ON UNIVERSAL GRAPHS FOR HOM-PROPERTIES

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Abstract

A graph property is any isomorphism closed class of simple graphs. For a simple finite graph H, let $\to H$ denote the class of all simple countable graphs that admit homomorphisms to H, such classes of graphs are called hom-properties. Given a graph property \mathcal{P} , a graph

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 $G \in \mathcal{P}$ is universal in \mathcal{P} if each member of \mathcal{P} is isomorphic to an induced subgraph of G. In particular, we consider universal graphs in $\rightarrow H$ and we give a new proof of the existence of a universal graph in $\rightarrow H$, for any finite graph H.

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1. INTRODUCTION

Let us denote by \mathcal{I} the class of all finite simple graphs and by $\mathcal{I}(\aleph_0)$ the class of all simple countable graphs. A graph property \mathcal{P} is any nonempty isomorphism-closed subclass of $\mathcal{I}(\aleph_0)$. We also say that a graph G has the property \mathcal{P} if $G \in \mathcal{P}$. A graph property \mathcal{P} is of *finite character* if a graph G has property \mathcal{P} if and only if each finite vertex-induced subgraph of G has property \mathcal{P} . We consider graph properties of finite character only. It is easy to see that if \mathcal{P} is of finite character and a graph has property \mathcal{P} then so does every induced subgraph.

A property \mathcal{P} is said to be *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$, that is \mathcal{P} is closed under taking subgraphs. A property \mathcal{P} is said to be *induced-hereditary* if $G \in \mathcal{P}$ and $H \leq G$ implies $H \in \mathcal{P}$, that is \mathcal{P} is closed under taking induced subgraphs. One can easily see that every hereditary property is induced-hereditary as well. On the other hand, the previous definitions yields that properties of finite character are induced-hereditary. However, not all induced-hereditary properties are of finite character; for example the graph property \mathcal{Q} of not containing a vertex of infinite degree is induced-hereditary but not of finite character. A property \mathcal{P} is called *additive* if it is closed under disjoint unions of graphs, which means that a graph has property \mathcal{P} providing all its connected components have this property. The interested reader can find more details about hereditary and induced-hereditary properties in [4].

Given a graph property \mathcal{P} , a graph $U \in \mathcal{P}$ is said to be *universal* in \mathcal{P} if each member of \mathcal{P} is isomorphic to an induced subgraph of U and every induced subgraph of U is a member of \mathcal{P} . R. Rado in [15] first remarked that among the countable graphs there exists a universal one, often called "the Rado graph" or "the infinite random graph" R (for details see [5]). A graph $W \in \mathcal{P}$ is called *weakly universal* in \mathcal{P} if each member of \mathcal{P} is isomorphic to a subgraph of W. In practice the two notions of universality

behave similarly. A universal graph is evidently weakly universal, and very often the proofs of the nonexistence of a universal graph can be made by excluding weakly universal graphs (see [6]). More information concerning universal graphs and their features can be found in [11].

For a finite graph H, the existence of a weakly universal graph W(H)in the class $\to H$ was in fact shown in [13]. In [1, 2] A. Bonato gave an explicit construction of the universal (pseudo-homogeneous) graph M(H)in $\to H$ as a deterministic limit of a chain of finite H-colourable graphs. In this paper we provide a new and explicit representation of a universal graph U(H) in the class $\to H$. The graph is presented by codes associated to its vertices. We shall show that this graph is isomorphic to M(H).

2. Hom-properties

All graphs considered in this paper are simple (without multiple edges or loops), finite or countable and we use the standard notation of [8].

A homomorphism of a graph G to a graph H is an edge-preserving mapping $f : V(G) \to V(H)$ satisfying $e = uv \in E(G)$ implies $f(e) = f(u)f(v) \in E(H)$. In this case we say that G is homomorphic to H and we write $G \to H$.

