Discussiones Mathematicae
Graph Theory 29 (2009) 385–400

ON LOCAL STRUCTURE OF 1-PLANAR GRAPHS OF MINIMUM DEGREE 5 AND GIRTH 4

Dávid Hudák and Tomáš Madaras*

Institute of Mathematics, Faculty of Sciences
University of P. J. Šafárik
Jesenná 5, 040 01 Košice, Slovak Republic

e-mail: davidh@centrum.sk, tomas.madaras@upjs.sk

Abstract

A graph is 1-planar if it can be embedded in the plane so that each edge is crossed by at most one other edge. We prove that each 1-planar graph of minimum degree 5 and girth 4 contains

(1) a 5-vertex adjacent to an ≤ 6-vertex,

(2) a 4-cycle whose every vertex has degree at most 9,

(3) a $K_{1,4}$ with all vertices having degree at most 11.

Keywords: light graph, 1-planar graph, star, cycle.

2000 Mathematics Subject Classification: 05C10.

1. Introduction

Throughout this paper, we consider connected graphs without loops or multiple edges. We use the standard graph terminology by [6]. By a *k-path* (a *k-cycle*) we mean a path P_k (a cycle C_k) on k vertices. A *k-star* S_k is the complete bipartite graph $K_{1,k}$. A vertex of degree k is called a *k-vertex*, a vertex of degree $\geq k$ ($\leq k$) an $\geq$ *k-vertex* (an $\leq$ k-vertex, respectively). For a plane graph G, the size of a face $\alpha \in G$ is the length of the minimal boundary walk of α; a face of size k (or $\geq k$) is called a *k-face* (an $\geq$ *k-face*, respectively).

*This work was supported by Science and Technology Assistance Agency under the contract No. APVV-0007-07, and by Slovak VEGA Grant 1/3004/06.

A graph is 1-*planar* if it can be embedded in the plane so that each edge is crossed by at most one other edge. 1-planar graphs were first considered by Ringel [11] in connection with the simultaneous vertex/face colouring of plane graphs (note that the graph of adjacency/incidence of vertices and faces of a plane graph is 1-planar); in the mentioned paper, he proved that each 1-planar graph is 7-colourable (in [5], a linear time algorithm for 7-colouring of 1-planar graphs is presented). Borodin [2, 3] proved that each 1-planar graph is 6-colourable (and the bound 6 is best possible), and in [4], it was proved that each 1-planar graph is acyclically 20-colourable. The global structural aspects of 1-planar graphs were studied in [12] and [10]. The local structure of 1-planar graphs was studied in detail in [7].

But, comparing to the family of all plane graphs, the family of 1-planar graphs is still only little explored. In particular, the complete information on dependence of girth of 1-planar graphs according to their minimum degree is not known. From Euler polyhedral formula, one obtains that each planar graph of minimum degree $\delta \geq 3$ has the girth at most 5, and if $\delta \geq 4$, then the girth is 3. For 1-planar graphs, it was proved in [7] that each 1-planar graph of minimum degree $\delta \geq 5$ has girth at most 4 (and there are graphs that reach this bound) and if $\delta \geq 6$, then the girth is 3. In addition, R. Soták (personal communication) found an example of 1-planar graph with $\delta = 4$ and girth 5, and in [7], there is an example of 1-planar graph of minimum degree 3 and girth 7; we conjecture that the values 5 and 7 are best possible.

The aim of this paper is to explore in deeper details the local properties of 1-planar graphs of minimum degree 5. Our motivation comes from the research of structure of plane graphs of minimum degree 5. It is known that such graphs contain a variety of small configurations having vertices of small degrees. For example, Borodin [1] proved that each plane graph of minimum degree 5 contains a triangular face of weight (that is, the sum of degrees of its vertices) at most 17 (the bound being sharp). Other similar results may be found in [8] or [9]. It appears that analogical results hold also for 1-planar graphs with sufficiently high minimum degree. In [7], it was proved that each 1-planar graph of minimum degree ≥ 6 contains a 3-cycle with all vertices of degree at most 10 as well as 4-star with all vertices of degrees at most 23, and if the minimum degree is 7, then it contains also a 6-star with all vertices of degree at most 15. On the other hand, such result do not apply in general on 1-planar graphs of minimum degree ≥ 5: in [7], there are examples of 1-planar graphs of minimum degree 5 such that all their 3-cycles have arbitrarily high degree-sum of vertices.

In this paper, we show that, under the additional requirement of girth being 4, 1-planar graphs of minimum degree ≥ 5 also contain certain small subgraphs with vertices of small degrees. We prove

Theorem 1. *Each* 1*-planar graph of minimum degree* 5 *and girth* 4 *contains a* 5*-vertex which is adjacent to an* ≤ 6*-vertex.*

Theorem 2. *Each* 1*-planar graph of minimum degree* 5 *and girth* 4 *contains a* 4*-cycle such that each its vertex has degree at most* 9.

Theorem 3. *Each* 1*-planar graph of minimum degree* 5 *and girth* 4 *contains a* 4*-star such that each its vertex has degree at most* 11.

In Theorems 1 and 2, the requirement of girth 4 cannot be avoided: in [7], it was shown that 4-cycles and 4-stars in 1-planar graphs of minimum degree 5 may reach arbitrarily high degrees. This requirement is also substantial for the first result, since there is an example of 1-planar graph of minimum degree 5 such that each its 5-vertex is adjacent only with ≥ 8-vertices (see Figure 1). An open question is whether each 1-planar graph of minimum degree 5 and girth 4 contains a pair of adjacent 5-vertices.

