

ON UNIVERSAL GRAPHS FOR HOM-PROPERTIES

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Abstract

A graph property is any isomorphism closed class of simple graphs. For a simple finite graph H , let $\rightarrow H$ denote the class of all simple countable graphs that admit homomorphisms to H , such classes of graphs are called hom-properties. Given a graph property $\mathcal{P}$, a graph

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$G \in \mathcal{P}$ is universal in $\mathcal{P}$ if each member of $\mathcal{P}$ is isomorphic to an induced subgraph of G . In particular, we consider universal graphs in $\rightarrow H$ and we give a new proof of the existence of a universal graph in $\rightarrow H$, for any finite graph H .

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1. INTRODUCTION

Let us denote by $\mathcal{I}$ the class of all finite simple graphs and by $\mathcal{I}(\aleph_0)$ the class of all simple countable graphs. A *graph property* $\mathcal{P}$ is any nonempty isomorphism-closed subclass of $\mathcal{I}(\aleph_0)$. We also say that a graph G has the property $\mathcal{P}$ if $G \in \mathcal{P}$. A graph property $\mathcal{P}$ is of *finite character* if a graph G has property $\mathcal{P}$ if and only if each finite vertex-induced subgraph of G has property $\mathcal{P}$. We consider graph properties of finite character only. It is easy to see that if $\mathcal{P}$ is of finite character and a graph has property $\mathcal{P}$ then so does every induced subgraph.

A property $\mathcal{P}$ is said to be *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$, that is $\mathcal{P}$ is closed under taking subgraphs. A property $\mathcal{P}$ is said to be *induced-hereditary* if $G \in \mathcal{P}$ and $H \leq G$ implies $H \in \mathcal{P}$, that is $\mathcal{P}$ is closed under taking induced subgraphs. One can easily see that every hereditary property is induced-hereditary as well. On the other hand, the previous definitions yields that properties of finite character are induced-hereditary. However, not all induced-hereditary properties are of finite character; for example the graph property $\mathcal{Q}$ of not containing a vertex of infinite degree is induced-hereditary but not of finite character. A property $\mathcal{P}$ is called *additive* if it is closed under disjoint unions of graphs, which means that a graph has property $\mathcal{P}$ providing all its connected components have this property. The interested reader can find more details about hereditary and induced-hereditary properties in [4].

Given a graph property $\mathcal{P}$, a graph $U \in \mathcal{P}$ is said to be *universal* in $\mathcal{P}$ if each member of $\mathcal{P}$ is isomorphic to an induced subgraph of U and every induced subgraph of U is a member of $\mathcal{P}$. R. Rado in [15] first remarked that among the countable graphs there exists a universal one, often called “the Rado graph” or “the infinite random graph” R (for details see [5]). A graph $W \in \mathcal{P}$ is called *weakly universal* in $\mathcal{P}$ if each member of $\mathcal{P}$ is isomorphic to a subgraph of W . In practice the two notions of universality

behave similarly. A universal graph is evidently weakly universal, and very often the proofs of the nonexistence of a universal graph can be made by excluding weakly universal graphs (see [6]). More information concerning universal graphs and their features can be found in [11].

For a finite graph H , the existence of a weakly universal graph $W(H)$ in the class $\rightarrow H$ was in fact shown in [13]. In [1, 2] A. Bonato gave an explicit construction of the universal (pseudo-homogeneous) graph $M(H)$ in $\rightarrow H$ as a deterministic limit of a chain of finite H -colourable graphs. In this paper we provide a new and explicit representation of a universal graph $U(H)$ in the class $\rightarrow H$. The graph is presented by codes associated to its vertices. We shall show that this graph is isomorphic to $M(H)$.

2. HOM-PROPERTIES

All graphs considered in this paper are simple (without multiple edges or loops), finite or countable and we use the standard notation of [8].

A *homomorphism* of a graph G to a graph H is an edge-preserving mapping $f : V(G) \rightarrow V(H)$ satisfying $e = uv \in E(G)$ implies $f(e) = f(u)f(v) \in E(H)$. In this case we say that G is homomorphic to H and we write $G \rightarrow H$.

