

## MONOCHROMATIC PATHS AND MONOCHROMATIC SETS OF ARCS IN BIPARTITE TOURNAMENTS

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### Abstract

We call the digraph  $D$  an  $m$ -coloured digraph if the arcs of  $D$  are coloured with  $m$  colours and all of them are used. A directed path is called monochromatic if all of its arcs are coloured alike. A set  $N$  of vertices of  $D$  is called a kernel by monochromatic paths if for every pair of vertices there is no monochromatic path between them and for every vertex  $v$  in  $V(D) \setminus N$  there is a monochromatic path from  $v$  to some vertex in  $N$ . We denote by  $A^+(u)$  the set of arcs of  $D$  that have  $u$  as the initial endpoint.

In this paper we introduce the concept of semikernel modulo  $i$  by monochromatic paths of an  $m$ -coloured digraph. This concept allow us to find sufficient conditions for the existence of a kernel by monochromatic paths in an  $m$ -coloured digraph. In particular we deal with bipartite tournaments such that  $A^+(z)$  is monochromatic for each  $z \in V(D)$ .

**Keywords:**  $m$ -coloured bipartite tournaments, kernel by monochromatic paths, semikernel of  $D$  modulo  $i$  by monochromatic paths.

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## 1. INTRODUCTION

For general concepts we refer the reader to [1]. A *kernel*  $N$  of a digraph  $D$  is an independent set of vertices of  $D$  such that for every  $w \in V(D) \setminus N$  there exists an arc from  $w$  to  $N$  (i.e.,  $N$  is dominating). So a kernel is a dominating independent set of vertices. For a deep wide study of domination the reader can see [15] or [16]. An  $m$ -coloured digraph  $D$  is called *quasi-monochromatic* if with at the most one exception all of its arcs are coloured alike (i.e., all arcs are coloured alike or all arcs but one are coloured alike). The problem of the existence of a kernel in a given digraph has been studied by several authors in particular Richardson [23, 24], Duchet and Meyniel [5], Duchet [3, 4], Galeana-Sánchez and V. Neumann-Lara [10, 11]. A survey on kernels can be found in [2] and in [6]. The concept of kernel by monochromatic paths is a generalization of the concept of kernel and it was introduced by Galeana-Sánchez [7]. In that work she obtained some sufficient conditions for an  $m$ -coloured tournament  $T$  has a kernel by monochromatic paths. More information about  $m$ -coloured digraphs can be found in [8, 9, 27, 28]. Another interesting generalization is the concept of  $(k, l)$ -kernel introduced by M. Kwaśnik [19]. Other results about  $(k, l)$ -kernels have been developed by M. Kucharska [17], M. Kucharska and M. Kwaśnik [18], M. Kwaśnik [20], and A. Włoch and I. Włoch [26].

In [25] Sands *et al.* have proved that any 2-coloured digraph has a kernel by monochromatic paths. In particular they proved that any 2-coloured tournament has a kernel by monochromatic paths. They also raised the following problem: Let  $T$  be a 3-coloured tournament such that every directed cycle of length 3 is quasi-monochromatic; must  $D$  have a kernel by monochromatic paths? In [21] S. Minggang proved that if in the problem we ask that every transitive tournament of order 3 be quasi-monochromatic, the answer will be yes. In [7] it was proved that if  $T$  is an  $m$ -coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic then  $T$  has a kernel by monochromatic paths. The known sufficient conditions for the existence of a kernel by monochromatic paths in  $m$ -coloured ( $m \geq 3$ ) tournaments, (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of small subdigraphs as directed cycles of length at most 4 or transitive tournaments of order 3. A related problem to the existence of kernels by monochromatic paths in tournaments is to find absorbing sets of minimum size, this problem has been studied by G. Hahn, P. Ille and R. Woodrow [14]. In [12] were proved that: Let  $D$  be an

$m$ -coloured bipartite tournament. If every directed cycle of length 4 in  $D$  is monochromatic, then  $D$  has a kernel by monochromatic paths. Another kind of condition is about the number of colours assigned to  $A^+(u)$  for each  $u$ , [13]. ( $A^+(u)$  denotes the sets of arcs with  $u$  as an endpoint).