As any 1-planar graph G with girth at least 5 has $\delta(G) \leq 4$, we obtain that G is 5-colourable. It is not known whether an analogical result holds for 1-planar graphs of girth at least 4. The example of infinite 1-planar graph of minimum degree 4 and girth 6 (constructed from hexagonal tiling of the plane, see Figure 2) together with the example of Soták concerning 1-planar graph of minimum degree 4 and girth 5 suggest to conjecture that each 1-planar graph with $\delta \geq 4$ has girth at most 5; in this case, any 1-planar graph of girth at least 6 would be of minimum degree at most 3, hence, it would be 4-colourable.

2. Proofs

2.1. Basic terms

The following definitions are taken from [7].

Let G be a 1-planar graph and let $D(G)$ be its 1-planar diagram (a drawing of G in which every edge is crossed at most once). Given two nonadjacent edges $xy, uv \in E(G)$, the *crossing* of xy, uv is the common point of two arcs $\widehat{xy}, \widehat{uv} \in D(G)$ (corresponding to edges xy and uv). Denote by $C = C(D(G))$ the set of all crossings in $D(G)$ and by E_0 the set of all non-crossed edges in $D(G)$. The *associated plane graph* $D^\times(G)$ of $D(G)$ is the plane graph such that $V(D^\times(G)) = V(D(G)) \cup C$ and $E(D^\times(G)) = E_0 \cup \{xz, yz | \ xy \in E(D(G)) - E_0, z \in C, z \in xy\}$. Thus, in $D^\times(G)$, the crossings of $D(G)$ become new vertices of degree 4; we call these vertices *crossing vertices*. The vertices of $D^\times(G)$ which are also vertices of G are called *true*.

Note that a 1-planar graph may have different 1-planar diagrams, which lead to nonisomorphic associated plane graphs. Among all possible 1-planar diagrams of a 1-planar graph G, we denote by $M(G)$ such a diagram that has the minimum number of crossings (it is not necessarily unique), and by $M^\times(G)$ its associated plane graph.

All results of this paper are proved in a common way: we proceed by contradiction, thus, we consider a hypothetical counterexample G which does not contain the specified graph. Further, on the plane graph $M^\times(G) = (V^\times, E^\times, F^\times)$, the Discharging method is used. We define the charge $c : V^\times \cup F^\times \to \mathbb{Z}$ by the assignment $c(v) = \deg_G(v) - 6$ for all $v \in V^\times$ and $c(\alpha) = 2\deg_G(\alpha) - 6$ for all $\alpha \in F^\times$. From the Euler polyhedral formula, it follows that $\sum_{x \in V^\times \cup F^\times} c(x) = -12$. Next, we define the local redistribution of charges between the vertices and faces of $M^\times(G)$ such that the sum of charges remain the same. This is performed by certain rules which specify the charge transfers from one element to other elements in specific situations. After such redistribution, we obtain a new charge $\widetilde{c} : \ V^\times \cup F^\times \to \mathbb{Q}$. To get the contradiction, we prove that for any element $x \in V^\times \cup F^\times$, $\widetilde{c}(x) \geq 0$ (hence, $\sum_{x \in V^\times \cup F^\times} \widetilde{c}(x) \geq 0$).

For the purposes of these proofs, we introduce some specialized notations. Given a d-vertex $x \in M^\times(G)$, by $x_1, \ldots, x_d$ we denote its neighbours in $M^\times(G)$ in the clockwise order. By $f_i, i = 1, \ldots, d$, we denote the face of $M^\times(G)$ which contains the path $x_i x x_{i+1}$ (index modulo d) as a part of its boundary walk. If f_i is a 4-face, then x'_i will denote the common neighbour of x_i and x_{i+1} which is different from x.

2.2. Proof of Theorem 1

The proof proceeds in the way described in Subsection 2.1 under the following discharging rules:

Rule 1. Each face $f \in F^\times$ sends $\frac{c(f)}{m(f)}$ to each incident 4- and 5-vertex, where $m(f)$ is the number of 4- and 5-vertices incident with f. If $m(f) = 0$, no charge is transferred.

Rule 2. Each ≥ 7-vertex v sends $\frac{c(v)}{\deg_G(v)}$ to each 5-vertex which is incident with v in G.

Let $\overline{c}(x)$ be the charge of a vertex $x \in V^\times$ after application of Rules 1 and 2.

Rule 3. Each 5-vertex v with $\overline{c}(v) > 0$ sends $\frac{\overline{c}(v)}{\overline{m}(v)}$ to each adjacent 4-vertex; $\overline{m}(v)$ is the number of 4-vertices adjacent to v (if $\overline{m}(v) = 0$, no charge is transferred).

We will check the final charge of vertices and faces after the charge redistribution. From Rule 1, it follows that $\widetilde{c}(f) \geq 0$ for each face $f \in F^\times$. Similarly, the Rule 2 ensures that $\widetilde{c}(v) \geq 0$ for any $\geq$ 7-vertex $v \in V^\times$. Since, for a 6-vertex v, $\widetilde{c}(v) = c(v) = 0$, it is enough to check the final charge of 5- and 4-vertices.

