.

Proof. Let $E_{\hat{e}} = \{\hat{P}_1, \hat{P}_2\}$. By assumption ($s = 2$) and Lemma 7 we can conclude that from every pair $\hat{P}_i$ ($i = 1, 2$) exactly one edge lies in $\hat{I}(e) = \hat{I}_1$ while the other belongs to $\hat{I}_2 \neq \hat{I}_1$. Let $e = ab \in I_1$ be transitive in $T_1 = I_1 + I_2 + J \in \mathcal{T}_G$ through the arcs ax and xb, both belonging to P_1. In addition, let e be transitive in a different transitive orientation, for example $T_2 = I_1 + I_2^{-1} + J \in \mathcal{T}_G$, through the arcs ay and yb, both belonging to P_2. Again, we may assume by Theorem 5 that both T_1 and T_2 are proper transitive orientations (whether or not $\hat{I}_1$ and $\hat{I}_2$ belong to a common maximal multiplex). We further may assume (w.l.o.g.) $ax \in I_2$ and $xb \in I_1$ (this assumption is, by symmetry, no limitation of the assumption

$e = ab$). Then we have either $\widehat{ya} \in \hat{I}_2$ and $\widehat{yb} \in \hat{I}_1$ (Case 1), or $\widehat{ya} \in \hat{I}_1$ and $\widehat{yb} \in \hat{I}_2$ (Case 2).

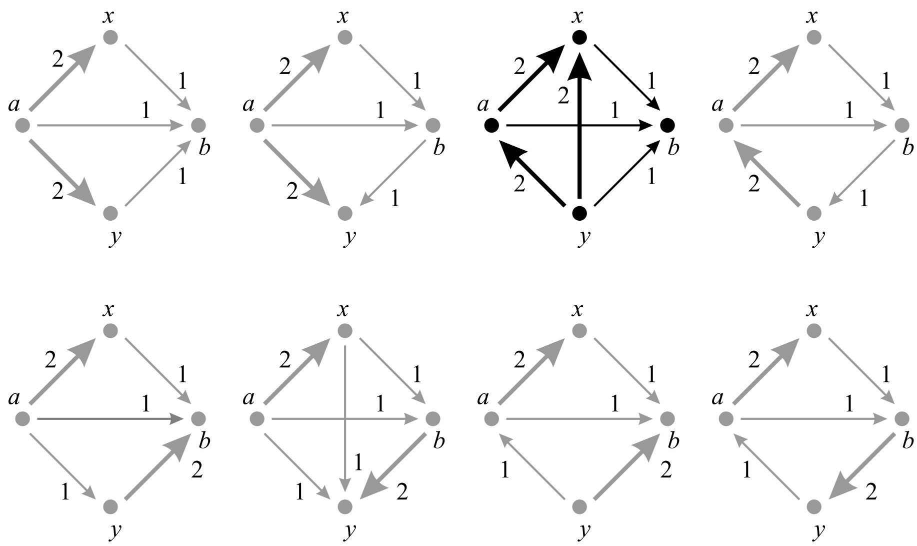


Figure 10. Cases 1.1 to 1.4 (first row) and 2.1 to 2.4 (second row).

For each case there are two possibilities for orienting each of the two edges $\widehat{ya}$ and $\widehat{yb}$. Thus there are 4 sub cases for each case (Figure 10).

(i) Suppose $\widehat{ya} \in \hat{I}_2$ and $\widehat{yb} \in \hat{I}_1$ (Case 1), and suppose further $ay \in I_2$ and $yb \in I_1$ (Case 1.1). Then the reversal of the orientation of $\hat{I}_2$ leads to an orientation T_2 that is transitive, but in which $e = ab$ is not transitive through edges of P_2—in contradiction to the assumption. Thus this case is irrelevant.

Let $ay \in I_2$ again, but suppose now $by \in I_1$ (Case 1.2). Then by Theorem 3 the transitive edge $ay \in I_2$ belongs to the same implication class as ab and $by \in I_1$—contradicting $\hat{I}_2 \neq \hat{I}_1$. Hence, this case does not occur.

Now let $ya \in I_2$ and $yb \in I_1$ (Case 1.3). Then we also have $yx \in I_2$ (by transitivity). In this case the reversal of any implication class I_1 or I_2 yields an orientation which is transitive and which contains e as always transitive edge as well. This case corresponds to the configuration mentioned in the statement.

Finally let $ya \in I_2$ again, but $by \in I_1$ (Case 1.4). Then we find a cycle (a, b, y) in T_1—contradicting $T_1 \in \mathcal{T}_G$. Thus this case does not occur.

(ii) Now suppose $\widehat{ya} \in \hat{I}_1$ and $\widehat{yb} \in \hat{I}_2$ (Case 2). Then we always have an edge of I_2 touching b. This is a contradiction to the Triangle Lemma, applied to (the reversed) $\triangle\, abc$. Hence this case cannot occur either. ■

From Lemma 6 we have learned that the color class of $\hat{e}$ is involved in every pair of every minimal set. By Lemma 7 we know that every minimal set of pairs of transitiving edges $E_{\hat{e}}^{(1)}, \ldots, E_{\hat{e}}^{(r)}$ consists of at most two pairs, involving at most one additional color class. Finally, Lemma 8 tells us that in case of a minimal set having two pairs of transitiving edges all involved edges have to satisfy a certain configuration.

Thus there are two possibilities for an edge e becoming always transitive (compare Figure 11). There may exist some minimal set consisting of only one pair of transitiving edges. Then e is transitive within its implication class and both edges of that pair belong to $\hat{I}(e)$. On the other hand, there may exist some other minimal set consisting of two pairs of transitiving edges. Then there are exactly two color classes involved with every pair containing an edge of each color class—one of which being $\hat{I}(e)$—and all these edges form the configuration displayed in Figure 9.

Note, that these two possibilities are not disjoint. Both can occur at the same time. The minimal sets $E_{\hat{e}}$ are minimal by inclusion. Thus, the existence of a set containing only one pair of transitiving edges does not forbid the existence of other minimal sets containing two pairs—and vice versa.

5.5. The sets $E_{T_{10}}$ and $E_{T_{11}}$ (2)

With the power of these properties in mind we are now able to close our gap in the proof of the main result. We already have seen that $G - \hat{e}$ is a comparability graph for $\hat{e} \in E_N$ and $\hat{e} \in E_R$, but not for $\hat{e} \in E_{T_0}$, where $E_T = E_{T_0} + E_{T_1}$. Thus the remaining cases are $\hat{e} \in E_{T_{10}}$ and $\hat{e} \in E_{T_{11}}$ with $E_{T_{10}} + E_{T_{11}} = E_{T_1}$.

We will then gather all these partial results in the next section.

Lemma 9 ($e \in E_{T_{10}}$)**.** *Let $G = (V, E)$ be a comparability graph and let $\hat{e} \in E_{T_{10}}$. Then $G - \hat{e}$ is no comparability graph.*

Proof. As e is an always transitive arc there exists a minimal set of pairs of transitiving edges $E_{\hat{e}}$. Suppose there is a minimal set $E_{\hat{e}} = \{\hat{P}\}$ where both arcs of $P = \{e', e''\}$ not only belong to $I(e)$, but are $\dot{\Gamma}$-connected ($e \in E_{T_{10}}$).

Clearly, e'^{-1} and e'' are directly Γ-connected in $G' = G - \hat{e}$ (Lemma 3) (see Figure 11 (left)). But then e' and e'^{-1} are Γ-connected in G' (through e''). Thus, there is an improper implication class, so G' is no comparability graph.

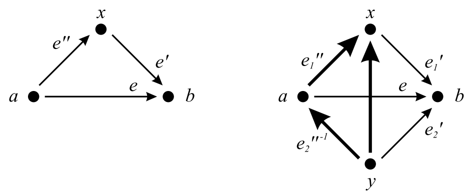


Figure 11. Configurations leading to e belonging to E_T.

Now suppose there is a minimal set $E_{\hat{e}} = \{\hat{P}_1, \hat{P}_2\}$, consisting of two pairs $P_1 = \{e'_1, e''_1\}$ and $P_2 = \{e'_2, e''_2\}$ (with Lemma 7), that makes e belong to E_T. By Lemma 6 and Lemma 8 we can assume $e = ab$, $e'_1 = xb$ and $e'_2 = yb$ to be arcs of $I_1 = I(e) \in \mathcal{I}_G$, and $e''_1 = ax$, $e''^{-1}_2 = ya$ and yx to be arcs of $I_2 \in \mathcal{I}_G$ with $\hat{I}_2 \neq \hat{I}_1$ (see Figure 11 (right)). Let $e'_1 \in P_1$ and $e'_2 \in P_2$ be $\dot{\Gamma}$-connected ($e \in E_{T_{10}}$). Then, clearly, e'_1 and e''^{-1}_1, as well as e'_2 and e''^{-1}_2, respectively, are directly Γ-connected in G'. But then the arc e'_1 is—through the arcs e''^{-1}_1 (direct Γ-connection), e''_2 (same implication class), and e'^{-1}_2 (direct Γ-connection)—Γ-connected to its reversal e'^{-1}_1 (same $\dot{\Gamma}$-component). Thus, there is an improper implication class and G' is therefore no comparability graph. ■

Lemma 10 ($e \in E_{T_{11}}$)**.** *Let $G = (V, E)$ be a comparability graph and let $\hat{e} \in E_{T_{11}}$. Then $G - \hat{e}$ is a comparability graph as well.*

Proof. As e is an always transitive arc there exists a minimal set of pairs of transitiving edges $E_{\hat{e}}$ that is either of shape $E_{\hat{e}} = \{\hat{P}\}$, where $P = \{e', e''t\} \subseteq I(e)$ (Case 1), or of shape $E_{\hat{e}} = \{\hat{P}_1, \hat{P}_2\}$, where $P_1 = \{e'_1, e''_1\}$ and $P_2 = \{e'_2, e''_2\}$ with e'_1 and e'_2 belonging to $I(e)$ (w.l.o.g.) (Case 2) (see Figure 11 and compare to the proof above).