A digraph  $T$  is called *bipartite tournament* if exists a partition  $\{V_1, V_2\}$  of  $V(T)$  such that every arc of  $A(T)$  joins a vertex of  $V_1$  to a vertex of  $V_2$  and between every vertex of  $V_1$  and every vertex of  $V_2$  exists one and only one arc.

We denoted by  $T_4$  the digraph such that  $V(T_4) = \{u, v, w, x\}$  and  $A(T_4) = \{(u, v), (v, w), (w, x), (u, x)\}$ . Let  $C_3$  be the 3-coloured directed cycle of length 3. We say that a digraph  $D$  is a  $(1, 1, 2)$  subdivision of  $C_3$  if  $V(D) = \{v_1, v_2, v_3, v_4\}$  and  $A(D) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$  such that  $(v_1, v_2)$  and  $(v_2, v_3)$  are coloured  $a$ ,  $(v_3, v_4)$  is coloured  $b$  and  $(v_4, v_1)$  is coloured  $c$ , with  $a \neq b$ ,  $a \neq c$ ,  $b \neq c$ , see Figure 1.

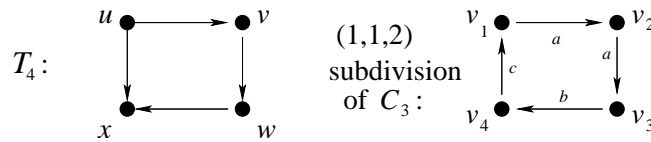


Figure 1

In this paper is proved that if  $T$  is an  $m$ -coloured bipartite tournament with  $m \geq 3$  such that for every vertex  $u$  of  $T$ ,  $A^+(u)$  is monochromatic (all of its elements have the same colour) and  $T$  contains no  $(1, 1, 2)$  subdivision of  $C_3$  and every  $T_4$  contained in  $T$  is at most 2-coloured. Then  $T$  has a kernel by monochromatic paths.

We will need the following results:

**Lemma 1.1.** *Every  $uv$ -monochromatic walk in a digraph contains a  $uv$ -monochromatic path.*

**Lemma 1.2** (H. Galeana-Sánchez and R. Rojas-Monroy [12]). *Let  $D$  be a bipartite tournament, if  $C = (u_0, u_1, \dots, u_n)$  is a directed walk in  $D$  then for  $i \in \{0, 1, \dots, n\}$  and  $j \in \{0, 1, \dots, n\}$ ,  $(u_i, u_j) \in A(D)$  or  $(u_j, u_i) \in A(D)$  if and only if  $j - i \equiv 1 \pmod{2}$ .*

**Theorem 1.3** (Sands, Sauer and Woodrow [25]). *Let  $D$  be a 2-coloured digraph, then  $D$  has a kernel by monochromatic paths.*

Let  $\alpha = (x_0, x_1, \dots, x_n)$  be a path, we will denote by  $\ell(\alpha)$  the length of  $\alpha$ , and we will denote by  $(x_i, \alpha, x_j)$  the path  $(x_i, x_{i+1}, \dots, x_j)$ .

## 2. SEMIKERNELS MODULO $i$ BY MONOCHROMATIC PATHS

An important concept in the study of the existence of kernels in digraphs has been semikernel that was given by V. Neumann-Lara in 1971 [22]. In this section we defined semikernel modulo  $i$  by monochromatic paths for an  $m$ -coloured digraph. This concept is a generalization of semikernel and will be useful to prove Theorem 3.1, our main result.

**Definition 2.1.** Let  $D$  be an  $m$ -coloured digraph, let  $1, 2, \dots, m$  be the distinct colours, and let  $i \in \{1, \dots, m\}$  be anyone but fixed. A set  $S \subseteq V(D)$  is a semikernel of  $D$  modulo  $i$  by monochromatic paths if the following conditions are fulfilled:

1.  $S$  is an independent set by monochromatic paths, i.e., for every pair of vertices there is no monochromatic path between them.
2. For each  $z \in V(D) \setminus S$  such that there exists a  $Sz$ -monochromatic path of different colour of  $i$ , then there exists a  $zS$ -monochromatic path in  $D$ . ( $Sz$ -monochromatic path denotes a  $sz$ -monochromatic path for some  $s \in S$ , similarly  $zS$ -monochromatic paths).