Case 1. Let x be a 5-vertex. Then all its neighbours in G are $\geq$ 7-vertices, thus, by Rule 2, x receives at least $5 \cdot \frac{1}{}$

$$\overline{c}(x) \geq -1 + 5 \cdot \frac{1}{}$$

$$\overline{c}(x_2) \geq -1 + \frac{5}{} \qquad \overline{c}(x_3) \geq \frac{17}{}$$

$$\frac{}{5} = \frac{67}{210} > 0.$$

Now, let one of x_1, x_4 be a 5-vertex, say x_1. Then x_3, x_4 are $\geq$ 7-vertices and $\widetilde{c}(x) \geq -2 + 1 + \frac{2}{}$

... both x_1, x_4 are 5-vertices. Then these vertices are

Case 2.3. Let x be incident with exactly two $\geq$ 4-faces. Note that the remaining 3-faces in the neighbourhood of x cannot be adjacent (otherwise their vertices form a 3-cycle); thus, assume that f_4 and f_2 are 3-faces. Moreover, at most two neighbours of x in $M^{\times}(G)$ are 5-vertices.

Case 2.3.1. If all vertices $x_1, \ldots, x_4$ are $\geq$ 6-vertices, then $\widetilde{c}(x) \geq -2 + 2 \cdot 1 = 0$.

Case 2.3.2. Let exactly one of $x_1, \ldots, x_4$ be a 5-vertex, say x_1. Then x_3, x_4 are $\geq$ 7-vertices, x_2 is a $\geq$ 6-vertex. If f_1 is an $\geq$ 5-face, then $\widetilde{c}(x) \geq -2 + 1 + \frac{2\cdot5-6}{5-1} = 0$; hence, assume that f_1 is a 4-face.

If x_1' is a true vertex, then it is an $\geq$ 7-vertex and $\widetilde{c}(x) \geq -2+1+1 = 0$; thus, assume that x_1' is a crossing-vertex. Denote by w and y the remaining neighbours of x_1 (such that the neighbours of x_1 in $M^{\times}(G)$ in clockwise ordering are w, y, x_1', x, x_4) and let f', f'', f''' be faces that contain, as a part of the boundary, the paths $x_1'xy, yxw$ and wxx_4, respectively.

Suppose first that f''' is an $\geq$ 4-face. If w is a true vertex, then it is $\geq$ 7-vertex and $\overline{c}(x_1) \geq -1+1+\frac{2}{\ }\ \ \frac{}{7} = \frac{29}{\ }$ $- \frac{}{3} = \frac{8}{63} > 0$. Thus, let w be a crossing-vertex. Now, if y is a true vertex, then it is also an $\geq$ 7-vertex and $\overline{m}(x_1) \leq 3$; we have $\overline{c}(x_1) \geq -1 + 5 \cdot \frac{1}{\ }$ $-$ $-$ $- \frac{}{3} = \frac{1}{63} > 0$. Hence, both y and w are crossing-vertices; but then f'' is necessarily an $\geq$ 4-face. In this case, we obtain $\overline{c}(x_1) \geq -1+5\cdot\frac{1}{\ }$ $-$ $-$ $-$ $- \frac{}{4} = \frac{3}{56} > 0$.

Now, let f''' be a 3-face. Then w is a crossing-vertex and f'' is an $\geq$ 4-face (otherwise either 3-cycle appears or 1-planarity of G is violated). If y is a true vertex, then it is an $\geq$ 7-vertex and we have $\overline{c}(x_1) \geq -1 + 5 \cdot \frac{1}{\ }$ $-$ $-$ $- \frac{}{3} = \frac{1}{63} > 0$; so, suppose that y is a crossing-vertex. But then f' is necessarily an $\geq$ 4-face and we obtain $\overline{c}(x_1) \geq -1+5\cdot\frac{1}{7}+\frac{2}{3}+2\cdot\frac{1}{\ }$ $-$ $- \frac{}{4} = \frac{1}{84} > 0$.

Case 2.3.3. Let two of $x_1, \ldots, x_4$ be 5-vertices, say x_1 and x_2. Then x_3, x_4 are $\geq$ 7-vertices. If f_1 is an $\geq$ 6-face, then $\widetilde{c}(x) \geq -2 + 1 + \frac{2\cdot6-6}{6} = 0$. If f_1 is a 5-face, then it is incident with at most two crossing-vertices, which implies that there exists an $\geq$ 7-vertex incident with f_1 which is adjacent with x_1 or x_2; this yields $m(f_1) \leq 4$ and $\widetilde{c}(x) \geq -2 + 1 + \frac{2\cdot5-6}{4} = 0$. So, let f_1 be a 4-face.

If x_1' is a true vertex, then it is an $\geq$ 7-vertex; in this case, $\overline{c}(x_1) \geq -1 + 5 \cdot \frac{1}{\ }$ $-$ $-$ $-$ $\overline{c}(x_2) \geq \frac{37}{\ }$ $-$ $- \frac{}{4} = \frac{3}{28} > 0$.

Hence, assume that x_1' is a crossing-vertex. Now, consider the charge of the vertex x_1 after application of Rules 1 and 2 (the situation for the vertex x_2 is symmetric). Let w, y, f', f'', f''' be the same elements as defined in the case 2.3.2.