Additionally, e belongs to $E_{T_{11}}$. Thus, for each $E_{\hat{e}}$ the involved edges belonging to $I(e)$ lie in different $\dot{\Gamma}$-components.

Suppose, $G' = G - \hat{e}$ is no comparability graph. Then G' contains an improper implication class. This implication class contains at least one arc of $I(e)$ by Corollary 1. This is only possible if there is a minimal set $E_{\hat{e}}$, where the relevant arcs, i.e., e' and e'', or e'_1 and e'_2, respectively, are Γ-connected not only in G but in G' as well. In either case we then would have e' be Γ-connected to e'^{-1}, or e'_1 be Γ-connected to e'^{-1}_1, respectively (compare to the proof of Lemma 9)—thus forming an improper implication class in G'. Starting by this assumption (G' is no comparability graph, i.e., e' and e'', or e'_1 and e'_2, respectively, are Γ-connected in G') we will show that every sub case arising leads to a contradiction.

(i) Let us first assume the existence of a minimal set $E_{\hat{e}} = \{\hat{P}\}$ with $P = \{e', e''\}$, where $e' = ax$ and $e'' = xb$ are Γ-connected in G' (Case 1). Then there is a Γ-chain $\mathcal{K} = \{l_1, \ldots, l_k\}$ in G' from $l_1 = e'$ to $l_k = e''$. As e' and e'' belong to different $\dot{\Gamma}$-components $\dot{I}'$ and $\dot{I}''$ of $I(e) = I_1$ $\mathcal{K}$ cannot be a Γ-chain in G. Thus, $\mathcal{K}$ must contain two arcs l_i and l_{i+1} that are directly Γ-connected in G', but not directly Γ-connected in G. W.l.o.g. we may assume that $\mathcal{K}$ is chosen with a minimal number of such transitions. By Lemma 3 we have $\hat{l}_i$ and $\hat{l}_{i+1}$ being either $\widehat{az}$ or $\widehat{bz}$, respectively.

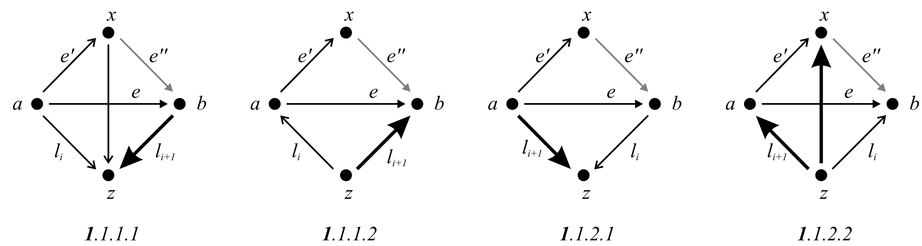


Figure 12. The 4 major subcases for Case 1.1.

Furthermore, we may assume that exactly one of these arcs l_i and l_{i+1} belongs to I_1. If neither arc belongs to I_1, then there must exist another pair of consecutive arcs in $\mathcal{K}$ that holds these properties (since $l_1, l_k \in I_1$). If, on the other hand, both arcs belong to I_1, then they form a triangle together with e, in which one of these two arcs is transitive. By Lemma 2 the transitive arc must belong to the same $\dot{\Gamma}$-component as the other arc. Then l_i and l_{i+1} are $\dot{\Gamma}$-connected in G, and this transition may be bypassed in $\mathcal{K}$ by a $\dot{\Gamma}$-chain between l_i and l_{i+1}, contradicting the minimality of $\mathcal{K}$. Therefore, we can always find a pair l_i, l_{i+1}, where exactly one arc belongs to I_1.

We now have to distinguish several sub cases (Figure 12) that arise from assigning $\hat{l}_i$ and $\hat{l}_{i+1}$ to $\widehat{az}$ and $\widehat{bz}$, respectively (2 possibilities), their respective orientations (2 possibilities), and their respective membership to

I_1 (2 possibilities). Thus there are 8 sub cases to be considered. We show that neither of them can occur. In each sub case we consider an arbitrary transitive orientation $T \in \mathcal{T}_G$ that contains all relevant arcs. (As G is a comparability graph and we only invoke constraints on arcs of one single implication class, such a T must exist.)

Suppose $l_i \in I_1$ (Case 1.1) at first. Then w.l.o.g. $l_i \in \dot{I}'$, where $\dot{I}'$ is the $\dot{\Gamma}$-component containing $e' = l_1$. Suppose further $\hat{l}_i = \widehat{az}$, and therefore $l_{i+1} = \widehat{bz}$ (Case 1.1.1). Let $l_i = az$ and $l_{i+1} = bz \in \dot{I}_2 \not\subseteq I_1$ (Case 1.1.1.1). Note, that $I_1 = I(e)$ is an implication class, while $\dot{I}_2$ is only a $\dot{\Gamma}$-component. Hence, statements like $\dot{I}_2 = I_1$ and $\dot{I}_2 \neq I_1$ are not well-defined.

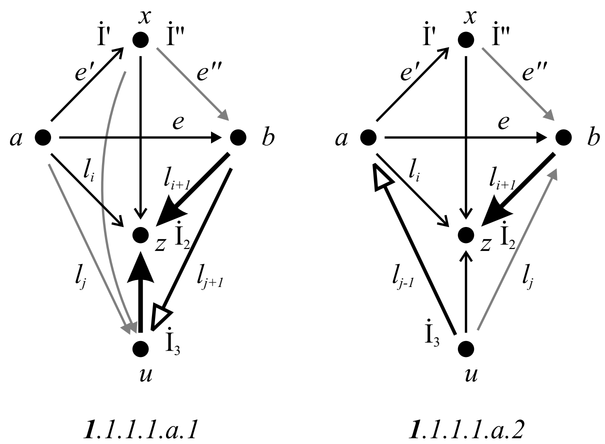


Figure 13. Subcase 1.1.1.1.a ($l_{i+1} \in \dot{I}_2$ with $\hat{I}_2 \neq \hat{I}_1$).

Unfortunately, we have no means to deduce $\dot{I}_2 \not\subseteq I_1^{-1}$. We therefore have to deal with both cases $\dot{I}_2 \not\subseteq I_1^{-1}$ and $\dot{I}_2 \subseteq I_1^{-1}$ seperately. We start by assuming $\dot{I}_2 \not\subseteq I_1^{-1}$ (Case 1.1.1.1.a), gaining $\hat{I}_2 \neq \hat{I}_1$. We then have $x \neq z$, and case (2) of the extended Triangle Lemma (Lemma 1) may be applied to $\triangle\ zba$ and arc $xb \in \dot{I}''$ (with I_1^{-1}, I_2^{-1}), yielding $xz \in \dot{I}'$.

$\mathcal{K}$ is a Γ-chain in G' between arcs of $I_1 \in \mathcal{I}_G$. In this Γ-chain we have a transition from I_1 into some other implication class $\dot{I}_2 = I_2 \in \mathcal{I}_G$ at l_{i+1}. Since the last arc in $\mathcal{K}$ belongs to I_1, there must be a transition back into I_1 as well. Let $l_{j-1}, l_j \in \mathcal{K}$ be a different pair of consecutive arcs with $l_j \in I_1$ and $l_{j-1} \in \dot{I}_3 \not\subseteq I_1$, where $\dot{I}_3$ is a $\dot{\Gamma}$-component not belonging to I_1. W.l.o.g. we have $l_j \in \dot{I}'' \subseteq I_1$, since the last arc in $\mathcal{K}$ belongs to $\dot{I}''$. By Lemma 3 this pair of arcs connects the vertices a and b with a new common neighbor $u \neq x, z$ (Figure 13).

