

ON $\mathcal{F}$ -INDEPENDENCE IN GRAPHS

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Abstract

Let $\mathcal{F}$ be a set of graphs and for a graph G let $\alpha_{\mathcal{F}}(G)$ and $\alpha_{\mathcal{F}}^*(G)$ denote the maximum order of an induced subgraph of G which does not contain a graph in $\mathcal{F}$ as a subgraph and which does not contain a graph in $\mathcal{F}$ as an induced subgraph, respectively. Lower bounds on $\alpha_{\mathcal{F}}(G)$ and $\alpha_{\mathcal{F}}^*(G)$ are presented.

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1. INTRODUCTION

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$ and refer to [8] for undefined notation.

A generalization of the well-studied concept of independent sets [12] in graphs was introduced in [4] and [7] (see also [3] and [11]). The following problem is considered there: *For two given graphs F and G , what is the maximum order of an induced subgraph of G that either does not contain F as a subgraph or does not contain F as an induced subgraph?*

The purpose of the present paper is to formalize the independence concept corresponding to this problem and to initiate its study. Therefore, for a graph G and a set $\mathcal{M}$ of graphs we denote by $f(G, \mathcal{M})$ the maximum order $|S|$ of a subgraph $G[S]$ of G induced by $S \subseteq V(G)$ such that $G[S]$ belongs to $\mathcal{M}$. Choosing $\mathcal{M}$ appropriately allows to capture the problem mentioned above. More precisely, let $\mathcal{F}$ be a set of graphs and for a graph G let $\alpha_{\mathcal{F}}(G)$ and $\alpha_{\mathcal{F}}^*(G)$ denote the maximum order of an induced subgraph of G which does not contain a graph in $\mathcal{F}$ as a subgraph and which does not contain a graph in $\mathcal{F}$ as an induced subgraph, respectively. Clearly, if we define $\mathcal{M}_{\mathcal{F}}$ as the set of all graphs which do not contain a graph in $\mathcal{F}$ as a subgraph and $\mathcal{M}_{\mathcal{F}}^*$ as the set of all graphs which do not contain a graph in $\mathcal{F}$ as an induced subgraph, then $\alpha_{\mathcal{F}}(G) = f(G, \mathcal{M}_{\mathcal{F}})$ and $\alpha_{\mathcal{F}}^*(G) = f(G, \mathcal{M}_{\mathcal{F}}^*)$. If $\mathcal{F} = \{F\}$, then we write $\alpha_F(G)$ and $\alpha_F^*(G)$ for short.

Several well-known graph parameters are special cases of these notions as shown in the following result which collects some obvious basic observations.

Proposition 1. *Let G be a graph.*

- (i) $\alpha_{K_2}(G)$ equals the independence number $\alpha(G)$ of G .
- (ii) $\alpha_{\bar{K}_2}(G)$ equals the clique number of G .
- (iii) $\alpha_{P_3}(G)$ equals the dissociation number of G [2].
- (iv) $\alpha_{K_r}(G) = \alpha_{K_r}^*(G)$.
- (v) $\alpha_{\bar{K}_r}(G) = \min\{|V(G)|, r - 1\}$.
- (vi) $\alpha_{\bar{K}_r}^*(G) = \max\{|S| \mid S \subseteq V(G), \alpha(G[S]) \leq r - 1\}$.
- (vii) $\alpha_{\mathcal{F}}^*(G) = \alpha_{\{\bar{F} \mid F \in \mathcal{F}\}}^*(\bar{G})$.

Our next result is a lower bound on $f(G, \mathcal{M})$ provided the set $\mathcal{M}$ has some natural properties.

Theorem 2. *Let $\mathcal{M}$ be a set of graphs and let G be a graph.*

(i) *If $\mathcal{M}$ is closed under taking induced subgraphs, then*

$$f(G, \mathcal{M}) \geq \sum_{S: S \subseteq V(G), G[S] \in \mathcal{M}} \binom{|V(G)|}{|S|}^{-1}.$$

(ii) *If $\mathcal{M}$ is closed under taking induced subgraphs and under forming the union of graphs, then*

$$f(G, \mathcal{M}) \geq \sum_{S: S \subseteq V(G), G[S] \in \mathcal{M}, G[S] \text{ is connected}} \binom{|N_G[S]|}{|S|}^{-1},$$

where $N_G[S] = \cup_{u \in S} N_G[u]$.