If f'' is a 3-face, then w is a crossing-vertex; subsequently, f'' has to be an $\geq$ 4-face. Independently of this, if f' is a 3-face, then y is not a crossing-vertex, hence, it is an $\geq$ 7-vertex. Thus, we consider four possibilities:

(a) f''', f' are 3-faces. Then $\overline{m}(x_1) = 3$ and $\overline{c}(x_1) \geq -1 + 5 \cdot \frac{1}{}$ – – —

$\overline{m}(x_1) \leq 4$

and $\overline{c}(x_1) \geq -1 + 5 \cdot \frac{1}{}$ – —

$\overline{m}(x_1) = 3$ and $\overline{c}(x_1) \geq -1 + 5 \cdot$

$\frac{1}{}$ – – —

$\overline{m}(x_1) \leq 4$ and $\overline{c}(x_1) \geq -1 + 5 \cdot$

$\frac{1}{}$ – – —

$\overline{126}$ (the bound being attained in the first and third case). The same considerations apply for the vertex x_2. Hence, in total, x receives at least $1 + \frac{1}{}$ $\overline{126} = \frac{263}{}$ ■

2.3. Proof of Theorem 2

The proof proceeds in the way described in Subsection 2.1 under the following discharging rules (for the purpose of this proof, a big vertex is one of degree ≥ 10):

Rule 1. Each face $f \in F^\times$ sends $\frac{c(f)}{m(f)}$ to each incident 4- and 5-vertex, where $m(f)$ is the number of 4- and 5-vertices incident with f. If $m(f) = 0$, no charge is transferred.

Rule 2. Each big vertex sends $\frac{2}{}$

$\overline{c}(x)$ be the charge of a vertex $x \in V^\times$ after application of Rules 1 and 2.

Rule 3. Each 5-vertex v with $\overline{c}(v) > 0$ sends $\frac{\overline{c}(v)}{\overline{m}(v)}$ to each adjacent 4-vertex; $\overline{m}(v)$ is the number of 4-vertices adjacent to v (if $\overline{m}(v) = 0$, no charge is transferred).

To show that the final charges of elements of $M^{\times}(G)$ are nonnegative, we consider several cases; due to the formulation of discharging rules, it is enough to check the final charges of 4-, 5- and big vertices.

Case 1. Let x be a 5-vertex. Then x is incident with at most three 3-faces (otherwise it is incident with two 3-faces with the common edge having one endvertex of degree 4; thus, x belongs to a 3-cycle of G, a contradiction). By Rule 1, $\widetilde{c}(x) \geq -1 + 2 \cdot \frac{2\cdot4-6}{4} = -1 + 2 \cdot \frac{1}{}$

$- \frac{}{6} = 0$; similarly, if at least two of them are $\geq$ 5-faces, then $\widetilde{c}(x) \geq -2 + 2 \cdot \frac{2\cdot5-6}{5} + \frac{1}{}$

$- \frac{}{5-1} = 0$. Further, if x_3 is an $\geq$ 6-vertex, then $\widetilde{c}(x) \geq -2 + \frac{4}{} \quad \frac{}{4-1} = \frac{2}{15} > 0$. Hence, we may assume that both x_2 and x_3 are 5-vertices. Then each of them is incident with at least three $\geq$ 4-faces, so $\overline{c}(x_2) \geq -1 + \frac{4}{} \quad - \quad - \; \overline{c}(x_3) \geq -1 + 3 \cdot \frac{1}{} \quad -$

$- \quad - \quad \frac{-}{5} + \frac{1}{5} = \frac{3}{50} > 0.$

Suppose next that f_2 is a 5-face. If at least one of x_2, x_3 is an $\geq$ 6-vertex, then $\widetilde{c}(x) \geq -2 + \frac{1}{} \quad \frac{}{4-1} + \frac{2\cdot5-6}{5-1} > 0$; thus, assume that both x_2, x_3 are 5-vertices. Then again, x_2 and x_3 are incident with at least three $\geq$ 4-faces and $\overline{c}(x_2) \geq -1 + 2 \cdot \frac{1}{} \quad - \quad - \; \overline{c}(x_3) \geq -1 + 2 \cdot \frac{1}{} \quad - \quad -$

$- \quad - \quad \frac{-}{5} = \frac{3}{25} > 0.$

Case 2.2.2. Let each of f_1, f_2, f_3 be a 4-face.
If both vertices x_2, x_3 are of degree ≥ 6, then $\widetilde{c}(x) \geq -2 + 2 \cdot \frac{1}{} \quad \frac{}{2} = 0$; also, it is easy to check that $\widetilde{c}(x) \geq 0$ if two of $x_1, \ldots, x_4$ (except of x_1, x_4)

are of degree ≥ 6. If both x_1, x_4 are ≥ 6-vertices, then $\overline{c}(x_2) \geq -1 + \frac{2}{}$

– – – – $\frac{-}{5} = \frac{1}{10} > 0$.

Moreover, if at least one of $x_1, \ldots, x_4$ is big, then $\widetilde{c}(x) \geq -2 + 2 \cdot \frac{1}{}$ –

–

$\overline{c}(x_1) \geq -1 + \frac{1}{}$ – – — $\overline{c}(x_2) \geq -1 + 2 \cdot \frac{1}{}$ – – — $\overline{m}(x_1) \leq$ 3, $\overline{m}(x_2) \leq 4$ and $\widetilde{c}(x) \geq -2 + 2 \cdot \frac{1}{}$ – $\frac{-}{3} + \frac{16}{4} = \frac{11}{}$

$\overline{c}(x_3) \geq -1 + 2 \cdot \frac{1}{}$ – – —

– – $\frac{-}{3} = \frac{17}{}$

$\overline{c}(x_3) \geq -1 + 3 \cdot \frac{1}{}$ – –

$\overline{m}(x_3) \leq 4$ and $\widetilde{c}(x) \geq -2 + 2 \cdot \frac{2}{}$ – $\frac{-}{4} > 0$. So, it remains to resolve the case when x_2' is not big and both x_2, x_3 are 5-vertices. Consider the charge of x_2 after application of the first two discharging rules (the case of x_3 is analogous). If x_2 is incident with at most one 3-face, then $\overline{c}(x_2) \geq -1 + 4 \cdot \frac{1}{}$ $\overline{m}(x_2) \leq 4$, thus, x_2 contributes at least $\frac{1}{}$