Suppose $\hat{l}_j = \widehat{au}$ (Case 1.1.1.1.a.1), and correspondingly $\hat{l}_{j-1} = \widehat{bu}$. Then we have $l_j = au \in \dot{I}''$ and $l_{j-1} = bu \in \dot{I}_3 \nsubseteq I_1$. (Otherwise the transitive arc l_{j-1} would belong to I_1 as well by Lemma 2.) From $\triangle\ zba$ with I_1^{-1} and I_2^{-1}, and $au \in \dot{I}''$ we get $uz \in I_2$ (case (2) in the extended Triangle Lemma). Then, case (1) may be applied to $\triangle\ xbz$ with respect to $uz \in I_2$. This yields $xu \in \dot{I}''$. But then, we may apply case (1) to $(\triangle\ uxa)^{-1}$ with respect to $za \in (\dot{I}')^{-1}$, gaining $zu \in \dot{I}''$. This contradicts $uz \in I_2$ with $\hat{I}_2 \neq \hat{I}_1$. Thus, Case a.1 cannot occur.

Let now $\hat{l}_j = \widehat{bu}$ (Case 1.1.1.1.a.2). Again, there is only one orientation for $\hat{l}_j$ feasible, namely $l_j = ub \in \dot{I}''$. (Otherwise, $l_{j-1} \in I_1$ by transitivity.) Hence, $l_{j-1} = ua \in \dot{I}_3 \nsubseteq I_1$. Now, we may apply case (1) of the extended Triangle Lemma to $(\triangle zbx)^{-1}$ and $bu \in (\dot{I}'')^{-1}$. This yields $uz \in \dot{I}'$. But then, case (2), applied to $\triangle\ uab$ with respect to $\dot{I}'$-arcs gains $u \notin V(\dot{I}')$—contradicting $uz \in \dot{I}'$. Hence, Case a.2 cannot occur either.

Let us now assume that there is no transition in $\mathcal{K}$ from I_1 to a different color class $\hat{I}_2 \neq \hat{I}_1$, but a transition from I_1 to its reversal, i.e., $\dot{I}_2 \subseteq I_1^{-1}$ (Case 1.1.1.1.b). Since $e \in E_{T_{11}}$, we have $\dot{I}_2 \neq (\dot{I}')^{-1}$. Otherwise, l_i and l_{i+1}^{-1} would be a pair of transitiving arcs belonging to the same $\dot{\Gamma}$-component of I_1.

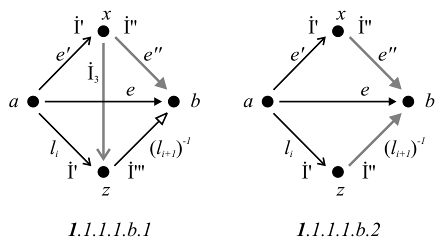


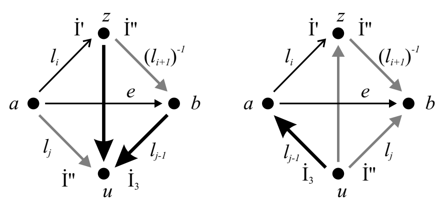
Figure 14. Subcase 1.1.1.1.b ($\dot{I}_2 = I_1^{-1}$ with $\dot{I}_2^{-1} = \dot{I}''' \neq \dot{I}''$ and $\dot{I}_2^{-1} = \dot{I}''$, respectively).

Suppose $\dot{I}_2^{-1} = \dot{I}''' \subseteq I_1$ with $\dot{I}''' \neq \dot{I}''$ and $\dot{I}''' \neq \dot{I}'$ first (Case 1.1.1.1.b.1) (Figure 14). From $\dot{I}''' \neq \dot{I}''$ we can conclude $x \neq z$ and $\widehat{xz} \in E$. Let $xz \in \dot{I}_3$ with $\dot{I}_3 \neq (\dot{I}''')^{-1}$ (Case b.1.1). Then, case (3) in Lemma 1 may be applied to $\triangle\ xzb$, yielding $xa \in \dot{I}_3$, i.e., $\dot{I}_3 = (\dot{I}')^{-1}$ and $zx \in \dot{I}'$. But then, we may apply case (3) once again to $\triangle\ zxb$ (considering $xb \in \dot{I}''$). This delivers $za \in \dot{I}'$ which contradicts our assumption $az \in \dot{I}' \subseteq I_1$, where I_1 is a proper implication class (G is a comparability graph). Thus, Case b.1.1 does not occur.

If, on the other hand $xz \in \dot{I}_3 = (\dot{I}''')^{-1}$ (Case b.1.2), i.e., $zx \in \dot{I}'''$, our last argument works as well. Here, we can conclude $za \in \dot{I}''' \subseteq I_1$ from $\triangle\ zxb$, contradicting $az \in \dot{I}' \subseteq I_1$. Thus, Case b.1.2 does not occur either.

Suppose now $\dot{I}_2^{-1} = \dot{I}''$ (Case 1.1.1.1.b.2). Here, we can neither expect the existence of $\widehat{zx} \in E$ nor even $z \neq x$. But we can make use of the existence of another transition in $\mathcal{K}$, again. Let l_{j-1} and l_j be arcs of $\mathcal{K}$ with $l_{j-1} \in \dot{I}_3 \nsubseteq I_1$ and $l_j \in I_1$. W.l.o.g. we may assume $l_j \in \dot{I}''$, since $l_k \in \dot{I}''$.

Although the first transition in $\mathcal{K}$ connects a $\dot{\Gamma}$-component of I_1, namely $\dot{I}'$, with some $\dot{\Gamma}$-component of I_1^{-1}, namely $(\dot{I}'')^{-1}$, we may not assume the last transition in $\mathcal{K}$ to be from I_1^{-1} into $\dot{I}''$. Therefore, we need to distinguish the sub cases $\dot{I}_3 \nsubseteq I_1^{-1}$ and $\dot{I}_3 \subseteq I_1^{-1}$. In either sub case we have to distinguish the sub cases $\hat{l}_j = \widehat{au}$ and $\hat{l}_j = \widehat{bu}$.



1.1.1.1.b.2.1.1 *1.1.1.1.b.2.1.2*

Figure 15. Subcases b.2.1.1 and b.2.1.2 ($\dot{I}_2^{-1} = \dot{I}''$ and $\hat{I}_3 \neq \hat{I}_1$, with $\hat{l}_j = \widehat{au}$ and $\hat{l}_j = \widehat{bu}$, respectively).

Suppose $\dot{I}_3 \nsubseteq I_1^{-1}$ first (Case 1.1.1.1.b.2.1), i.e., $\hat{I}_3 \neq \hat{I}_1$. Furthermore, let $\hat{l}_j = \widehat{au}$ and $\hat{l}_{j-1} = \widehat{bu}$ (Case b.2.1.1) (Figure 15). By the transitivity argument we then have $l_j = au \in \dot{I}''$ and $l_{j-1} = bu \in I_3$ (otherwise $l_{j-1} \in I_1$). From $\hat{I}_3 \neq \hat{I}_1$ follows $z \neq u$ and $\widehat{zu} \in E$. Then, $(\triangle\ uba)^{-1}$ and $za \in (\dot{I}')^{-1}$ imply $zu \in I_3$ (case (2)). But then follows $u \notin V(\dot{I}'')$ from $(\triangle\ ubz)^{-1}$ (Case (1)), contradicting our assumption in this very sub case. Therefore, suppose $\hat{l}_j = \widehat{bu}$ and $\hat{l}_{j-1} = \widehat{au}$ (Case b.2.1.2). By transitivity of $\dot{I}'' + e$ and $\hat{I}_3 \neq \hat{I}_1$ we can only have $l_j = ub \in \dot{I}''$ and $l_{j-1} = ua \in I_3$. From $\hat{I}_3 \neq \hat{I}_1$ follows $z \neq u$ and $\widehat{zu} \in E$. Then, $\triangle\ uab$, with respect to $az \in \dot{I}'$, implies $uz \in \dot{I}''$ (Case (2)).

Unfortunately, we cannot conclude some contradiction from $\triangle\ auz$ (with I_3^{-1}) and arc $zb \in \dot{I}''$, since the extended Triangle Lemma (Lemma 1) cannot be applied (because of $a \in V(e)$). Therefore we consider a new triangle containing a further arc from I_3. No arc in our Γ-chain $\mathcal{K}$ can be part of two different transitioning pairs of arcs at the same time (by Lemma 3). Hence, there exists $l_{j-2} \in \mathcal{K}$ with $l_{j-2} \in I_3$. Now, this new arc l_{j-2} is either of shape $l_{j-2} = uv$ or of shape $l_{j-2} = va$, since it is directly Γ-connected to l_{j-1}. Suppose the former (Case b.2.1.2.a) (Figure 16). Then, $\triangle\ zua$ (with I_1^{-1}), with respect to $uv \in I_3$, implies $vz \in \dot{I}'$ (case (1)). But then, we can conclude $zv \in I_3$ from $\triangle\ vuz$ (with I_3^{-1}) and $zb \in \dot{I}''$ (case (1)). This contradicts $\hat{I}_3 \neq \hat{I}_1$.