A core of a finite graph G, denoted by C(G), is any subgraph G' of G such that $G \to G'$ while G fails to be homomorphic to any proper subgraph of G'. A finite graph G is called a core if G is a core of itself, so that $G \cong C(G)$. A graph G homomorphic to a given graph H is also said to be H-colourable. It can be easily seen that up to isomorphism every finite graph has a unique core (see [9]). A hom-property is any class $\to H = \{G \in \mathcal{I}(\aleph_0) | G \to H\}$. The properties $\to H, H \in \mathcal{I}$, are called hom-properties or colour classes (see [14]). Graph homomorphisms and their structure were extensively investigated (see [9, 12, 13, 17]), more references can be found in the survey [14] and in the book [10].

Let us mention some known results concerning hom-properties. Homproperties can be given in various ways, for example the property $\rightarrow C_6$ is the same as the property $\rightarrow C_{38}$ and/or $\rightarrow K_2$. Let us say that a graph Ggenerates the hom-property $\rightarrow H$ whenever $\rightarrow H = \rightarrow G$.

A standard way to describe hom-properties is by cores (see [13]):

Proposition 1. For any finite graph H and its core C(H) it holds \rightarrow $H = \rightarrow (C(H))$.

The next result follows directly from the definitions:

Proposition 2. For any graph $H \in \mathcal{I}$, the hom-property $\rightarrow H$ is hereditary and additive.

For any graph $G \in \mathcal{I}$ with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, we define a *multiplication* $G^{::}(W_1; W_2; \ldots; W_n)$ of G in the following way:

- 1. $V(G^{::}) = W_1 \cup W_2 \cup \cdots \cup W_n$,
- 2. for each $1 \leq i \leq n$: $|W_i| \geq 1$,
- 3. for any pair $1 \leq i < j \leq n$: $W_i \cap W_j = \emptyset$,
- 4. for any $1 \le i \le j \le n$, $u \in W_i, v \in W_j$: $\{u, v\} \in E(G^{::})$ if and only if $\{v_i v_j\} \in E(G)$.

The sets W_1, W_2, \ldots, W_n are called the *multivertices* corresponding to vertices v_1, v_2, \ldots, v_n , respectively. The condition 4 immediately yields that W_1, W_2, \ldots, W_n are independent sets and any two vertices belonging to the same multivertex have identical neighbourhoods. Furthermore, it is not difficult to see that $G^{::}$ is homomorphic to G. In order to emphasize the structure of $G^{::}$ we also use the notation $G^{::}(W_1, W_2, \ldots, W_n)$.

Let us recall some important properties of multiplications presented in [12, 13]:

Lemma 1. Let $G^{::}$ be a multiplication of a graph G. If w, w^* are two distinct vertices belonging to the same multivertex W of $G^{::}$, then there exists a homomorphism $\psi: G \longrightarrow G - w^*$.

The multiplication operation strongly copies the structure of the original graph H. This can be expressed in the language of uniquely H-colourable graphs. This concept was introduced in [17]. We say that a graph G is uniquely H-colourable if there is a surjective homomorphism φ from G to H, such that any other homomorphism from G to H is the composition $\varphi \circ \alpha$ of φ and an automorphism α of H.

According to Lemma 1 one can rather easily see the following fact.

Theorem 1. Let H be a core. Then any multiplication $H^{::}(W_1, W_2, \ldots, W_n)$ of H is uniquely H-colourable.

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3. Main Results

By the definition of $\to H$, it is easy to see that for a given finite core H, the graph $W(H) = H^{::}(W_1, W_2, \ldots, W_n)$, with $|W_i| = \aleph_0$ for $i = 1, 2, \ldots, n$ is a weakly universal graph in the class $\to H$.

In this section we shall show, how to derive a universal graph U(H)in $\rightarrow H$ from the graph W(H). As was already pointed out, its existence was proved by A. Bonato in [2]. Some of its properties were investigated in [1].