$\overline{c}(x_2) \geq -1 + 3 \cdot \frac{1}{}$ – $\frac{}{10}$, $\overline{m}(x_2) \leq 3$ and x_2 may contribute at least $\frac{\frac{9}{10}}{3} > \frac{1}{}$

$\overline{c}(x_2) \geq -1 + 4 \cdot \frac{1}{}$ $\overline{m}(x_2) \leq 4$ and x_2 contributes at least $\frac{1}{}$

–

– –

Suppose now that x_2' is a big vertex. If some of x_2, x_3 is an $\geq$ 6-vertex, then $\widetilde{c}(x) \geq -2 + \frac{1}{}$ –

$\overline{c}(x_2) \geq -1 + 3 \cdot \frac{1}{}$ – $\overline{10}, \overline{c}(x_2) \geq -1 + 3 \cdot \frac{1}{}$ – $\overline{10}$ and

$\widetilde{c}(x) \geq -2 + 2 \cdot \frac{1}{}$ – $\frac{10}{4} = \frac{7}{60} > 0$.

Case 2.2.2.2.2. Let each of x_1', x_2', x_3' be a crossing-vertex.

Case 2.2.2.2.2.1. Suppose first that exactly one of $x_1, \ldots, x_4$ is an $\geq$ 6-vertex; due to the symmetry, we will consider x_1 or x_2. Let x_1 be an $\geq$ 6-vertex. If x_4 is incident with at most two 3-faces, then $\overline{c}(x_4) \geq -1 + 3 \cdot \frac{1}{}$ – $\overline{m}(x_4) \leq 4$ and x_4 may contribute to x at least $\frac{1}{}$

$\overline{c}(x_4) \geq -1 +$

$2 \cdot \frac{1}{}$ – – $\overline{m}(x_4) \leq 3$ and x_4 may contribute to x at least $\frac{2}{15} > \frac{1}{}$

$\overline{c}(x_2) \geq -1 + \frac{2}{}$ – – $\overline{c}(x_3) \geq -1 + 3 \cdot \frac{1}{}$ –

– – – $\overline{15} + \frac{1}{10} = \frac{1}{40} > 0$.

Now, let x_2 be an $\geq$ 6-vertex. Then $\overline{c}(x_1) \geq -1 + \frac{1}{}$ – – $\overline{c}(x_3) \geq$

$-1 + 2 \cdot \frac{1}{}$ – – – – $\frac{-}{5} + \frac{2}{5} = 0$.

Case 2.2.2.2.2.2. Let all neighbours of x be 5-vertices. Note that each of x_2, x_3 is incident with at least three $\geq$ 4-faces; this implies that each of them may contribute at least $\frac{-1+3\cdot\frac{1}{}}{4} = \frac{1}{}$

$\frac{-}{5} =$

$\frac{1}{}$ –

– –

$\overline{m}(x_1) = 3, \overline{c}(x_1) \geq -1 + 2 \cdot \frac{1}{}$ – – $\frac{-}{3} = \frac{2}{15} > \frac{1}{}$

– – $\overline{15} > 0$.

Case 2.3. Let x be incident with exactly two 3-faces (note that they are not adjacent). Then (to avoid a light 4-cycle) one of $x_1, \ldots, x_4$ is big, say x_1. If x has also another big neighbour then $\widetilde{c}(x) \geq -2 + 2 \cdot \frac{2}{}$ –

– –

– –

Case 2.3.1. Let x_2, x_3, x_4 be 5-vertices. If f_1 is an $\geq$ 5-face, then $\overline{c}(x_2) \geq -1+1+\frac{1}{}$ – $\overline{m}(x_2) \leq 4, \overline{c}(x_4) \geq -1+2\cdot\frac{1}{}$ – – $\overline{m}(x_4) \leq 4$ and $\widetilde{c}(x) \geq -2+1+\frac{1}{}$ – $\frac{-}{4} + \frac{1}{4} = \frac{1}{}$
$\overline{c}(x_4) \geq -1+\frac{4}{}$ – – $\frac{}{10}, \overline{m}(x_4) \leq 4, \overline{c}(x_2) \geq -1+\frac{2}{}$ – – $\overline{m}(x_2) \leq 4$ and $\widetilde{c}(x) \geq -2+\frac{2}{}$ – – $\frac{\overline{10}}{4} + \frac{1}{4} = \frac{1}{12} > 0$. Hence, we may suppose that both f_1 and f_3 are 4-faces. In addition, we may also suppose that x_3' is a crossing vertex (otherwise x_3' is big, since $x_3'x_3x_2x_4$ is a 4-cycle; then $\overline{c}(x_3) \geq -1+\frac{1}{}$ – – — $\overline{c}(x_4) \geq -1+\frac{1}{}$ – – — $\overline{m}(x_3) \leq 3, \overline{m}(x_4) \leq 3$ and $\widetilde{c}(x) \geq -2+2\cdot\frac{2}{}$ – $\frac{-}{3} + \frac{29}{3} = \frac{56}{}$