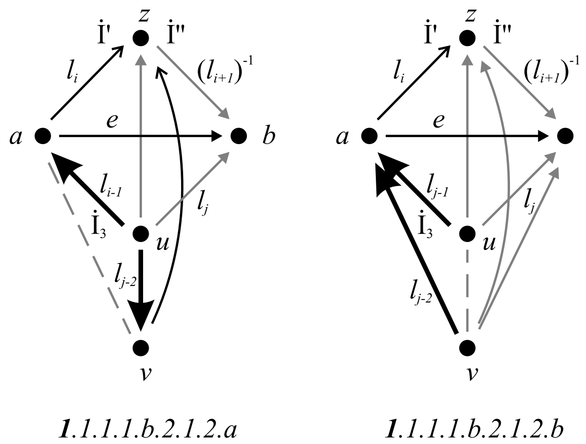
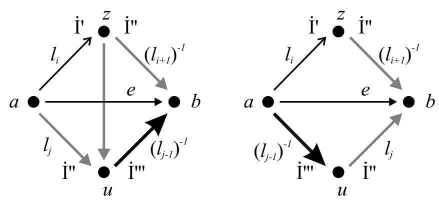


Figure 16. Subcase b.2.1.2 ($l_{j-2} = uv$ or $l_{j-2} = va$).

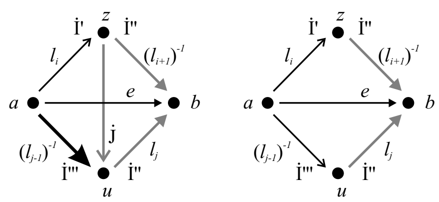
Thus, suppose that l_{j-2} is of shape $l_{j-2} = va \in I_3$ (Case b.2.1.2.b). Here, we can conclude $vz \in \dot{I}''$ from $\triangle\ zua$ (with I_1^{-1}) and $va \in I_3$ (case (1)). By transitivity of $\dot{I}'' + e$ follows the existence of $vb \in \dot{I}''$. Now, the partial graph spanned by the vertices v, a, z, b is isomorphic to that spanned by u, a, z, b. Thus, again, we cannot apply the extended Triangle Lemma to $\triangle\ avz$ (with I_3^{-1}), since $a \in V(e)$. On the other hand we can conclude that l_{j-2} can be no transitioning arc in $\mathcal{K}$. Otherwise we here would have a transition from $\dot{I}'' \subseteq I_1$ ($vb = l_{j-3} \in \dot{I}''$) into I_3 ($l_{j-2} = va \in I_3$) in $\mathcal{K}$, contradicting the minimality of such transitions (both transitions, at l_{j-2} and at l_j could be bypassed). Hence, there must exist some arc in I_3 that does not touch vertex a (particularly, there exists some $uv \in I_3$). For any such arc we may apply our argumentation from Case b.2.1.2.a, yielding $vz \in \dot{I}'$ on the one hand, and $zv \in I_3$ on the other. This completes the consideration of Case 1.1.1.1.b.2.1.



1.1.1.1.b.2.2.1 *1.1.1.1.b.2.2.2*

Figure 17. Subcases b.2.2.1 and b.2.2.2 ($\dot{I}_2^{-1} = \dot{I}''$ and $\dot{I}_3 \subseteq I_1^{-1}$, with $\hat{l}_j = \widehat{au}$ and $\hat{l}_j = \widehat{bu}$, respectively).

Suppose now $\dot{I}_3 \subseteq I_1^{-1}$ (Case 1.1.1.1.b.2.2). Let $\hat{l}_j = \widehat{au}$ and $\hat{l}_{j-1} = \widehat{bu}$ (Case b.2.2.1) (Figure 17). For $l_j = ua \in \dot{I}''$ and $l_{j-1} = ub \in I_1^{-1}$ we would have a cycle in our transitive orientation $T \in \mathcal{T}_G$. Hence, $l_j = au \in \dot{I}''$ and $l_{j-1} = bu \in I_1^{-1}$. Since $e \in E_{T_{11}}$, $l_{j-1}^{-1} = ub$ belongs to some $\dot{\Gamma}$-component $\dot{I}''' \subseteq I_1$ with $\dot{I}''' \neq \dot{I}''$. From $\dot{I}' \neq \dot{I}''$ we have $z \neq u$ and $\widehat{zu} \in E$. Let $zu \in \dot{J}$. Suppose $\dot{J} \neq (\dot{I}')^{-1}$. Then, $\triangle\, uza$ with I_1^{-1} and J^{-1} meets the conditions of case (3) in the extended Triangle Lemma, yielding $bu \in \dot{J}$. Hence, $\dot{J}^{-1} = \dot{I}'''$, i.e., $uz \in \dot{I}'''$. But then, we can conclude $u \notin V(\dot{I}'')$ from $\triangle\, uzb$ (case (1)), which contradicts $au \in \dot{I}''$. Hence, $\dot{J} = (\dot{I}')^{-1}$, i.e., $uz \in \dot{I}'$. Here, $(\triangle\, zua)^{-1}$ meets the conditions of case (1), yielding $z \notin V(\dot{I}'')$. But this contradicts $zb \in \dot{I}''$. Therefore, Case b.2.2.1 cannot occur.



1.1.1.1.b.2.2.2.1 *1.1.1.1.b.2.2.2.2*

Figure 18. Subcase b.2.2.2 ($l_{j-1}^{-1} = au \in \dot{I}'''$ with $\dot{I}''' \neq \dot{I}'$ and $\dot{I}''' = \dot{I}'$, respectively).

Finally, suppose $\hat{l}_j = \widehat{bu}$ and hence $\hat{l}_{j-1} = \widehat{au}$ (Case b.2.2.2). We then have $l_j = ub \in \dot{I}''$ and $l_{j-1} = ua \in I_1^{-1}$ (otherwise $l_{j-1} \in I_1$ by transitivity), i.e., $l_{j-1}^{-1} = au \in \dot{I}'''$ with $\dot{I}''' \neq \dot{I}''$ ($e \in E_{T_{11}}$). Now we have to distinguish two possible cases concerning the relation between $\dot{I}'''$ and $\dot{I}'$. Suppose $\dot{I}''' \neq \dot{I}'$ (Case b.2.2.2.1). Then we have $z \neq u$ and $\widehat{zu} \in E$. Let $zu \in \dot{J}$ (Figure 18). For $\dot{J}^{-1} \neq \dot{I}'$, $\triangle\, uza$ with $\dot{I}_1^{-1}$ and $\dot{J}^{-1}$ meets the conditions of case

(3), yielding $bu \in \dot{J}$. Hence $uz \in \dot{J}^{-1} = \dot{I}''$. But then, $(\triangle\, zua)^{-1}$ meets these conditions as well, yielding $bz \in \dot{I}''$, contradicting $zb \in \dot{I}''$ (I_1 proper implication class). Hence, $\dot{J}^{-1} = \dot{I}'$, i.e., $uz \in \dot{I}'$. But then again, we can apply case (3) to $(\triangle\, zua)^{-1}$, getting $bz \in \dot{I}'$ (contradiction).

Finally, let $l_{j-1}^{-1} = au \in \dot{I}''' = \dot{I}'$ (Case b.2.2.2.2). Then we have the following situation. There is a transition in $\mathcal{K}$ from $\dot{I}' \subseteq I_1$ to $(\dot{I}'')^{-1}$ at l_i, l_{i+1} (Case b.2) and some other transition from $(\dot{I}')^{-1}$ to $\dot{I}'' \subseteq I_1$ at l_{j-1}, l_j (Case b.2.2.2.2).

$$\begin{array}{ccccccccc} e'\,\Gamma & \ldots & \Gamma\, l_i\, \Gamma_{G'}\, l_{i+1}\, \Gamma & & \ldots & & \Gamma\, l_{j-1}\, \Gamma_{G'}\, l_j\, \Gamma & \ldots & \Gamma\, e'' \\ \dot{I}' & & \dot{I}' \quad \left(\dot{I}''\right)^{-1} & & \uparrow & & \left(\dot{I}'\right)^{-1} \quad \dot{I}'' & & \dot{I}''. \end{array}$$

By assumption, any such transitioning pair relevant for our considerations contains exactly one arc of I_1. Hence, there is no direct transition between two different $\dot{\Gamma}$-components $\dot{I}^*$ and $\dot{I}^{**}$ of I_1. From $e \in E_{T_{11}}$ furthermore follows that there is no direct transition from any $\dot{\Gamma}$-component $\dot{I}^*$ of I_1 to its reversal $(\dot{I}^*)^{-1}$. That means—as we need to construct a transition from $(\dot{I}'')^{-1}$ to $(\dot{I}')^{-1}$ in our situation—that there have to be at least *two* other transitioning pairs. One for a transition away from I_1^{-1}, and a second into I_1^{-1}, again. We will explore the first of these two options. Let l_p, l_{p+1} be a pair in $\mathcal{K}$ with $l_p \in (\dot{I}'')^{-1}$ and $l_{p+1} \in \dot{I}_4 \nsubseteq I_1^{-1}$, where $\dot{I}_4 \neq \dot{I}', \dot{I}''$.