**Theorem 2.2.** Let  $D$  be an  $m$ -coloured digraph such that for every  $u \in V(D)$ ,  $A^+(u)$  is monochromatic. Then  $D$  has a non empty semikernel modulo  $i$  by monochromatic paths for every  $i \in \{1, 2, \dots, m\}$ .

**Proof.** Let  $i \in \{1, \dots, m\}$  and  $z_0 \in V(D)$  such that  $A^+(u)$  is coloured  $i$ , clearly  $\{z_0\}$  is a semikernel modulo  $i$  by monochromatic paths. ■

Let  $i \in \{1, \dots, m\}$  and  $\Gamma_i = \{S \mid S \text{ is a non empty semikernel mod } i \text{ by monochromatic paths of } D\}$ . Notice that  $\Gamma_i \neq \emptyset$  by Theorem 2.2.

We define a partial order in  $\Gamma_i$  as follows:

**Definition 2.3.** Let  $\{S_1, S_2\} \subseteq \Gamma_i$ . We called  $S_1 \leq S_2$  if for each  $x_1 \in S_1$  exists  $x_2 \in S_2$  such that one the following properties is satisfies:

1.  $x_1 = x_2$ ,
2. There exists a  $x_1x_2$ -monochromatic path coloured  $i$  and there exists no  $x_2x_1$ -monochromatic path coloured  $i$ .

We will denote  $x_1 \xrightarrow{i} x_2$  if there exists an  $x_1x_2$ -monochromatic path coloured  $i$ , and  $x_2 \not\xrightarrow{i} x_1$  if there is no monochromatic path coloured  $i$ .

**Theorem 2.4.**  $\Gamma_i$  is a partially ordered set by  $\leq$ .

**Proof.** 1.  $S \leq S$  for every  $S \in \Gamma_i$  is immediate.

2.  $S_1 = S_2$  whenever  $\{S_1, S_2\} \subseteq \Gamma_i$ ,  $S_1 \leq S_2$ ,  $S_2 \leq S_1$ .

We will prove  $S_1 \subseteq S_2$ . Let  $x_1 \in S_1$ , we have  $S_1 \leq S_2$  then there exists  $x_2 \in S_2$  such that  $x_1 = x_2$  or  $x_1 \xrightarrow{i} x_2$  and  $x_2 \not\xrightarrow{i} x_1$ . We analyze two cases.

- (a) If  $x_1 = x_2$  then  $x_1 \in S_2$ .
- (b) If  $x_1 \xrightarrow{i} x_2$  and  $x_2 \not\xrightarrow{i} x_1$ . Since  $x_2 \in S_2$ ,  $S_2 \leq S_1$  there exists  $u_1 \in S_1$  such that  $x_2 = u_1$  or  $x_2 \xrightarrow{i} u_1$  and  $u_1 \not\xrightarrow{i} x_2$ . Then we have the following cases
  - (i) Let  $x_2 = u_1$  then  $x_1 \xrightarrow{i} x_2 = u_1$ ,  $\{x_1, u_1\} \subseteq S_1$ , contradicting that  $S_1$  is independent by monochromatic paths.
  - (ii) If  $x_2 \xrightarrow{i} u_1$  and  $u_1 \not\xrightarrow{i} x_2$  then  $x_1 \xrightarrow{i} x_2 \xrightarrow{i} u_1$ , from Lemma 1.1  $x_1 \xrightarrow{i} u_1$ , contradicting that  $S_1$  is independent by monochromatic paths.

Thus (b) is not possible.

The proof of  $S_2 \subseteq S_1$  is similar.

Thus  $S_1 = S_2$ .

3. If  $S_1 \leq S_2$ ,  $S_2 \leq S_3$  we prove that  $S_1 \leq S_3$ ,  $\{S_1, S_2, S_3\} \subseteq \Gamma_i$ .

Let  $x_1 \in S_1$ . Since  $S_1 \leq S_2$ , then there exists  $x_2 \in S_2$  such that  $x_1 = x_2$  or  $x_1 \xrightarrow{i} x_2$  and  $x_2 \not\xrightarrow{i} x_1$ . We consider two possible cases.