$\overline{c}(x_3) \geq -1+3\cdot\frac{1}{}$ – $\overline{c}(x_4) \geq -1+3\cdot\frac{1}{}$ – $\frac{}{10}, \overline{m}(x_3) \leq 4, \overline{m}(x_4) \leq 4$. Now, if $\overline{m}(x_4) \leq 3$, then $\widetilde{c}(x) \geq -2+\frac{1}{}$ – – $\frac{\overline{10}}{3} + \frac{1}{4} + \frac{1}{4} = \frac{1}{30} > 0$. Thus, let $\overline{m}(x_4) = 4$. Further, if $\overline{m}(x_3) \leq 3$, then $\widetilde{c}(x) \geq -2+\frac{1}{}$ – – $\frac{\overline{10}}{4} + \frac{1}{3} + \frac{1}{4} = 0$; hence, we may assume that also $\overline{m}(x_3) = 4$. Consider now the faces $\alpha, \beta, \gamma, f_2, f_3$ that appear around x_3 in the counter-clockwise order. We may assume that γ is a 3-face (otherwise $\widetilde{c}(x) \geq -2+\frac{1}{}$ – – $\frac{\overline{10}}{4} + \frac{1}{}$ $\frac{-}{4} = \frac{1}{12} > 0$). If some of α, β is an $\geq$ 5-face, then $\overline{c}(x_3) \geq -1+2\cdot\frac{1}{2}+\frac{4}{}$ – $\frac{}{2}+\frac{2}{3}+\frac{2}{}$ $\frac{\overline{10}}{4}+\frac{4}{4}+\frac{1}{4} = \frac{1}{30} > 0$. Hence, let $\alpha = [x_3x_3'yw], \beta = [x_3wzq]$ be 4-faces. As x_2x_4yz is a 4-cycle in G, one of y, z must be big. Then $\overline{c}(x_3) \geq -1+2\cdot\frac{1}{}$ – – – – – $\frac{\overline{10}}{+}\frac{2}{4} + \frac{1}{4} = 0$.

Case 2.3.1.2. Let $x_3'x_4$ be incident with exactly one $\geq$ 4-face f_3. Then there exists a 3-face $[x_3'x_4w]$. Since $x_4wx_3x_2$ is a 4-cycle in G, w is a big vertex. Moreover, at least one of edges wx_4, x_1x_4 is incident with an $\geq$ 4-face. This yields $\overline{c}(x_4) \geq -1+\frac{1}{}$ – – — $\overline{m}(x_4) \leq 3$. Now, if the edge x_3x_3' is incident with two $\geq$ 4-faces, then $\overline{c}(x_3) \geq -1+3\cdot\frac{1}{}$ – $\overline{m}(x_3) \leq 4$ and $\widetilde{c}(x) \geq -2+\frac{1}{}$ $\frac{}{3}+\frac{2}{}$ $\frac{-}{3}+\frac{1}{4}+\frac{1}{4} = \frac{1}{18} > 0$. So, let x_3x_3' be incident with exactly one $\geq$ 4-face f_3. Then there exists a 3-face $[x_3x_3'w']$. As $x_4x_2x_3w$ is a 4-cycle in G, w' is big. Therefore, we have $\overline{c}(x_3) \geq -1+2\cdot\frac{1}{}$ – – $\overline{m}(x_3) \leq 3$ and $\widetilde{c}(x) \geq -2+\frac{1}{}$ – – $\frac{-}{3} + \frac{2}{3} + \frac{1}{4} = \frac{23}{360} > 0$.

Case 2.3.1.3. Let $x_3'x_3$ be incident with exactly one $\geq$ 4-face f_3. Then there exists a 3-face $[x_3'x_3w]$. Again, $wx_4x_2x_3$ is a 4-cycle in G, so w is big.

Now, if the edge x_3w is incident with an $\geq$ 4-face, then $\overline{c}(x_3) \geq -1 + \frac{1}{\ }$ – – — $\overline{m}(x_3) \leq 3, \overline{c}(x_4) \geq -1 + 3 \cdot \frac{1}{\ }$ – $\overline{10}, \overline{m}(x_4) \leq 4$ and $\widetilde{c}(x) \geq -2 + \frac{1}{\ }$ – – $\frac{-}{3} + \frac{\frac{9}{10}}{4} + \frac{\frac{1}{\ }}{4} = \frac{1}{45} > 0$. In the opposite case, consider the face $\gamma \neq f_2$ incident with the edge x_2x_3. If γ is an $\geq$ 5-face, then $\overline{c}(x_3) \geq -1 + \frac{1}{\ }$ – – $\overline{10}, \overline{c}(x_2) \geq -1 + \frac{1}{\ }$ – $\overline{10}, \overline{m}(x_3) \leq 3, \overline{m}(x_4) \leq 4$ and $\widetilde{c}(x) \geq -2 + \frac{1}{\ }$ – – $\frac{\overline{10}}{4} + \frac{\frac{7}{10}}{3} + \frac{\frac{3}{10}}{4} = \frac{1}{10} > 0$. Hence, suppose that γ is a 4-face. If $\overline{m}(x_3) = 2$, then $\widetilde{c}(x) \geq -2 + \frac{1}{\ }$ – – $\frac{\overline{10}}{4} + \frac{2}{2} + \frac{1}{4} = \frac{1}{30} > 0$; so, we may assume that $\overline{m}(x_3) = 3$. Now, if $\overline{m}(x_2) \leq 2$, then $\widetilde{c}(x) \geq -2 + \frac{1}{\ }$ – – $\frac{\overline{10}}{4} + \frac{2}{3} + \frac{1}{2} = \frac{1}{120} > 0$. So, assume that $\overline{m}(x_2) = 3$. Then x_2 is adjacent to two crossing vertices lying in the face $\omega \neq f_1$ that is incident with the edge $x_1'x_2$, which implies that ω is an $\geq$ 4-face. Then we have $\overline{c}(x_2) \geq -1 + 2 \cdot \frac{1}{\ }$ – – – – – $\frac{\overline{10}}{4} + \frac{2}{3} + \frac{2}{3} = \frac{53}{360} > 0$.