Consider a graph H of order d. We are going to construct a graph U(H) – the candidate for a universal graph for the property $\rightarrow H$. Let η be a bijection $\eta : \mathbb{N}^d \rightarrow \mathbb{N}$. Let us denote the vertices of H by v_1, v_2, \ldots, v_d . For each $i = 1, 2, \ldots, d$ take a countable set of independent vertices $W_i = \{v_i^1, v_i^2, \ldots, v_i^k, \ldots\}$ and for a fixed $i \in \{1, 2, \ldots, d\}$ and for each $k \in \mathbb{N}$, let us assign to v_i^k a d + 1-tuple $(u_1, u_2, \ldots, u_d, u_{d+1})$ such that $k = u_{d+1} = \eta(u_1, u_2, \ldots, u_d, u_{d+1})$. One can immediately see that in such a way the vertices in W_i obtain different ordered (d+1)-tuples, while the codes of v_i^k and v_j^k are the same.

Now we are going to describe the structure of the universal graph U = U(H) in $\to H$. Put $V(U) = W_1 \cup W_2 \cup \cdots \cup W_d$. If $u = v_i^r = (u_1, u_2, \ldots, u_d, u_{d+1}) \in W_i$ and $u' = v_j^s = (u'_1, u'_2, \ldots, u'_d, u'_{d+1}) \in W_j$ are vertices of U, then uu' is an edge of U if and only if i < j and $v_i v_j \in E(H)$ and 2^r occurs in the unique base 2 expansion of u'_i (the *i*-th element of the code of the vertex $u' \in W_j$). Note that for each *i* the set W_i is independent.

Now we are going to prove the main result. The proof of the theorem follows the idea of the proof of Rado in [15] (see also [3]).

Theorem 2. Let H be a graph. Then U(H) is an universal graph for the property $\rightarrow H$.

Proof. Let us fix a positive integer $k \in \{2, 3, ..., d\}$ and for $j = \{1, 2, ..., k-1\}$ let $A_j, B_j \subseteq W_j$ be arbitrary finite disjoint sets. We shall show that there exists a vertex $w \in W_k$ such that w is joined to all vertices from A_j 's but it is joined to no vertex from B_j 's. This property provides a variation of the property called "e.c. – existentially closed" (see e.g. [3]). We referred it briefly EC*. For each j = 1, 2, ..., k-1 let us put $z_j = \max\{u_{d+1} : u = (u_1, u_2, ..., u_{d+1}) \in A_j \cup B_j\}$. Now define

$$a_{j} = \begin{cases} 2^{z_{j}+1} + \sum_{u \in A_{j}} 2^{u_{d+1}} & \text{ for all } j \in \{1, \dots, k-1\}, \\ 0 & \text{ for } k \leq j \leq d, \\ \eta(a_{1}, a_{2}, \dots, a_{d}) & \text{ for } j = d+1. \end{cases}$$

We claim that for each $j \in \{1, \ldots, k-1\}$ the vertex $w \in W_k$ with the code $(a_1, a_2, \ldots, a_{d+1})$ is joined to all vertices from A_j but with no vertex from B_j providing that $v_j v_k \in E(H)$. Indeed, by the definition of U = U(H) and the construction of the code of w, the vertex w is joined to each vertex of A_j . To see, that w is joined to no vertex of B_j , note that for all vertices $u' = (u'_1, u'_2, \ldots, u'_{d+1})$ of $B_j \ 2^{u'_{d+1}}$ is not in the base 2 expansion of a_j .

It remains to prove that for any countable graph G belonging to the class $\rightarrow H$ there exists a graph G', induced subgraph of U, isomorphic to the graph G.

Since the property $\rightarrow H$ is additive and hereditary we can represent a countable graph G in $\rightarrow H$ as a limit of finite graphs from $\rightarrow H$ (see [16, 3]). Thus it is sufficient to provide the embeddings of all finite graphs to U = U(H). Let us remark here that the property $\rightarrow H$ is of finite character, hence the compactness can also be used (see [7]). In order to prove that if G is a fixed finite graph belonging to $\rightarrow H$ then there exists a graph G', the induced subgraph of U, isomorphic to G we follow the idea of the proof of Theorem 6.7 of [3] and we omit some technical details.