- (a) If  $x_1 = x_2$ . Since  $S_2 \leq S_3$  then there exists  $x_3 \in S_3$  such that  $x_2 = x_3$  or  $x_2 \xrightarrow{i} x_3$  and  $x_3 \not\xrightarrow{i} x_2$ . Then we have:
  - (i) If  $x_2 = x_3$ , then  $x_1 = x_2 = x_3$  with  $x_3 \in S_3$ .
  - (ii) If  $x_2 \xrightarrow{i} x_3$  and  $x_3 \not\xrightarrow{i} x_2$ . Since  $x_1 = x_2$ , we have that  $x_2 = x_1 \xrightarrow{i} x_3$  and  $x_3 \not\xrightarrow{i} x_2 = x_1$ .
- (b) If  $x_1 \xrightarrow{i} x_2$  and  $x_2 \not\xrightarrow{i} x_1$ . We have the next cases:
  - (i) If  $x_2 = x_3$  then  $x_1 \xrightarrow{i} x_2 = x_3$  and  $x_3 \not\xrightarrow{i} x_1$ .

- (ii) If  $x_2 \xrightarrow{i} x_3$  and  $x_3 \not\xrightarrow{i} x_2$ , then  $x_1 \xrightarrow{i} x_2$  and  $x_2 \xrightarrow{i} x_3$  then from Lemma 1.1  $x_1 \xrightarrow{i} x_3$ .

We will prove  $x_3 \not\xrightarrow{i} x_1$ . By contradiction, we assume there exists  $x_3 \xrightarrow{i} x_1$  and we have  $x_1 \xrightarrow{i} x_2$ , then from Lemma 1.1 there exists  $x_3 \xrightarrow{i} x_2$ , contradicting that  $x_3 \not\xrightarrow{i} x_2$ .

Thus  $S_1 \leq S_3$ . ■

Since  $D$  is a finite digraph, we claim that  $(\Gamma_i, \leq)$  has maximal elements.

### 3. THE MAIN RESULT

Now we show a sufficient condition for an  $m$ -coloured bipartite tournament has a kernel by monochromatic paths.

**Theorem 3.1.** *Let  $T$  be an  $m$ -coloured bipartite tournament with  $m \geq 3$ , such that  $A^+(u)$  is monochromatic for every  $u \in V(T)$ . If every  $T_4$  contained in  $T$  is at most 2-coloured and  $T$  has not  $(1, 1, 2)$  subdivision of  $C_3$ , then  $T$  has a kernel by monochromatic paths.*

**Proof.** By induction on  $|V(T)|$ .

If  $|V(T)| \leq 3$ ,  $T$  has at most 2 arcs and  $T$  can not be 3-coloured, so by Theorem 1.3  $T$  has a kernel by monochromatic paths.

If  $|V(T)| = 4$ , suppose that  $V(T) = \{u, v, w, x\}$  w.l.o.g. suppose that  $(X, Y)$  is a bipartition of  $T$  and we have the next cases:

*Case I.*  $X = \{u\}$ ,  $Y = \{v, w, x\}$ .

I.a.  $|A^+(u)| \geq 2$ , it contradicts the 3-coloring.

I.b.  $|A^+(u)| \leq 1$ .

i.  $|A^+(u)| = 0$ , in this case  $N = \{u\}$  is a kernel by monochromatic paths.

ii.  $|A^+(u)| = 1$ , in this case  $N = \{v, w, x\}$  is a kernel by monochromatic paths of  $T$ .

*Case II.*  $X = \{u, v\}$ ,  $Y = \{w, x\}$ .

II.a.  $|A^+(u)| = 2 = |A^+(v)|$ , this is not possible,  $T$  is 3-coloured.