Proof. We only prove (ii) and leave the very similar proof of (i) to the reader. We choose a permutation $v_1, v_2, \dots, v_n$ of the vertices of G uniformly at random. Let $S_0 = \emptyset$ and for $1 \leq i \leq n$ let $S_i = S_{i-1} \cup \{v_i\}$ if $G[S_{i-1} \cup \{v_i\}] \in \mathcal{M}$ and $S_i = S_{i-1}$ otherwise. Clearly, $f(G, \mathcal{M}) \geq |S_n|$ and $v_i \in S_n$ if and only if $v_i \in S_i$ and the component H_i of $G[S_i]$ containing v_i belongs to $\mathcal{M}$. Therefore, for a set $S \subseteq V(G)$ with $v_i \in S$ such that $G[S] \in \mathcal{M}$ and $G[S]$ is connected, a lower bound for the probability that $H_i = G[S]$ is the probability that in the chosen permutation the vertices $S \setminus \{v_i\}$ precede v_i while v_i precedes the vertices in $N_G[S] \setminus S$ which equals $\frac{1}{|S|} \binom{|N_G[S]|}{|S|}^{-1}$. Therefore, by linearity of expectation

$$\begin{aligned} f(G, \mathcal{M}) &\geq \mathbf{E}(|S_n|) = \sum_{i=1}^n \mathbf{P}(v_i \in S_n) \\ &\geq \sum_{i=1}^n \sum_{S: v_i \in S \subseteq V(G), G[S] \in \mathcal{M}, G[S] \text{ is connected}} \frac{1}{|S|} \binom{|N_G[S]|}{|S|}^{-1} \\ &= \sum_{S: S \subseteq V(G), G[S] \in \mathcal{M}, G[S] \text{ is connected}} \sum_{i: v_i \in S} \frac{1}{|S|} \binom{|N_G[S]|}{|S|}^{-1} \\ &= \sum_{S: S \subseteq V(G), G[S] \in \mathcal{M}, G[S] \text{ is connected}} \binom{|N_G[S]|}{|S|}^{-1} \end{aligned}$$

and the proof is complete. ■

Corollary 3. *Let G be a graph. Then*

- (i) $\alpha(G) \geq \sum_{u \in V(G)} \frac{1}{1+d_G(v)}$ (Caro [5], Wei [14]).
- (ii) *The dissociation number satisfies*

$$\alpha_{P_3}(G) \geq \sum_{u \in V(G)} \frac{1}{1+d_G(v)} + \sum_{uv \in E(G)} \frac{2}{|N_G[u] \cup N_G[v]| (|N_G[u] \cup N_G[v]| - 1)}.$$

Proof. Note that $\mathcal{M}_{\{K_2\}} = \{\bar{K}_r \mid r \in \mathbb{N}\}$ and $\mathcal{M}_{\{P_3\}} = \mathcal{M}_{\{K_2\}} \cup \{K_2 \cup \bar{K}_r \mid r \in \mathbb{N}\}$. Both statements follow immediately from Theorem 2(ii) and the observation that the only connected graph in $\mathcal{M}_{\{K_2\}}$ is K_1 and the only connected graphs in $\mathcal{M}_{\{P_3\}}$ are K_1 and K_2 . ■

The famous bound due to Caro [5] and Wei [14] from Corollary 3 has yet another generalization in this context.

Theorem 4. *If G is a graph and $r \in \mathbb{N}$, then*

$$\alpha_{K_{r+1}}(G) \geq \sum_{v \in V(G)} \frac{1}{1+d_G(v)-\alpha_{K_r}(G[N_G(v)])}.$$

Proof. We mimic a proof from [1]. For every vertex $v \in V(G)$ let the set $X_v \subseteq N_G(v)$ be such that $|X_v| = d_G(v) - \alpha_{K_r}(G[N_G(v)])$ and $G[N_G(v) \setminus X_v]$ does not contain K_r as a subgraph. Let $v_1, v_2, \dots, v_n$ be a permutation of the vertices of G chosen uniformly at random and let $v_i \in S$ if and only if $X_{v_i} \cap \{v_1, v_2, \dots, v_i\} = \emptyset$, i.e., v_i is the first vertex of $\{v_i\} \cup X_{v_i}$ that appears within the permutation. Clearly, $G[S]$ does not contain K_{r+1} as a subgraph and

$$\alpha_{K_{r+1}}(G) \geq \mathbf{E}(|S|) = \sum_{v \in V(G)} \mathbf{P}(v \in S) = \sum_{v \in V(G)} \frac{1}{1+d_G(v)-\alpha_{K_r}(G[N_G(v)])}. \quad \blacksquare$$

The next result relies on methods proposed in [10].