Case 2.3.2. Let x_4 be an $\geq$ 6-vertex and x_2, x_3 be 5-vertices; moreover, we can suppose that both f_1, f_3 are 4-faces (otherwise $\widetilde{c}(x) \geq -2 + 1 + \frac{2}{\ }$ –

$\overline{c}(x_2) \geq -1 + 2 \cdot \frac{1}{\ }$ – – $\overline{m}(x_2) \leq 4$; the same holds for x_3 and we get $\widetilde{c}(x) \geq -2 + 2 \cdot \frac{2}{\ }$ – $\frac{-}{4} > 0$.

Case 2.3.2.2. Let x_3 be incident with precisely two $\geq$ 4-faces. Then there exists a 3-face $[x_3x_3'w]$. As $x_4x_2x_3w$ is a 4-cycle in G, w is a big vertex; consequently, $\overline{m}(x_1) = 3$. Now, if the edge x_3w is incident with an $\geq$ 4-face, then $\overline{c}(x_3) \geq -1 + 2 \cdot \frac{2}{\ }$ – — $\overline{m}(x_2) \leq 4$ and $\overline{c}(x_2) \geq -1 + \frac{1}{\ }$ – – – – $\frac{-}{3} + \frac{\frac{1}{\ }}{4} = \frac{7}{360} > 0$. Thus, assume that x_3w is incident only with 3-faces $[x_3x_3'w], [x_3wy]$. If y is a true vertex, then $\overline{m}(x_3) = 2, \overline{c}(x_3) \geq -1 + \frac{1}{\ }$ – – — $\frac{-}{4}$ from x_2) $\widetilde{c}(x) \geq -2 + 2 \cdot \frac{2}{\ }$ – $\frac{-}{2} + \frac{\frac{1}{\ }}{4} = \frac{7}{120} > 0$. So, let y be a crossing-vertex. Consider now the local neighbourhood of vertex x_2; let $\alpha, \beta, \gamma, f_1, f_2$ be faces incident with x_2 in the counter-clockwise order. If α is an $\geq$ 5-face, then $\overline{c}(x_3) \geq -1 + \frac{2}{\ }$ – – — $\overline{c}(x_2) \geq -1 + \frac{2}{\ }$ – $\overline{15}, \overline{m}(x_3) = 3, \overline{m}(x_2) \leq 4$ and $\widetilde{c}(x) \geq -2 + 2 \cdot \frac{2}{\ }$ – $\frac{-}{3} + \frac{\frac{7}{15}}{4} = \frac{5}{36} > 0$; hence, let $\alpha = [x_2x_3yz]$ be a 4-face. Then z is a true vertex, which implies

$\overline{m}(x_2) \leq 3$. If x_2 is incident with at least three $\geq$ 4-faces, the $\overline{c}(x_2) \geq -1+2\cdot$ $\frac{1}{}$ – – – – $\frac{}{3} + \frac{2}{3} = \frac{13}{}$

$\overline{m}(x_2) \leq 2$. In this case, we obtain $\widetilde{c}(x) \geq -2+2\cdot\frac{2}{}$ – $\frac{}{3}+\frac{1}{2} = \frac{1}{180} > 0$.

Case 2.3.3. Let x_3 be an $\geq$ 6-vertex and x_2, x_4 be 5-vertices; again, we can suppose that both f_1, f_3 are 4-faces (otherwise $\widetilde{c}(x) \geq -2+1+\frac{2}{}$ –

$\overline{c}(x_4) \geq -1+2\cdot\frac{1}{}$

– – — $\overline{m}(x_4) \leq 4$ and $\widetilde{c}(x) \geq -2+2\cdot\frac{2}{}$ – $\frac{}{4}+\frac{1}{4} = \frac{1}{24} > 0$; so, suppose that x_4 is incident with precisely two $\geq$ 4-faces. Then there exists a 3-face $[x_3'x_4w]$. As $x_4x_2x_3w$ is a 4-cycle, w must be big. Hence, $\overline{c}(x_4) \geq -1+\frac{1}{}$ – – — $\overline{m}(x_4) \leq 3$ and $\widetilde{c}(x) \geq -2+2\cdot\frac{2}{}$ – $\frac{}{3}+\frac{1}{4} = \frac{7}{72} > 0$.

Case 3. Let x be a big d-vertex. Then $\widetilde{c}(x) \geq d-6-\frac{2}{}$ $\frac{}{5} - 6 \geq 0$ for $d \geq 10$. ■

2.4. Proof of Theorem 3

The proof proceeds in the way described in Subsection 2.1 under the following discharging rules (here, the big vertex is one of degree ≥ 12):

Rule 1. Each face $f \in F^\times$ sends $\frac{c(f)}{m(f)}$ to each incident 4- and 5-vertex, where $m(f)$ is the number of 4- and 5-vertices incident with f. If $m(f) = 0$, no charge is transferred.

Rule 2. Each big vertex v sends $\frac{1}{}$

$\overline{c}(x)$ be the charge of a vertex $x \in V^\times$ after application of Rules 1 and 2.