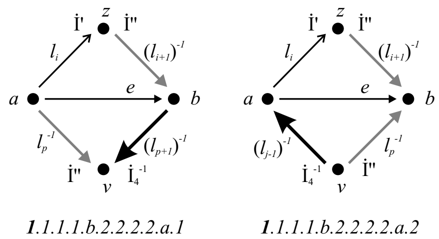


Figure 19. Subcases b.2.2.2.2.a.1 and b.2.2.2.2.a.2 ($\hat{I}_4 \neq \hat{I}_1$, with $\hat{l}_p = \widehat{av}$ and $\hat{l}_p = \widehat{bv}$, respectively).

Let us assume $\dot{I}_4 \nsubseteq I_1$, i.e., $\hat{I}_4 \neq \hat{I}_1$ (Case b.2.2.2.2.a) (Figure 19). Then either $\hat{l}_p = \widehat{av}$ or $\hat{l}_p = \widehat{bv}$. Suppose the former (Case a.1). We then have $l_p^{-1} = av \in \dot{I}''$ and $l_{p+1}^{-1} = bv \in I_4^{-1}$ (otherwise $l_{p+1}^{-1} \in \dot{I}''$). This case is symmetric to Case 1.1.1.1.b.2.1.1 and thus cannot occur (compare Figures 15 and 19). Now, suppose the latter (Case a.2). Here, we have $l_p^{-1} = vb \in \dot{I}''$

and $l_{p+1}^{-1} = va \in I_4^{-1}$ (otherwise $l_{p+1}^{-1} \in \dot{I}''$). This case is symmetric to Case 1.1.1.1.b.2.1.2 and thus canot occur either (Figures 15 and 19).

Hence, $\dot{I}_4 \subseteq I_1$ (Case b.2.2.2.2.b), with $\dot{I}_4 \neq \dot{I}', \dot{I}''$. Again, either $\hat{l}_p = \widehat{av}$ or $\hat{l}_p = \widehat{bv}$ (Figure 20). Suppose the former (Case b.2.2.2.2.b.1). Then $l_p^{-1} = av \in \dot{I}''$ and $l_{p+1} = vb \in \dot{I}_4$ (otherwise $\triangle\ abv$ would be a cycle). This situation is symmetric (even slightly more restrictive) to Case 1.1.1.1.b.2.2.1 and thus cannot occur (Figures 17 and 20). Assuming the latter (Case b.2.2.2.2.b.2), we get $l_p^{-1} = vb \in \dot{I}''$ and $l_{p+1} = av \in \dot{I}_4$ (otherwise, again, $\triangle\ abv$ would be a cycle). Now, this situation is symmetric to Case 1.1.1.1.b.2.2.2.1 (slightly more restrictive, again) and thus cannot occur either (Figures 18 and 20). Hence sub Case b.2.2.2 cannot occur, completing the consideration of Case 1.1.1.1.

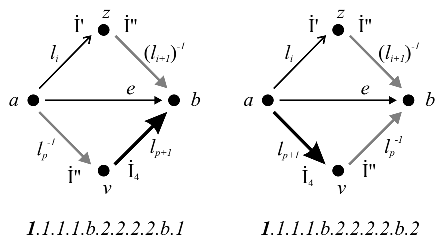


Figure 20. Subcase b.2.2.2.2.b ($l_{p+1} \in \dot{I}_4 \subseteq I_1, \neq \dot{I}', \dot{I}''$ with $\hat{l}_p = \widehat{av}$ and $\hat{l}_p = \widehat{bv}$, respectively).

Let us now consider the next major sub case (Figure 12), $l_i = za \in I_1$ (Case 1.1.1.2). As we have already used this implication several times by now, we can safely argue that this case cannot occur, since then the transitive arc $l_{i+1} = zb \in \dot{I}_2 \nsubseteq I_1$ would belong to I_1 as well. Thus, Case 1.1.1 cannot occur.

Let us next assume $\hat{l}_i = \widehat{bz}$ (Case 1.1.2) and hence $\hat{l}_{i+1} = \widehat{az}$ ($l_{i+1} \in \dot{I}_2 \nsubseteq I_1$). Here, the sub case $l_i = bz \in \dot{I}' \subseteq I_1$ (Case 1.1.2.1) with $l_{i+1} = az \in \dot{I}_2$ (transitive arc) yields a contradiction to $\dot{I}_2 \nsubseteq I_1$.

Hence, $l_i = zb \in \dot{I}' \subseteq I_1$ and $l_{i+1} = za \in \dot{I}_2$ (Case 1.1.2.2). If $\dot{I}_2 \neq (\dot{I}'')^{-1}$ (Case 1.1.2.2.a), then we get from $\triangle\ zab$ and $xb \in \dot{I}''$ (case (2)) $zx \in \dot{I}_2$ (Figure 21). But then, case (1) for $\triangle\ zax$ implies $z \notin V(\dot{I}')$—contradicting $l_i = zb \in \dot{I}'$. Therefore, $\dot{I}_2 = (\dot{I}'')^{-1}$ (Case 1.1.2.2.b), i.e., $l_{i+1}^{-1} = az \in \dot{I}''$ (Figure 21). From $\dot{I}' \neq \dot{I}''$ follows $x \neq z$ and $\widehat{xz} \in E$. Let $xz \in \dot{J}$. For $\dot{J} \neq (\dot{I}')^{-1}$ we can apply the extended Triangle Lemma (case (1)) to $\triangle\ zxa$ with I_1^{-1} and $\dot{J}^{-1}$. This yields $z \notin V(\dot{I}')$, contradicting

$zb \in \dot{I}'$. Hence, $\dot{J}^{-1} = \dot{I}''$, i.e., $zx \in \dot{I}''$. But then the transitive arcs ax and zb belong to $\dot{I}''$ as well (contradiction to $ax, zb \in \dot{I}' \neq \dot{I}''$). Thus, this case does not occur either, completing the consideration of Case 1.1.

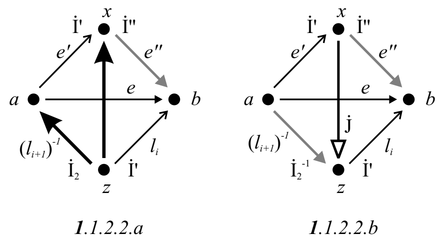


Figure 21. Subcase 1.1.2.2 ($l_i = zb \in \dot{I}'$, $l_{i+1} = za \in \dot{I}_2 \nsubseteq I_1$).

Up to now we only have considered the case where $l_i \in \dot{I}' \subseteq I_1$ is the edge belonging to I_1. Let now $l_{i+1} \in I_1$ and $l_i \in \dot{I}_2 \nsubseteq I_1$ (Case 1.2). W.l.o.g. we then may assume $l_{i+1} \in \dot{I}''$ ($l_k \in \dot{I}''$). Analogously to Case 1.1 we know have to consider all 4 sub cases concerning the position and orientation of l_i and l_{i+1}. However, each of these 4 sub cases is symmetric to one of the sub Cases 1.1.1.1 through 1.1.2.2, when reversed and $\dot{I}'$ and $\dot{I}''$ being swapped (compare Figures 12 and 20). Therefore the whole Case 1 does not occur.

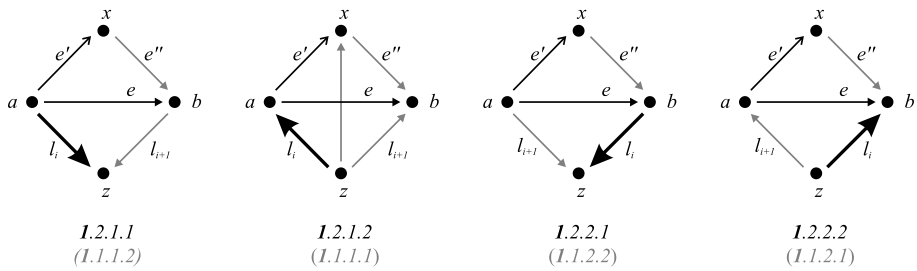


Figure 22. Subcases for Case 1.2 and their symmetric correspondents (in brackets). in Case 1.1.

(ii) Let us now assume the existence of a minimal set $E_{\hat{e}} = \{\hat{P}_1, \hat{P}_2\}$ (with Lemma 7) with $P_1 = \{e_1', e_1''\}$ and $P_2 = \{e_2', e_2''\}$ (Case 2). By Lemma 6 we know that one arc of each P_i, say e_i' ($i = 1, 2$), belongs to $I_1 = I(e)$, respectively. By our general assumption (G' is no comparability graph) we have e_1' and e_2' being Γ-connected in G'. As in Case 1 there exists a Γ-chain $\mathcal{K} = \{l_1, \ldots, l_k\}$ from $l_1 = e_1'$ to $l_k = e_2'$. By Lemma 8 (configuration $(*)$)

we may further assume $e = ab$, $e_1' = xb$ and $e_2' = yb$ to be arcs of $I_1 \in \mathcal{I}_G$, and $e_1'' = ax$, $e_2''^{-1} = ya$ and yx to be arcs of I_2 with $\hat{I}_2 \neq \hat{I}_1$ (Figure 11 on page 449).