It is obvious that K_1 is and induced subgraph of U. Now let G be a finite graph belonging to $\to H$. Then there exists a homomorphism $\varphi : G \to H$. For an arbitrary vertex $v \in V(G)$ the graph S = G - v has order smaller than G and therefore, according to the induction hypothesis, there exists an induced subgraph S' of the graph U that is isomorphic to S. Moreover, it is not difficult to see, that there exists such an embedding that a vertex $u \in V(G)$ with $\varphi(u) = j$ is mapped to a vertex of $W_j \subseteq V(U)$.

According to the labeling of the vertices of H (see the description of the construction above), let k be the largest index such that $v_i \in V(H)$ (i = 1, 2, ..., d) is an image of some vertex of G, i.e., $k = \max\{i : \varphi(x) = v_i, x \in V(G)\}$. Let us choose a vertex $u^* \in V(G)$ such that $\varphi(u) = k$ (note that the set of vertices of G with $\varphi(u) = k$ is independent in G). According to the previous, using EC* property and taking an appropriate vertex of $W_k \subseteq V(U)$ we can now extend the embedding of the graph $S = G - u^*$ to an embedding of the whole graph G of U and the proof is complete.

A. Bonato in [1] investigated universal pseudo-homogeneous graphs, that were defined in the following way:

Definition 1. Let \mathcal{C} be a class of countable graphs closed under isomorphisms. A countable graph $M \in \mathcal{C}$ is called *universal pseudo-homogeneous* if there is a subclass \mathcal{C}' of finite graphs from \mathcal{C} such that:

- (PH1) The graph M embeds each graph in C' as an induced subgraph.
- (PH2) Each finite $S \leq M$ is contained in $T \leq M$ with $T \in \mathcal{C}'$.
- (PH3) For each $G \leq M$ with $G \in \mathcal{C}'$ and for each graph $H \in \mathcal{C}'$ so that $G \leq H$, there is an $H' \leq M$ and an isomorphism $f: H \to H'$ such that f restricted to G is and identity mapping.

A. Bonato in [1, 2] proved that for each finite core graph H there is a countable universal pseudo-homogeneous H-colourable graph M(H), that is unique up to isomorphism. If we consider the class of graphs that are H-colourable and as the class C' we take the class of finite uniquely H-colourable graphs, then we immediately have the following result.

Theorem 3. Let H be a finite core. Then U(H) is the unique universal pseudo-homogeneous graph for the property $\rightarrow H$ with respect to the family of finite uniquely H-colourable graphs.

Proof. In order to prove the assertion of the theorem we have to verify properties (PH1)–(PH3) from Definition 1. We remind that in our case the set C' is the class of uniquely *H*-colourable graphs.

Since U(H) is universal in $\to H$, the property (PH1) is evidently satisfied. As all the induced subgraphs of U(H) belongs to $\to H$, the condition (PH2) is evidently satisfied as well. Now we focus on the condition (PH3). Firstly we fix the graph G. Let $G \leq X, X \in \to H$ and let $V(X) \setminus V(G) = \{v_1^1, \ldots, v_1^{i_1}, \ldots, v_k^1, \ldots, v_k^{i_k}\}$. Since X and G are uniquely H-colourable, we can find an extension $X' \leq U(H)$ of X. Observe that there exists a vertex $w_1^1 \in U(H)$ that is an image of v_1^1 . Thus by induction hypothesis we obtain that such images exist for all vertices in $V(X) \setminus V(G)$ (we can apply similar arguments as in the proof of Theorem 2, but using also the "or" statement in the construction of U(H)). Indeed, by the consecutive selection of vertices with a suitable structure of neighbours (because of selection of the vertex w with respect to the structure of A_j 's and B_j 's) we can find the desired graph X'. **Corollary 1.** Let H be a finite core and let $G \in \to H$, then $U(H) \cong U(G)$.

Proof. Both universal graphs U(H) and U(G) are universal pseudohomogeneous graphs for $\rightarrow H$ and thus they are isomorphic to the universal pseudo-homogeneous graph M(H), the existence of which have been proved by A. Bonato in [1].

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