Rule 3. Each 5-vertex v with $\overline{c}(v) > 0$ sends $\frac{\overline{c}(v)}{\overline{m}(v)}$ to each adjacent 4-vertex; $\overline{m}(v)$ is the number of 4-vertices adjacent to v (if $\overline{m}(v) = 0$, no charge is transferred).

We will check final charges of elements of $M^\times(G)$. Due to the formulation of Rule 1, the final charge of each face of $M^\times(G)$ is nonnegative. Also, if v is a big d-vertex, then it has at most d neighbours of degree 5 in G, thus $\widetilde{c}(v) \geq d - 6 - \frac{d}{2} \geq 0$ since $d \geq 12$. As vertices of degree between 6 and 11 do not change their initial charge, it is enough to check just the final charges of 5- and 4-vertices.

Case 1. Let x be a 5-vertex. Recall that x is incident with at least two $\geq$ 4-faces. Also, in G, x has at least two big neighbours (otherwise a light 4-star is found). Thus, $\overline{c}(x) \geq -1 + 2 \cdot \frac{1}{}$

$\frac{\overline{c}(x)}{\overline{m}(x)}$ from Rule 3 according to type of the neighbourhood of x.

(a) If x is incident only with $\geq$ 4-faces, then $\overline{c}(x) \geq -1 + 5 \cdot \frac{1}{}$

$\frac{}{5} = \frac{1}{}$

$\overline{c}(x) \geq -1 + 4 \cdot \frac{1}{}$

$\overline{c}(x) \geq -1 + 3 \cdot \frac{1}{}$

$\frac{}{3} = \frac{1}{}$

$\overline{c}(x) \geq -1 + 3 \cdot$

$\frac{1}{}$ $\frac{}{4} = \frac{3}{}$

$\overline{c}(x) \geq -1 + 2 \cdot \frac{1}{}$

if both these vertices are 5-vertices, then f_1 contributes $\frac{1}{\,}$

■

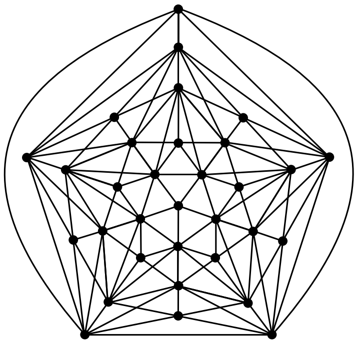

Figure 1

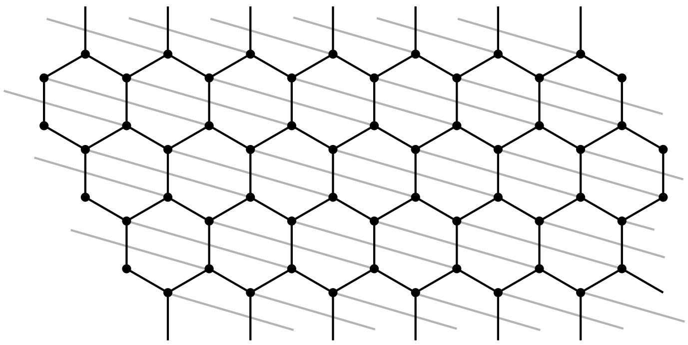

Figure 2

Acknowledgement

The authors wish to thank anonymous referee for valuable comments and suggestions which helped to decrease the upper bound in Theorem 2.

References

[1] O.V. Borodin, *Solution of Kotzig-Grünbaum problems on separation of a cycle in planar graphs* (Russian), Mat. Zametki **46** (1989) 9–12; translation in Math. Notes **46** (1989) 835–837.

total contribution of these

[2] O.V. Borodin, *Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of* 1*-planar graphs*, Metody Diskret. Analiz. **41** (1984) 12–26 (in Russian).

[3] O.V. Borodin, *A new proof of the* 6 *color theorem*, J. Graph Theory **19** (1995) 507–521.

[4] O.V. Borodin, A.V. Kostochka, A. Raspaud and E. Sopena, *Acyclic coloring of* 1*-planar graphs* (Russian), Diskretn. Anal. Issled. Oper. **6** (1999) 20–35; translation in Discrete Appl. Math. **114** (2001) 29–41.

[5] Z.-Z. Chen and M. Kouno, *A linear-time algorithm for* 7*-coloring* 1*-plane graphs*, Algorithmica **43** (2005) 147–177.

[6] R. Diestel, Graph Theory (Springer, New York, 1997).

[7] I. Fabrici and T. Madaras, *The structure of* 1*-planar graphs*, Discrete Math. **307** (2007) 854–865.

[8] S. Jendrol' and T. Madaras, *On light subgraphs in plane graphs of minimum degree five*, Discuss. Math. Graph Theory **16** (1996) 207–217.

[9] S. Jendrol', T. Madaras, R. Soták and Z. Tuza, *On light cycles in plane triangulations*, Discrete Math. **197/198** (1999) 453–467.

[10] V.P. Korzhik, *Minimal non-*1*-planar graphs*, Discrete Math. **308** (2008) 1319–1327.

[11] G. Ringel, *Ein Sechsfarbenproblem auf der Kügel*, Abh. Math. Sem. Univ. Hamburg **29** (1965) 107–117.

[12] H. Schumacher, *Zur Structur* 1*-planar Graphen*, Math. Nachr. **125** (1986) 291–300.

Received 13 November 2007
Revised 28 July 2008
Accepted 28 July 2008