Similar to Case 1 the transitiving arcs e_1' and e_2' both belong to I_1, but each to a different $\dot{\Gamma}$-component $\dot{J}_1$ or $\dot{J}_2$, respectively ($e \in E_{T_{11}}$). Therefore, $\mathcal{K}$ can be no Γ-chain in G ($e \in E_{T_{11}}$). Hence, there exists a pair of consecutive arcs $l_i, l_{i+1} \in \mathcal{K}$ which are directly Γ-connected in G', but not in G. Assuming $\mathcal{K}$ to be chosen with a minimal number of such transitions and following the arguments of Case 1 we have exactly one of these arcs belonging to I_1. We furthermore either have $\hat{l}_i = \widehat{az}$ or $\hat{l}_i = \widehat{bz}$, as well as the existence of a similar pair of consecutive arcs $l_{j-1}, l_j \in \mathcal{K}$ realising a transition back into I_1.

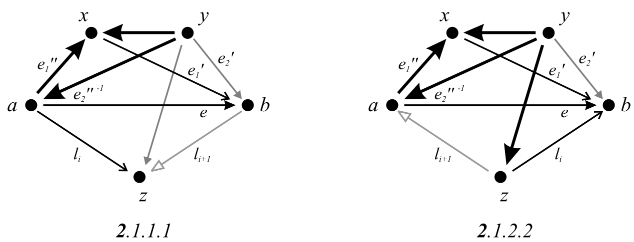


Figure 23. Nontrivial sub cases of Case 2.1.

Therefore we have to consider the same 8 sub cases concerning the respective membership to I_1, the position, and orientation of l_i and l_{i+1}. As above, half of these sub cases are trivial, while most of the others are lengthy.

We start by assuming $l_i \in I_1$ with $l_i \in \dot{J}_1$ (w.l.o.g., since $l_1 \in \dot{J}_1$), again (Case 2.1). Then we have $l_{i+1} \in \dot{I}_3$ with $\dot{I}_3 \not\subseteq I_1$. Suppose $\hat{l}_i = \widehat{az}$ and hence $\hat{l}_{i+1} = \widehat{bz}$ (Case 2.1.1). Furthermore, suppose $l_i = az \in \dot{J}_1$ and $l_{i+1} = bz \in \dot{I}_3$ (Case 2.1.1.1) (Figure 23).

For $\triangle\ yxb$ with respect to $az \in \dot{J}_1$ case (1) of the extended Triangle Lemma (Lemma 1) implies the existence of $yz \in \dot{J}_2$. But then we find $az \in \dot{J}_2$ from $\triangle\ zya$ (with I_1^{-1}) with respect to $ax \in I_2$ (case (1)). This contradicts $az \in \dot{J}_1 \neq \dot{J}_2$. This sub case thus does not occur.

The case where $l_i = za \in \dot{J}_1$ and $l_{i+1} = zb \in \dot{I}_3 \neq I_1$ (Case 2.1.1.2) cannot occur either, since then the transitive edge l_{i+1} would belong to I_1. Thus Case 2.1.1 cannot occur.

Therefore suppose now $\hat{l}_i = \widehat{bz}$ (Case 2.1.2). The sub case with $l_i = bz$ and $l_{i+1} = az$ (Case 2.1.2.1) can be ruled out. Transitivity of I_1 would imply once more $l_{i+1} \in I_1$. Hence, $l_i = zb \in \dot{J}_1$ and $l_{i+1} = za \in \dot{I}_3 \nsubseteq I_1$ (Case 2.1.2.2) (Figure 23).

Here, we find $yz \in I_2$ from $\triangle\ yxb$ with respect to $zb \in \dot{J}_1$ (case (1)). On the other hand, we will find $zy \in \dot{I}_3$, leading to $\dot{I}_3 = I_3 = I_2^{-1}$. To see this, observe first from $\triangle\ yza$ that $\widehat{az} \in \hat{I}_3$ cannot belong to $\hat{I}_1$. Otherwise the original Triangle Lemma would deliver $y \notin V(I_1)$ (contradiction). Hence, not only $\dot{I}_3 \nsubseteq I_1$, but $\dot{I}_3 \nsubseteq I_1^{-1}$ as well, i.e., $\dot{I}_3 = I_3$ is a proper implication class. But then, we may apply case (2) of the extended Triangle Lemma to $\triangle\ zab$ with respect to $yb \in \dot{J}_2$. This yields $zy \in I_3$. Hence $I_3 = I_2^{-1}$.

From now on we have to make several further assumptions, again, to construct contradictions. In addition to $l_i \in I_1$ and $l_{i+1} \in I_2^{-1}$ we need to consider several other such pairs in $\mathcal{K}$. We first consider the pair $l_{j-1}, l_j \in \mathcal{K}$ realizing the (last) transition back into I_1. Let $l_{j-1} \in \dot{I}_4 \nsubseteq I_1$ and $l_j \in \dot{J}_2 \subseteq I_1$ (since $l_k \in \dot{J}_2$).

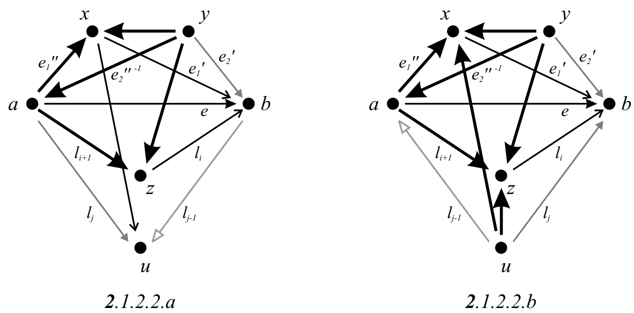


Figure 24. Subcases 2.1.2.2.a and 2.1.2.2.b ($\hat{l}_j = \widehat{au}$ and $\hat{l}_j = \widehat{bu}$, respectively).

Suppose $\hat{l}_j = \widehat{au}$ (Case 2.1.2.2.a). Then $l_j = au \in \dot{J}_2$ (otherwise $l_{j-1} \in I_1$) and $l_{j-1} = bu \in \dot{I}_4 \nsubseteq I_1$ (Figure 24). Now, $\triangle\ xyb$ (with I_2^{-1}) and $au \in \dot{J}_2$ imply $xu \in \dot{J}_1$ (case (1)). But then $(\triangle\ uxa)^{-1}$ with respect to $ya \in I_2$ implies $au \in \dot{J}_1$—contradicting $au \in \dot{J}_2 \neq \dot{J}_1$. Thus, this sub case cannot occur.

Hence $\hat{l}_j = \widehat{bu}$ (Case 2.1.2.2.b). We then may savely assume $l_j = ub \in \dot{J}_2$ and $l_{j-1} = ua \in \dot{I}_4 \nsubseteq I_1$ (otherwise $l_{j-1} \in I_1$). Here, $\triangle\ xyb$ with I_2^{-1}, when applied to $ub \in \dot{J}_2$, implies $ux \in I_2$ (case (1)). From $\triangle\ uxb$ with respect to $zb \in \dot{J}_1$ follows $uz \in I_2$ (case (1)). Now, comparing $\triangle\ uzb$ with $\triangle\ uab$ yields $ua \in I_2 = \dot{I}_4$ (case (3)). Now, we need to make further assumptions, again.

Up to now we have two transitions in $\mathcal{K}$, a first one from I_1 into I_2^{-1} at l_{j+1} and a last one from I_2 back into I_1 at l_j. Since a direct transition from

I_2^{-1} to I_2 is not possible (any transitioning pair of arcs of I_2 then would form a path from a over some vertex v to b in G, having $e \in I_2 \neq I_1$ as their transitive arc). Therefore, we consider a third transition in $\mathcal{K}$, from $l_{p-1} \in I_2^{-1}$ to some $l_p \in \dot{I}_5$ with $\dot{I}_5 \neq I_2, I_2^{-1}$, i.e. $\hat{I}_5 \neq \hat{I}_2$. Furthermore we may assume $\dot{I}_5 \neq \dot{J}_1, \dot{J}_2$ by minimality of $\mathcal{K}$.

From the original Triangle Lemma follows for $\triangle\, byx$ (with I_1^{-1}) $b \notin V(I_2)$. Thus, the sub case, where $\hat{l}_{p-1} = \widehat{bv} \in \hat{I}_2$ and $\hat{l}_p = \widehat{av}$ can be ruled out. Hence, $\hat{l}_p = \widehat{bv}$. Suppose $l_p = vb \in \dot{I}_5$ and $l_{p-1}^{-1} = av \in I_2$ first (Case b.1) (Figure 25). By the original Triangle Lemma for $\triangle\, byx$ (with I_1^{-1}) with respect to $av \in I_2$ follows that $vb \in \dot{I}_5$ belongs to I_1. Since vb cannot belong to $\dot{J}_1$ or $\dot{J}_2$ by assumption, we have $\dot{I}_5 = \dot{J}_3 \subseteq I_1$. We then may apply case (2) of the extended Triangle Lemma to $\triangle\, vab$ (with I_2^{-1}) and $xb \in \dot{J}_1$. This yields $xv \in I_2$. But then case (3) for $\triangle\, xvb$ reveals $xa \in I_2$. This contradicts $ax \in I_2$ (I_2 is proper).