- II.b.  $|A^+(u)| = 2$ ,  $|A^+(v)| = 1$ , this is not possible,  $T$  contains no 3-coloured  $T_4$ .
- II.c.  $|A^+(u)| = 2$ ,  $|A^+(v)| = 0$ . In this case,  $N = \{u, v\}$  is a kernel by monochromatic paths (we are using 3 colours).
- II.d.  $|A^+(u)| = 1 = |A^+(v)|$ .
  - i.  $|A^+(x)| = 2$ ,  $|A^+(w)| = 0$ , analogous at the case (II.c).
  - ii.  $|A^+(x)| = 1 = |A^+(w)|$  is not possible,  $T$  contains no  $(1,1,2)$  subdivision of  $C_3$ , and we suppose that  $m \geq 3$  then there is no monochromatic paths of length greater or equal than 1. Thus  $T$  has a kernel by monochromatic paths, let  $N = \{u, v\}$ .

Now, let  $T$  be an  $m$ -coloured bipartite tournament with  $|V(T)| = n$ ,  $n \geq 5$ .

By contradiction. We assume that  $T$  has no kernel by monochromatic paths. From Theorem 2.2 we know that  $T$  has a non empty semikernel modulo  $i$  by monochromatic paths. Let  $S$  be a maximal element of  $(\Gamma_i, \leq)$ . Then  $S$  is not a kernel by monochromatic paths of  $T$ . Let  $X_0 = \{x \in V(T) \setminus S \mid \text{there is no } xS\text{-monochromatic path}\}$ . Since  $X_0 \neq \emptyset$  we have that  $T[X_0]$  is a proper induced subdigraph of  $T$ . And by inductive hypothesis  $T[X_0]$  has a kernel by monochromatic paths we call  $N_0$ . Let  $B = \{x \in S \mid \text{there is no } xN_0\text{-monochromatic path coloured } i \text{ in } T\}$ . Then we have the following assertions:

1.  $B \cup N_0 \in \Gamma_i$ .

1.1. First we will prove  $B \cup N_0$  is independent by monochromatic paths. We have that:

1.1.1.  $B$  is independent by monochromatic paths,  $B \subseteq S$ .

1.1.2.  $N_0$  is independent by monochromatic paths in  $T$ . We observe that  $N_0$  is independent by monochromatic paths in  $T[X_0]$ . We proceed by contradiction. We assume that there exists  $\{x, y\} \subseteq N_0$  such that there exists a  $xy$ -monochromatic path in  $T$ , we call it  $\alpha$ . Then we have that: (i)  $V(\alpha) \cap (V(T) \setminus X_0) \neq \emptyset$  (otherwise  $\alpha \subseteq D[X_0]$  contradicting that  $S$  is independent by monochromatic paths in  $D[X_0]$ ), and (ii)  $V(\alpha) \cap S = \emptyset$  (otherwise we have an  $X_0S$ -monochromatic path, contradiction to the definition of  $X_0$ ).

Since  $V(\alpha) \cap (V(T) \setminus X_0) \neq \emptyset$ , then exists  $z \in V(\alpha) \cap (V(T) \setminus X_0)$  and we can suppose that  $z \notin S$  for (ii). Thus exists a  $zS$ -monochromatic path we call it  $\gamma$ . Since  $z \in V(\alpha) \cap V(\gamma)$  and  $A^+(z)$  is monochromatic we have that  $\alpha$  and  $\gamma$  have the same colour. Then  $(x, \alpha, z) \cup \gamma$  is a walk and by Theorem

1.1 it contains a  $xS$ -monochromatic path, contradicting  $x \in N_0 \subseteq X_0$ . Thus  $N_0$  is independent by monochromatic paths in  $T$ .

1.1.3. There is no  $BN_0$ -monochromatic paths. We assume that there exists  $v \in B$  and  $u \in N_0$  such that there is a  $vu$ -monochromatic path, we call it  $\alpha$ , and  $\alpha$  is coloured distinct of  $i$  because  $v \in B$ . Since  $S \in \Gamma_i$ ,  $S$  is a non empty semikernel by monochromatic paths modulo  $i$ , then by definition of  $S$  exists a  $uS$ -monochromatic path in  $T$ , contradicting  $u \in N_0 \subseteq X_0$ .

1.1.4. There is no  $N_0B$ -monochromatic paths. It is immediately from  $B \subseteq S$ ,  $N_0 \subseteq X_0$  and definition of  $X_0$ .