Theorem 5. *If G is a graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and $r \in \mathbb{N}$, then*

$$\alpha_{K_{1,r}}(G) = \max \sum_{v_i \in V(G)} p_i \sum_{Y: Y \subseteq N_G(v_i), |Y| < r} \left(\prod_{v_j \in Y} p_j \prod_{v_k \in N_G(v_i) \setminus Y} (1 - p_j) \right),$$

where the maximum is taken over all $(p_1, p_2, \dots, p_n) \in [0, 1]^n$.

Proof. Let $p_i \in [0, 1]$ for $1 \leq i \leq n$. We consider a random subset X of $V(G)$ formed by choosing every vertex v_i independently with probability p_i . If $S = \{v \in X \mid d_{G[X]}(v) < r\}$, then

$$\alpha_{K_{1,r}}(G) \geq \mathbf{E}(S) = \sum_{v_i \in V(G)} p_i \sum_{Y: Y \subseteq N_G(v_i), |Y| < r} \left(\prod_{v_j \in Y} p_j \prod_{v_k \in N_G(v_i) \setminus Y} (1 - p_j) \right).$$

Conversely, if $S \subseteq V$ is such that $\alpha_{K_{1,r}}(G) = |S|$ and $G[S]$ has maximum degree less than r , then setting $p_i^* = 1$ for all $v_i \in S$ and $p_i^* = 0$ for all $v_i \notin S$ yields

$$\alpha_{K_{1,r}}(G) = \mathbf{E}(S) = \sum_{v_i \in V(G)} p_i^* \sum_{Y: Y \subseteq N_G(v_i), |Y| < r} \left(\prod_{v_j \in Y} p_j^* \prod_{v_k \in N_G(v_i) \setminus Y} (1 - p_j^*) \right)$$

which completes the proof. $\blacksquare$

It is trivial that for several specific choices of $\mathcal{M}$ and $\mathcal{F}$ the decision problems associated with $f(G, \mathcal{M})$, $\alpha_{\mathcal{F}}(G)$ and $\alpha_{\mathcal{F}}^*(G)$ are NP-complete. In view of Mihók's original problem, we consider the case that $\mathcal{F}$ consists of just one graph in more detail.

Theorem 6. *If F is a graph containing at least one edge, then the following problems are NP-complete.*

- (i) *For a given graph G and $k \in \mathbb{N}$, decide whether $\alpha_F(G) \geq k$.*
- (ii) *For a given graph G and $k \in \mathbb{N}$, decide whether $\alpha_F^*(G) \geq k$.*

Proof. Let uv be an arbitrary edge of F . For a graph G let the graph G' arise as follows: For every edge xy of G add a copy F_{xy} of F and identify the copy of the edge uv in F_{xy} with xy (in any orientation).

It is obvious that for every set $T \subseteq V(G')$ of minimum cardinality such that $G'[V(G') \setminus T]$ does not contain F as a subgraph (or induced subgraph), T must intersect every copy F_{xy} of F in G' . Since deleting either x or y from F_{xy} clearly deletes this copy of F , we can assume that $T \subseteq V(G)$ and that $T \cap \{x, y\} \neq \emptyset$ for all $xy \in E(G)$. Hence T is exactly a vertex cover of G . This implies $\alpha(G) = \alpha_F(G') = \alpha_F^*(G')$ and the desired statement follows from the NP-completeness of the corresponding decision problem for the independence number [9]. $\blacksquare$

Note that in view Proposition 1(vii), the decision problem “ $\alpha_{\mathcal{F}}^*(G) \geq k$?” remains NP-complete even if F is edgeless.