A *core* of a finite graph G , denoted by $C(G)$, is any subgraph G' of G such that $G \rightarrow G'$ while G fails to be homomorphic to any proper subgraph of G' . A finite graph G is called a core if G is a core of itself, so that $G \cong C(G)$. A graph G homomorphic to a given graph H is also said to be *H -colourable*. It can be easily seen that up to isomorphism every finite graph has a unique core (see [9]). A *hom-property* is any class $\rightarrow H = \{G \in \mathcal{I}(\aleph_0) \mid G \rightarrow H\}$. The properties $\rightarrow H$, $H \in \mathcal{I}$, are called hom-properties or colour classes (see [14]). Graph homomorphisms and their structure were extensively investigated (see [9, 12, 13, 17]), more references can be found in the survey [14] and in the book [10].

Let us mention some known results concerning hom-properties. Hom-properties can be given in various ways, for example the property $\rightarrow C_6$ is the same as the property $\rightarrow C_{38}$ and/or $\rightarrow K_2$. Let us say that a graph G *generates* the hom-property $\rightarrow H$ whenever $\rightarrow H = \rightarrow G$.

A standard way to describe hom-properties is by cores (see [13]):

Proposition 1. *For any finite graph H and its core $C(H)$ it holds $\rightarrow H = \rightarrow (C(H))$.*

The next result follows directly from the definitions:

Proposition 2. *For any graph $H \in \mathcal{I}$, the hom-property $\rightarrow H$ is hereditary and additive.*

For any graph $G \in \mathcal{I}$ with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, we define a *multiplication* $G^{\ddot{}}(W_1; W_2; \dots; W_n)$ of G in the following way:

1. $V(G^{\ddot{}}) = W_1 \cup W_2 \cup \dots \cup W_n$,
2. for each $1 \leq i \leq n$: $|W_i| \geq 1$,
3. for any pair $1 \leq i < j \leq n$: $W_i \cap W_j = \emptyset$,
4. for any $1 \leq i \leq j \leq n$, $u \in W_i, v \in W_j$: $\{u, v\} \in E(G^{\ddot{}})$ if and only if $\{v_i v_j\} \in E(G)$.

The sets $W_1, W_2, \dots, W_n$ are called the *multivertices* corresponding to vertices $v_1, v_2, \dots, v_n$, respectively. The condition 4 immediately yields that $W_1, W_2, \dots, W_n$ are independent sets and any two vertices belonging to the same multivertex have identical neighbourhoods. Furthermore, it is not difficult to see that $G^{\ddot{}}$ is homomorphic to G . In order to emphasize the structure of $G^{\ddot{}}$ we also use the notation $G^{\ddot{}}(W_1, W_2, \dots, W_n)$.

Let us recall some important properties of multiplications presented in [12, 13]:

Lemma 1. *Let $G^{\ddot{}}$ be a multiplication of a graph G . If w, w^* are two distinct vertices belonging to the same multivertex W of $G^{\ddot{}}$, then there exists a homomorphism $\psi : G \rightarrow G - w^*$.*

The multiplication operation strongly copies the structure of the original graph H . This can be expressed in the language of uniquely H -colourable graphs. This concept was introduced in [17]. We say that a graph G is *uniquely H -colourable* if there is a surjective homomorphism φ from G to H , such that any other homomorphism from G to H is the composition $\varphi \circ \alpha$ of φ and an automorphism α of H .

According to Lemma 1 one can rather easily see the following fact.

Theorem 1. *Let H be a core. Then any multiplication $H^{\ddot{}}(W_1, W_2, \dots, W_n)$ of H is uniquely H -colourable.*

3. MAIN RESULTS

By the definition of $\rightarrow H$, it is easy to see that for a given finite core H , the graph $W(H) = H \cdot (W_1, W_2, \dots, W_n)$, with $|W_i| = \aleph_0$ for $i = 1, 2, \dots, n$ is a weakly universal graph in the class $\rightarrow H$.

In this section we shall show, how to derive a universal graph $U(H)$ in $\rightarrow H$ from the graph $W(H)$. As was already pointed out, its existence was proved by A. Bonato in [2]. Some of its properties were investigated in [1].