Thus  $B \cup N_0$  is independent by monochromatic paths.

1.2. Let  $z \in V(T) \setminus (B \cup N_0)$  such that there exists a  $wz$ -monochromatic path coloured distinct of  $i$ , with  $w \in B \cup N_0$ . We will prove that there exists a  $z(B \cup N_0)$ -monochromatic path. Since  $N_0$  is a kernel by monochromatic paths of  $T[X_0]$ , then  $N_0 \neq \emptyset$ , so  $B \cup N_0 \neq \emptyset$ .

Now we proceed by contradiction. We assume that there is no  $z(B \cup N_0)$ -monochromatic path in  $T$ . Let  $\alpha$  be a  $wz$ -monochromatic path coloured  $j$ , with  $j \neq i$ , we may assume  $j = 2$ ,  $\alpha = (w = x_0, x_1, \dots, x_{n-1}, z)$ . We have the following cases:

*Case 1.* Suppose  $w \in B$ . Since  $w \in B \subseteq S$  and  $S \in \Gamma_i$  then there exists a  $zs$ -monochromatic path in  $T$  with  $s \in S$ , we call it  $\alpha'$ , let  $\alpha' = (z, y_1, y_2, \dots, y_{m-1}, s)$ . Thus  $s \in S \setminus B$ ,  $\{w, s\} \subseteq S$  and  $S$  is independent by monochromatic paths, then  $\alpha$  and  $\alpha'$  have distinct colour, we may assume that  $\alpha'$  is coloured  $b \neq 2$ .

Since  $s \in S \setminus B$  and the definition of  $B$  implies that for some  $u \in N_0$  there exists a  $su$ -monochromatic path coloured  $i$ , we call it  $\alpha''$  let  $\alpha'' = (s, z_1, z_2, \dots, z_\ell = u)$ .

If  $b = i$  then  $\alpha' \cup \alpha''$  is a  $zu$ -walk, by Theorem 1.1 it contains a  $zu$ -monochromatic path with  $u \in N_0$ . Thus there exists a  $z(B \cup N_0)$ -monochromatic path, a contradiction.

We may assume that  $b \neq i$ , we remember that  $b \neq i$ ,  $b \neq 2$ , and  $i \neq 2$  we may assume that  $b = 3$ .

*Case 1.1.*  $\ell(\alpha') = 1$ .

Then  $(x_{n-1}, z, s, z_1)$  is a path of length 3, Lemma 1.2 implies  $(x_{n-1}, z_1) \in A(T)$  or  $(z_1, x_{n-1}) \in A(T)$ . If  $(x_{n-1}, z_1) \in A(T)$ , then  $\{x_{n-1}, z, s, z_1\}$  induces a 3-coloured  $T_4$ , contradicting the hypothesis. So we will assume  $(z_1, x_{n-1}) \in A(T)$ , now we consider two possible cases:



*Case 1.1.1.*  $\ell(\alpha'') \geq 2$ .

In this case  $(z_1, x_{n-1})$  is coloured  $i$  thus  $\{x_{n-1}, z, s, z_1\}$  induces a  $(1, 1, 2)$  subdivision of  $C_3$ , a contradiction.

*Case 1.1.2.*  $\ell(\alpha'') = 1$ .

In this case  $z_1 = u$ . If  $(u, x_{n-1})$  is coloured  $i$  or 2, then  $\{u, x_{n-1}, z, s\}$  is a  $(1, 1, 2)$  subdivision of  $C_3$ , a contradiction. We may assume  $(u, x_{n-1})$  is coloured  $c$ , with  $c \neq 2$  and  $c \neq i$ . And we analyze the following cases:

*Case 1.1.2.1.*  $\ell(\alpha) \geq 2$ .

We have  $(x_{n-2}, x_{n-1}, z, s)$  is a path of length 3 and Lemma 1.2 implies that  $x_{n-2}$  and  $s$  are adjacent. If  $(x_{n-2}, s) \in A(T)$  then it is coloured 2, so  $\{x_{n-2}, s, u, x_{n-1}\}$  induces a 3-coloured  $T_4$ , a contradiction. Then we may assume  $(s, x_{n-2}) \in A(D)$  then it is coloured  $i$  and  $\{s, x_{n-2}, x_{n-1}, z\}$  induces a  $(1, 1, 2)$  subdivision of  $C_3$  a contradiction.