Tuza [13] observed the following nice relation between the independence number and the domination number $\gamma(G)$ of a graph G [10]:

$$\alpha(G) = \max\{\gamma(H) \mid H \text{ is an induced subgraph of } G\}.$$

We close with a generalization of this equality. For a set $\mathcal{F}$ of graphs and a graph G let $\gamma_{\mathcal{F}}(G)$ ($\gamma_{\mathcal{F}}^*(G)$) denote the minimum cardinality $|D|$ of a set $D \subseteq V(G)$ such that for every vertex $u \in V(G) \setminus D$ there is a graph $F \in \mathcal{F}$ and a set $D' \subseteq D$ with $|D'| = |V(F)| - 1$ such that $G[D' \cup \{u\}]$ contains a graph in $\mathcal{F}$ as an (induced) subgraph (see also [4]).

Theorem 7. *If $\mathcal{F}$ is a set of graphs and G is a graph, then*

$$\begin{aligned}\alpha_{\mathcal{F}}(G) &= \max\{\gamma_{\mathcal{F}}(H) \mid H \text{ is an induced subgraph of } G\}, \\ \alpha_{\mathcal{F}}^*(G) &= \max\{\gamma_{\mathcal{F}}^*(H) \mid H \text{ is an induced subgraph of } G\}.\end{aligned}$$

Proof. We only prove the first equality and leave the very similar proof of the second equality to the reader.

If $S \subseteq V(G)$ such that $|S| = \alpha_{\mathcal{F}}(G)$ and $G[S]$ does not contain a graph in $\mathcal{F}$ as a subgraph, then $\gamma_{\mathcal{F}}(G[S]) = |S| \geq \alpha_{\mathcal{F}}(G)$.

Conversely, if $G[S]$ is an induced subgraph of G for which $\gamma_{\mathcal{F}}(G[S])$ is maximum, then let $S' \subseteq S$ be of maximum cardinality such that $G[S']$ does not contain a graph in $\mathcal{F}$ as a subgraph. We obtain $\gamma_{\mathcal{F}}(G[S]) \leq |S'| = \alpha_{\mathcal{F}}(G[S]) \leq \alpha_{\mathcal{F}}(G)$ and the proof is complete. ■

REFERENCES

- [1] N. Alon and J.H. Spencer, *The probabilistic method* (2nd ed.), (Wiley, 2000).
- [2] R. Boliac, C. Cameron and V. Lozin, *On computing the dissociation number and the induced matching number of bipartite graphs*, *Ars Combin.* **72** (2004) 241–253.
- [3] M. Borowiecki, I. Broere, M. Frick, P. Mihok and G. Semanisin, *Survey of hereditary properties of graphs*, *Discuss. Math. Graph Theory* **17** (1997) 5–50.
- [4] M. Borowiecki, D. Michalak and E. Sidorowicz, *Generalized domination, independence and irredundance in graphs*, *Discuss. Math. Graph Theory* **17** (1997) 147–153.

- [5] Y. Caro, New results on the independence number, Technical Report (Tel-Aviv University, 1979).
- [6] Y. Caro and Y. Roditty, *On the vertex-independence number and star decomposition of graphs*, Ars Combin. **20** (1985) 167–180.
- [7] G. Chartrand, D. Geller and S. Hedetniemi, *Graphs with forbidden subgraphs*, J. Combin. Theory **10** (1971) 12–41.
- [8] G. Chartrand and L. Lesniak, *Graphs and Digraphs* (Chapman & Hall, 2005).
- [9] M.R. Garey and D.S. Johnson, *Computers and Intractability* (W.H. Freeman and Company, San Francisco, 1979).
- [10] J. Harant, A. Pruchnewski and M. Voigt, *On Dominating Sets and Independent Sets of Graphs*, Combinatorics, Probability and Computing **8** (1999) 547–553.
- [11] S. Hedetniemi, *On hereditary properties of graphs*, J. Combin. Theory (B) **14** (1973) 349–354.
- [12] V. Vadim and D. de Werra, *Special issue on stability in graphs and related topics*, Discrete Appl. Math. **132** (2003) 1–2.
- [13] Zs. Tuza, Lecture at Conference of Hereditarnia (Zakopane, Poland, September 2006).
- [14] V.K. Wei, A lower bound on the stability number of a simple graph, Bell Laboratories Technical Memorandum 81-11217-9 (Murray Hill, NJ, 1981).

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