Consider a graph H of order d . We are going to construct a graph $U(H)$ – the candidate for a universal graph for the property $\rightarrow H$. Let η be a bijection $\eta : \mathbb{N}^d \rightarrow \mathbb{N}$. Let us denote the vertices of H by $v_1, v_2, \dots, v_d$. For each $i = 1, 2, \dots, d$ take a countable set of independent vertices $W_i = \{v_i^1, v_i^2, \dots, v_i^k, \dots\}$ and for a fixed $i \in \{1, 2, \dots, d\}$ and for each $k \in \mathbb{N}$, let us assign to v_i^k a $d+1$ -tuple $(u_1, u_2, \dots, u_d, u_{d+1})$ such that $k = u_{d+1} = \eta(u_1, u_2, \dots, u_d)$ (if there is no danger of confusion we shall write and use $v_i^k = (u_1, u_2, \dots, u_d, u_{d+1})$). One can immediately see that in such a way the vertices in W_i obtain different ordered $(d+1)$ -tuples, while the codes of v_i^k and v_j^k are the same.

Now we are going to describe the structure of the universal graph $U = U(H)$ in $\rightarrow H$. Put $V(U) = W_1 \cup W_2 \cup \dots \cup W_d$. If $u = v_i^r = (u_1, u_2, \dots, u_d, u_{d+1}) \in W_i$ and $u' = v_j^s = (u'_1, u'_2, \dots, u'_d, u'_{d+1}) \in W_j$ are vertices of U , then uu' is an edge of U if and only if $i < j$ and $v_i v_j \in E(H)$ and 2^r occurs in the unique base 2 expansion of u'_i (the i -th element of the code of the vertex $u' \in W_j$). Note that for each i the set W_i is independent.

Now we are going to prove the main result. The proof of the theorem follows the idea of the proof of Rado in [15] (see also [3]).

Theorem 2. *Let H be a graph. Then $U(H)$ is an universal graph for the property $\rightarrow H$.*

Proof. Let us fix a positive integer $k \in \{2, 3, \dots, d\}$ and for $j = \{1, 2, \dots, k-1\}$ let $A_j, B_j \subseteq W_j$ be arbitrary finite disjoint sets. We shall show that there exists a vertex $w \in W_k$ such that w is joined to all vertices from A_j 's but it is joined to no vertex from B_j 's. This property provides a variation of the property called “e.c. – existentially closed” (see e.g. [3]). We referred it briefly EC*. For each $j = 1, 2, \dots, k-1$ let us put $z_j = \max\{u_{d+1} : u = (u_1, u_2, \dots, u_{d+1}) \in A_j \cup B_j\}$. Now define

$$a_j = \begin{cases} 2^{z_j+1} + \sum_{u \in A_j} 2^{u_{d+1}} & \text{for all } j \in \{1, \dots, k-1\}, \\ 0 & \text{for } k \leq j \leq d, \\ \eta(a_1, a_2, \dots, a_d) & \text{for } j = d+1. \end{cases}$$

We claim that for each $j \in \{1, \dots, k-1\}$ the vertex $w \in W_k$ with the code $(a_1, a_2, \dots, a_{d+1})$ is joined to all vertices from A_j but with no vertex from B_j providing that $v_j v_k \in E(H)$. Indeed, by the definition of $U = U(H)$ and the construction of the code of w , the vertex w is joined to each vertex of A_j . To see, that w is joined to no vertex of B_j , note that for all vertices $u' = (u'_1, u'_2, \dots, u'_{d+1})$ of B_j $2^{u'_{d+1}}$ is not in the base 2 expansion of a_j .

It remains to prove that for any countable graph G belonging to the class $\rightarrow H$ there exists a graph G' , induced subgraph of U , isomorphic to the graph G .

Since the property $\rightarrow H$ is additive and hereditary we can represent a countable graph G in $\rightarrow H$ as a limit of finite graphs from $\rightarrow H$ (see [16, 3]). Thus it is sufficient to provide the embeddings of all finite graphs to $U = U(H)$. Let us remark here that the property $\rightarrow H$ is of finite character, hence the compactness can also be used (see [7]). In order to prove that if G is a fixed finite graph belonging to $\rightarrow H$ then there exists a graph G' , the induced subgraph of U , isomorphic to G we follow the idea of the proof of Theorem 6.7 of [3] and we omit some technical details.