*Case 1.1.2.2.*  $\ell(\alpha) = 1$ .

In this case  $(w, z, s, u)$  is a  $wu$ -path of length 3, by Lemma 1.2  $u$  and  $w$  are adjacent. The definition of  $X_0$  implies  $(u, w) \notin A(D)$ , so  $(w, u) \in A(D)$ . Then  $\{w, z, s, u\}$  induces a 3-coloured  $T_4$ , a contradiction.

*Case 1.2.*  $\ell(\alpha') > 1$ .

We have that  $S$  is an independent set and  $\{w, s\} \subseteq S$  then  $w$  and  $s$  are not adjacent in  $T$ . Since  $T$  is a bipartite tournament then  $w$  and  $y_{m-1}$  are adjacent. If  $(y_{m-1}, w) \in A(T)$ , then it is coloured 3, so  $(z, \alpha', y_{m-1}) \cup (y_{m-1}, w)$  is a  $zw$ -monochromatic path, a contradiction. We may assume that  $(w, y_{m-1}) \in A(T)$  and we have that it is coloured 2. Now,  $(w, y_{m-1}, s, z_1)$  is a path of length 3, by Lemma 1.2 we have that  $w$  and  $z_1$  are adjacent. If  $(w, z_1) \in A(T)$  then it is coloured 2, then  $\{w, y_{m-1}, s, z_1\}$  induces a 3-coloured  $T_4$ , a contradiction. We may assume that  $(z_1, w) \in A(T)$ .

If  $\ell(\alpha'') \geq 2$  then  $(z_1, w) \in A(T)$  is coloured  $i$ . Hence  $\{w, y_{m-1}, s, z_1, w\}$  induces a  $(1, 1, 2)$  subdivision of  $C_3$ , a contradiction. If  $\ell(\alpha'') = 1$ , then  $z_1 = u$  and  $(u, w) \in A(T)$ , and we have a contradiction with the definition of  $X_0$ . If  $(w, u) \in A(T)$  is coloured 2, then  $\{w, y_{m-1}, s, u\}$  induces a 3-coloured  $T_4$ , a contradiction.

We conclude that case 1 is not possible.

*Case 2.* Suppose that  $w \in N_0$ .

If  $z \in X_0$ , since  $N_0$  is kernel by monochromatic paths of  $T[X_0]$ , then there

exists a  $zN_0$ -monochromatic path, a contradiction. Assume that  $z \notin X_0$ . Then there is a  $zs$ -monochromatic path  $\alpha'$ , for some  $s \in S$ . Let  $\alpha' = (z, y_1, y_2, \dots, y_{m-1}, s)$ . If  $s \in B$  we have done. Assume that  $s \notin B$ . The definition of  $B$  implies that there is a  $sx$ -path  $\alpha''$  coloured  $i$ , for some  $x \in N_0$ . Let  $\alpha'' = (s, z_1, z_2, \dots, z_\ell = x)$ . If  $\alpha'$  is coloured 2, then  $\alpha \cup \alpha'$  is a  $ws$ -monochromatic walk and Theorem 1.1 implies that it contains a  $ws$ -monochromatic path, but this is a contradiction with the definition of  $X_0$ . We may assume that  $\alpha'$  is not coloured 2. If  $\alpha'$  is coloured  $i$  then  $\alpha' \cup \alpha''$  contains a  $zx$ -monochromatic path with  $x \in N_0 \subset B \cup N_0$ , a contradiction. Assume that  $\alpha'$  is coloured 3, with  $3 \neq 2$  and  $3 \neq i$ . Since  $T$  is a bipartite tournament then  $w$  and  $s$  are adjacent or  $y_{m-1}$  and  $w$  are adjacent, thus we have the following cases.