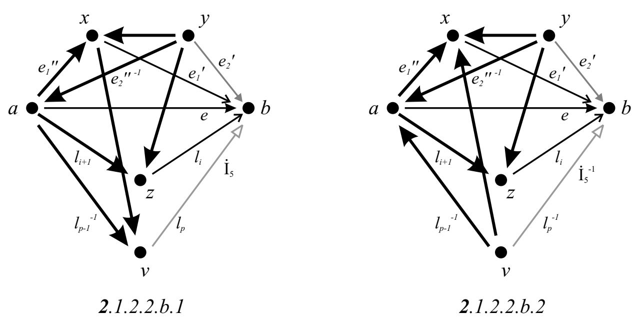


Figure 25. Subcases b.1 and b.2 ($l_p = vb$ and $l_p = bv$, respectively).

Thus, suppose $l_p = bv \in I_5$ and $l_{p-1}^{-1} = va \in I_2$ (Case b.2). Here, again, we have $vb \in \dot{I}_5^{-1} \subseteq I_1$ by the original Triangle Lemma for $\triangle\, byx$ (with I_1^{-1}) with respect to $va \in I_2$. Suppose $vb \in \dot{J}_3 \neq \dot{J}_2$ (Case b.2.1). Then we find $yv \in I_2$ from $\triangle\, yab$ and $vb \in \dot{J}_3$ (case (2)). On the other hand, we have $vy \in I_2$ (contradiction) by $\triangle\, vab$ and $yb \in \dot{J}_2$ (case (2)). Hence, $vb \in \dot{I}_5^{-1} = \dot{J}_3 = \dot{J}_2$ (Case b.2.2).

We thus have a transition from I_2^{-1} into $\dot{J}_2^{-1} \subseteq I_1^{-1}$ at l_p. Together with the already established transitions from $\dot{J}_1 \subseteq I_1$ into I_2^{-1} at l_{i+1} and from I_2 back into I_1, again, at l_j, we have the following situation,

$$\dot{J}_1 \;\rightarrow\; I_2^{-1} \;\rightarrow\; \left(\dot{J}_2\right)^{-1} \quad \ldots \quad I_2 \;\rightarrow\; \dot{J}_2\;.$$

Therefore there must exist a *fourth* pair of consecutive arcs $l_q, l_{q+1} \in \mathcal{K}$ with $l_q \in (\dot{J}_2)^{-1}$ and $l_{q+1} \in \dot{I}_6$ with $\dot{I}_6 \nsubseteq I_1^{-1}$, $\dot{I}_6 \nsubseteq I_2^{-1}$, and $\dot{I}_6 \neq \dot{J}_1, \dot{J}_2$ (minimality of $\mathcal{K}$).

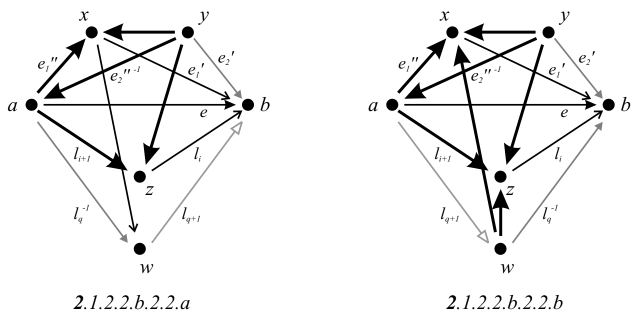


Figure 26. Subcases b.2.2.a and b.2.2.b ($\hat{l}_q = \widehat{aw}$ and $\hat{l}_q = \widehat{bw}$, respectively).

Let $\hat{l}_q = \widehat{aw}$ (Case b.2.2.a) (Figure 26). Once more, by transitivity of I_1, there is only one orientation feasible. We have $l_q^{-1} = aw \in \dot{J}_2$ and $l_{q+1} = wb \in \dot{I}_6$ ($\dot{I}_6 \nsubseteq I_1^{-1}, I_2^{-1}, \neq \dot{J}_1, \dot{J}_2$). Here, we find $xw \in \dot{J}_1$ from $\triangle\ xyb$ (with I_2^{-1}) and $aw \in \dot{J}_2$ (case (1)). But then we find $aw \in \dot{J}_1$ from $\triangle\ wax$ (with I_1^{-1}) and $ya \in I_2$ as well (case (1)). This contradicts our current assumption ($aw \in \dot{J}_2 \neq \dot{J}_1$).

Finally, suppose $\hat{l}_q = \widehat{bw}$ (Case b.2.2.b). Then we find $l_q^{-1} = wb \in \dot{J}_2$ and $l_{q+1} = aw \in \dot{I}_6$ ($\dot{I}_6 \nsubseteq I_1^{-1}, I_2^{-1}, \neq \dot{J}_1, \dot{J}_2$) (otherwise $l_{q+1}^{-1} \in I_1$ by transitivity). Here, from $\triangle\ xyb$ (with I_2^{-1}), with respect to $wb \in \dot{J}_2$, follows $wx \in I_2$ (case (1)). For $\dot{I}_6 \neq I_2$, i.e., $\hat{I}_6 \neq \hat{I}_2$ (Case b.2.2.b.1), $\triangle\ xaw$ (with I_2^{-1}) yields $x \notin V(\dot{I}_6)$ (original Triangle Lemma) and thus $\hat{I}_6 \neq \hat{I}_1$. Hence we have $\dot{I}_6 = I_6$ with $\hat{I}_6 \neq \hat{I}_1, \hat{I}_2$. But then we find from $\triangle\ wab$ (with I_6^{-1}) and $xb \in \dot{J}_1$ (case (2)) $xw \in I_6 = I_2^{-1}$ (contradiction). Hence, $\dot{I}_6 = I_2$ (Case b.2.2.b.2). But then we find $xw \in I_2$ (contradiction to $wx \in I_2$) from $\triangle\ wab$ (with I_2^{-1}) and $xb \in \dot{J}_1$ (case (2)). So none of these sub cases can occur. This completes the consideration of Case 2.1.2.2 and hence that of case 2.1 as well.

Suppose now $l_{i+1} \in I_1$ (Case 2.2) with $l_{i+1} \in \dot{J}_2$ (since $l_k \in \dot{J}_2$) and $l_i \in \dot{I}_3$ with $\dot{I}_3 \nsubseteq I_1$. We first assume $\hat{l}_{i+1} = \widehat{az}$ and $\hat{l}_i = \widehat{bz}$ (Case 2.2.1). Suppose furthermore $l_{i+1} = az \in \dot{J}_2$ and $l_i = bz \in \dot{I}_3 \nsubseteq I_1$ (Case 2.2.1.1) (Figure 27). From $\triangle\ xyb$ (with I_2^{-1}) with respect to $az \in \dot{J}_2$ then follows $xz \in \dot{J}_1$ (case (1)). But then, $\triangle\ zax$ (with I_1^{-1}) implies for $ya \in I_2$ (case (1)) $az \in \dot{J}_1$, contradicting $az \in \dot{J}_2 \neq \dot{J}_1$. Hence this case cannot occur.

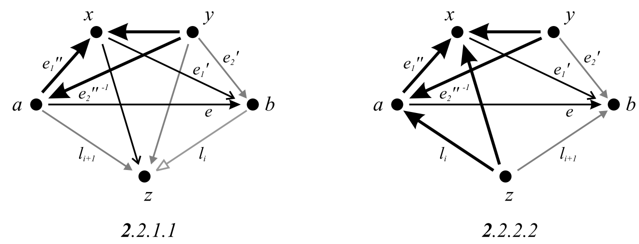


Figure 27. Nontrivial sub cases of Case 2.2.

The case, where $l_{i+1} = za$ belongs to $\dot{J}_1 \subseteq I_1$ (Case 2.2.1.2) cannot occur, since then we would have $l_i = zb$ belonging to I_1 as well. Thus the whole Case 2.2.1 does not occur.

Now let $\hat{l}_{i+1} = \widehat{bz}$ and $\hat{l}_i = \widehat{az}$ (Case 2.2.2). Then there must be $l_{i+1} = zb \in \dot{J}_2$ and $l_i = za \in \dot{I}_3 \nsubseteq I_1$ (Case 2.2.2.2), since $l_{i+1} = bz$ (Case 2.2.2.1) would imply $l_i \in I_1$, again (Figure 27). From $\triangle\ xyb$ (with I_2^{-1}) with respect to $zb \in \dot{J}_2$ follows $zx \in I_2$ (case (1)). From $\triangle\ zxb$ follows $za \in I_2 = \dot{I}_3$ by case (3) of the extended Triangle Lemma. Thus, the last transition in $\mathcal{K}$ is one from I_2 into $\dot{J}_2 \subseteq I_1$. Once more, we now have to consider another transition in this chain. Here, we explore the very first one—from $\dot{J}_1 \subseteq I_1$ ($l_1 \in \dot{J}_1$) to some $\dot{\Gamma}$-component $\dot{I}_4 \nsubseteq I_1$. Let $l_j \in \dot{J}_1$ and $l_{j+1} \in \dot{I}_4$.