It is obvious that K_1 is an induced subgraph of U . Now let G be a finite graph belonging to $\rightarrow H$. Then there exists a homomorphism $\varphi : G \rightarrow H$. For an arbitrary vertex $v \in V(G)$ the graph $S = G - v$ has order smaller than G and therefore, according to the induction hypothesis, there exists an induced subgraph S' of the graph U that is isomorphic to S . Moreover, it is not difficult to see, that there exists such an embedding that a vertex $u \in V(G)$ with $\varphi(u) = j$ is mapped to a vertex of $W_j \subseteq V(U)$.

According to the labeling of the vertices of H (see the description of the construction above), let k be the largest index such that $v_i \in V(H)$ ($i = 1, 2, \dots, d$) is an image of some vertex of G , i.e., $k = \max\{i : \varphi(x) = v_i, x \in V(G)\}$. Let us choose a vertex $u^* \in V(G)$ such that $\varphi(u) = k$ (note that the set of vertices of G with $\varphi(u) = k$ is independent in G). According to the previous, using EC* property and taking an appropriate vertex of $W_k \subseteq V(U)$ we can now extend the embedding of the graph $S = G - u^*$ to an embedding of the whole graph G of U and the proof is complete. ■

A. Bonato in [1] investigated universal pseudo-homogeneous graphs, that were defined in the following way:

Definition 1. Let $\mathcal{C}$ be a class of countable graphs closed under isomorphisms. A countable graph $M \in \mathcal{C}$ is called *universal pseudo-homogeneous* if there is a subclass $\mathcal{C}'$ of finite graphs from $\mathcal{C}$ such that:

- (PH1) The graph M embeds each graph in $\mathcal{C}'$ as an induced subgraph.
- (PH2) Each finite $S \leq M$ is contained in $T \leq M$ with $T \in \mathcal{C}'$.
- (PH3) For each $G \leq M$ with $G \in \mathcal{C}'$ and for each graph $H \in \mathcal{C}'$ so that $G \leq H$, there is an $H' \leq M$ and an isomorphism $f : H \rightarrow H'$ such that f restricted to G is an identity mapping.

A. Bonato in [1, 2] proved that for each finite core graph H there is a countable universal pseudo-homogeneous H -colourable graph $M(H)$, that is unique up to isomorphism. If we consider the class of graphs that are H -colourable and as the class $\mathcal{C}'$ we take the class of finite uniquely H -colourable graphs, then we immediately have the following result.

Theorem 3. *Let H be a finite core. Then $U(H)$ is the unique universal pseudo-homogeneous graph for the property $\rightarrow H$ with respect to the family of finite uniquely H -colourable graphs.*

Proof. In order to prove the assertion of the theorem we have to verify properties (PH1)–(PH3) from Definition 1. We remind that in our case the set $\mathcal{C}'$ is the class of uniquely H -colourable graphs.

Since $U(H)$ is universal in $\rightarrow H$, the property (PH1) is evidently satisfied. As all the induced subgraphs of $U(H)$ belongs to $\rightarrow H$, the condition (PH2) is evidently satisfied as well. Now we focus on the condition (PH3). Firstly we fix the graph G . Let $G \leq X$, $X \in \rightarrow H$ and let $V(X) \setminus V(G) = \{v_1^1, \dots, v_1^{i_1}, \dots, v_k^1, \dots, v_k^{i_k}\}$. Since X and G are uniquely H -colourable, we can find an extension $X' \leq U(H)$ of X . Observe that there exists a vertex $w_1^1 \in U(H)$ that is an image of v_1^1 . Thus by induction hypothesis we obtain that such images exist for all vertices in $V(X) \setminus V(G)$ (we can apply similar arguments as in the proof of Theorem 2, but using also the “or” statement in the construction of $U(H)$). Indeed, by the consecutive selection of vertices with a suitable structure of neighbours (because of selection of the vertex w with respect to the structure of A_j ’s and B_j ’s) we can find the desired graph X' . ■

Corollary 1. *Let H be a finite core and let $G \in \rightarrow H$, then $U(H) \cong U(G)$.*

Proof. Both universal graphs $U(H)$ and $U(G)$ are universal pseudo-homogeneous graphs for $\rightarrow H$ and thus they are isomorphic to the universal pseudo-homogeneous graph $M(H)$, the existence of which have been proved by A. Bonato in [1]. ■

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