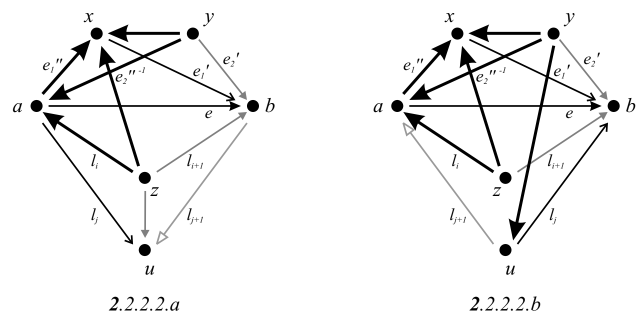


Figure 28. Subcase 2.2.2.2 ($l_{i+1} = zb \in \dot{J}_2$, $l_i = za \in \dot{I}_3 = I_2$, with $l_j \in \dot{J}_1$ and $l_{j+1} \in \dot{I}_4 \nsubseteq I_1$).

Suppose $\hat{l}_j = \widehat{au}$ (Case 2.2.2.2.a), i.e., $\hat{l}_{j+1} = \widehat{bu}$ (Figure 28). Then we have $l_j = au \in \dot{J}_1$ and $l_{j+1} = bu \in \dot{I}_4 \nsubseteq I_1$ (otherwise $l_{j+1} \in I_1$). Then, $\triangle\ zab$ implies, with respect to $au \in \dot{J}_1$ (case (2)), $zu \in \dot{J}_2$. But then follows from $\triangle\ uza$ (with I_1^{-1}) with respect to $ax \in I_2$ (case (1)) $au \in \dot{J}_2$, contradicting

$au \in \dot{J}_1 \neq \dot{J}_2$. Hence, this sub case cannot occur. (This case is symmetric to Case 2.1.1.1.)

This leaves $\hat{l}_j = \widehat{bu}$ (Case 2.2.2.2.b). We then find $l_j = ub \in \dot{J}_1$ and $l_{j+1} = ua \in \dot{I}_4 \nsubseteq I_1$ (otherwise $l_{j+1} \in I_1$). From $\triangle\, yxb$, with respect to $ub \in \dot{J}_1$, follows the existence of $yu \in I_2$ (case (1)). But then case (3) of the extended Triangle Lemma reveals $au \in I_2$, when applied to $\triangle\, uyb$ (with I_2^{-1}). Hence, $\dot{I}_4 = I_2^{-1}$. We thus have a transition from $\dot{J}_1$ to I_2^{-1} between l_j and l_{j+1}, and another from I_2 to $\dot{J}_2$ between l_i and l_{i+1}. Now, this case identical to Case 2.1.2.2.b with u and z being swapped (and the names of the arcs of $\mathcal{K}$ being ignored) (compare Figures 24 and 28, and remember $ua \in I_2$ for Case 2.1.2.2.b). For this case we have already shown that it leads to contradiction in any sub case. This completes the consideration of the whole Case 2.

Thus, every single case and sub case emerging from the assumption that G' is *no* comparability graph, leads to a contradiction. Hence, G' is a comparability graph. ∎

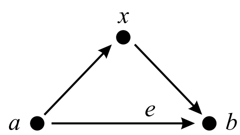
5.6. Result

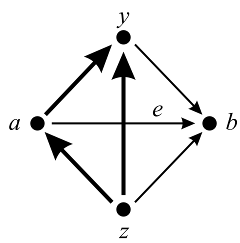
Our partial results explored in sections 5.3. and 5.5. can now be gathered by the following theorem stating a complete mathematical characterization of those edges $\hat{e} \in E$ that may be removed from a comparability graph $G = (V, E)$ without leading to a graph $G - \hat{e}$ that is no comparability graph.

Theorem 6 (transitive orientations of $G - e$). *Let $G = (V, E)$ be a comparability graph, and let $e = ab \in E$ be an arc. Then $G - \hat{e}$ is no comparability graph if and only if there exists on of the following constellations in G*

(i) *$\exists\ x \in V$ with $ax, xb \in I(e)$, such that ax and xb belong to a common $\dot{\Gamma}$-component;*

(ii) *$\exists\ y, z \in V$ with $yb, zb \in I(e) = I_1$, and $za, ay, zy \in I_2$ with $\hat{I}_2 \neq \hat{I}_1$, such that yb and zb belong to a common $\dot{\Gamma}$-component.*





Proof. E may be partitioned into E_N, E_T and E_R, with $E_T = E_{T_0} + E_{T_1}$ and $E_{T_1} = E_{T_{10}} + E_{T_{11}}$. Thus, $E = [E_{T_0} + (E_{T_{10}} + E_{T_{11}})] + E_N + E_R$. The theorem states that $G - \hat{e}$ is a comparability graph if and only if $\hat{e}$ neither belongs to E_{T_0}, nor to $E_{T_{10}}$. We have proved in Lemma 4 and Lemma 10 that $G - \hat{e}$ can be oriented transitively for $\hat{e} \in E_N + E_R$ and $\hat{e} \in E_{T_{11}}$, respectively. On the other hand we have shown in Lemma 5 and Lemma 9 the opposite for $\hat{e} \in E_{T_0}$ and $\hat{e} \in E_{T_{10}}$. ■

Thus, the only cases for which $G - \hat{e}$ is no comparability graph arise for always transitive arcs e whose implication classes are *either* not split at all by the removal of $\hat{e}$ ($\hat{e} \in E_{T_0}$, either case in Theorem 6 may apply), *or* split into several $\dot{\Gamma}$-components ($\hat{e} \in E_{T_1}$), where some transitiving edges are left Γ-connected ($\hat{e} \in E_{T_{10}}$, either case may apply).

6. CONCLUSIONS

We have solved the problem whether the graph obtained by deleting some given edge $\hat{e} \in E$ from a comparability graph $G = (V, E)$ is still a comparability graph or not. We have done this by exploring the properties of the implication class containing e. Therefore we have partitioned the edge set E into the sets of never transitive edges E_N, always transitive edges E_T, and all remaining edges E_R. While for edges from E_N or E_R the (positive) answer to our stated problem was already given by Willenius [16], it remained open for always transitive edges. Therefore, we have introduced the notion of $\dot{\Gamma}$-components as a substructure of the implication class $I(e)$. We have partitioned E_T further into several subsets and subsubsets regarding to the properties of the respective $\dot{\Gamma}$-components of $I(e)$. For each subset we then were able to show its respective behavior, resulting in the statement in Theorem 6. By exploring always transitive edges we furthermore have gained some new insights into the structure of comparability graphs (Lemma 7 and Lemma 8).

From our main result it is easy to develop a sufficient condition for each possible outcome.

Remark 1 (sufficient condition, $G - e$ comparability)**.** Let $G = (V, E)$ be a comparability graph and $e \in E$ an arbitrary arc. If all arcs $e', e'' \in \Gamma(e)$ from the Γ-neighborhood of e belong to different $\dot{\Gamma}$-components ($e' \not\dot{\Gamma}^{+} e''$), then $G - \hat{e}$ is a comparability graph.

Remark 2 (sufficient condition, $G-e$ not comparability)**.** Let $G=(V,E)$ be a comparability graph and let $e \in E$ be an arc that is transitive within its implication class ($e \in E_T$). If the transitiving arcs e_1 and e_2 belong to the same $\dot{\Gamma}$-component ($e_1 \dot{\Gamma}^+ e_2$), then $G-\hat{e}$ is no comparability graph.

Furthermore, it is clear that $G-\hat{e}$ has to be a comparability graph, if for some transitive orientation $T \in \mathcal{T}_G$ it is known that T does not contain e as a transitive arc.

Note, that the knowledge whether or not $\hat{e} \in E$ belongs to E_T is indeed a necessary information. The sets E_N and E_R may as well as E_T be partitioned into subsets E_{N_0} and E_{N_1}, or E_{R_0} and E_{R_1}, respectively, regarding to the number of $\dot{\Gamma}$-components of $I(e)$. But only for always transitive edges this piece of information is relevant. Thus, it is surprising that the membership of $\hat{e}$ to E_T can be determined in polynomial time, while the number of transitive orientations is exponentially bounded by the number of color classes.

The reason for this is that by Lemma 7 and Lemma 8 every always transitive edge is characterized by one of only two possible configurations. It is either transitive within its implication class, or it satisfies configuration $(*)$ from Lemma 8 (see Figure 9 on page 446). By searching for these two configurations it is possible to identify always transitive edges without computing every transitive orientation itself. These configurations can be found in time $\mathcal{O}(n^2m)$, where n is the number of vertices of G and m the number of edges.

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