*Case 2.1.* Suppose that  $w$  and  $s$  are adjacent. Since  $w \in N_0 \subseteq X_0$ , then the definition of  $X_0$  implies that  $(s, w) \in A(T)$ . Hence  $(s, w)$  is coloured  $i$ . Now,  $(y_{m-1}, s, w, x_1)$  is a path of length 3, so Lemma 1.2, implies that  $y_{m-1}$  and  $x_1$  are adjacent. If  $(y_{m-1}, x_1) \in A(T)$  then it is coloured 3 and  $\{y_{m-1}, s, w, x_1\}$  induces a 3-coloured  $T_4$ , a contradiction. We may assume that  $(x_1, y_{m-1}) \in A(T)$ . If  $\ell(\alpha) \geq 2$  then  $(x_1, y_{m-1})$  is coloured 2, so  $\{y_{m-1}, s, w, x_1\}$  induces a  $(1, 1, 2)$  subdivision of  $C_3$ , a contradiction. When  $\ell(\alpha) = 1$  then  $z = x_1$  and  $(x_1, y_{m-1})$  is coloured 3, so  $\{z, y_{m-1}, s, w\}$  induces a  $(1, 1, 2)$  subdivision of  $C_3$ , a contradiction.

*Case 2.2.* Suppose that  $(y_{m-1}, w) \in A(T)$ . In this case  $(y_{m-1}, w)$  is coloured 3, then  $(z, \alpha', y_{m-1}) \cup (y_{m-1}, w)$  contains a  $zw$ -monochromatic path, a contradiction.

*Case 2.3.* Suppose that  $(w, y_{m-1}) \in A(T)$ . We have that  $(w, y_{m-1})$  is coloured 2. Then  $(w, y_{m-1}, s, z_1)$  is a path of length 3, by Lemma 1.2  $w$  and  $z_1$  are adjacent. If  $(w, z_1) \in A(T)$  then it is coloured 2, so  $\{w, y_{m-1}, s, z_1\}$  induces a 3-coloured  $T_4$ , a contradiction. If  $(z_1, w) \in A(T)$ , then it is coloured  $i$ . Thus  $\{w, y_{m-1}, s, z_1\}$  induces a  $(1, 1, 2)$  subdivision of  $C_3$ , a contradiction.

Then case 2 is not possible and we conclude that for every  $z \in V(T) \setminus (B \cup N_0)$  there is a  $(B \cup N_0)z$ -monochromatic path.

Therefore  $B \cup N_0 \in \Gamma_i$ .

2.  $S < B \cup N_0$ .

Let  $u \in S$ , we will prove that there exists  $v \in B \cup N_0$  such that  $u = v$  or there is a  $uv$ -monochromatic path coloured  $i$  and there is no a  $vu$ -monochromatic path coloured  $i$ .

If  $u \in B$ , we have done. Assume that  $u \in S \setminus B$  then there is a  $uv$ -monochromatic path coloured  $i$ , for some  $v \in N_0$  and the definition of  $X_0$  implies that there is not a  $vu$ -monochromatic path coloured  $i$ .

Then  $S < B \cup N_0$  contradicting of maximality of  $S$  in  $(\Gamma_i, \leq)$ .

We conclude that  $T$  has a kernel by monochromatic paths. ■

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### REFERENCES

- [1] C. Berge, *Graphs* (North Holland, Amsterdam, New York, 1985).
- [2] E. Boros and V. Gurvich, *Perfect graphs, kernels, and cores of cooperative games*, Discrete Math. **306** (2006) 2336–2354.
- [3] P. Duchet, *Graphes noyau-parfaits*, Ann. Discrete Math. **9** (1980) 93–101.
- [4] P. Duchet, *Classical Perfect Graphs, An introduction with emphasis on triangulated and interval graphs*, Ann. Discrete Math. **21** (1984) 67–96.
- [5] P. Duchet and H. Meyniel, *A note on kernel-critical graphs*, Discrete Math. **33** (1981) 103–105.
- [6] A.S. Fraenkel, *Combinatorial games: selected bibliography with a succinct gourmet introduction*, Elec. J. Combin. (surveys) #DS2.
- [7] H. Galeana-Sánchez, *On monochromatic paths and monochromatic cycles in edge coloured tournaments*, Discrete Math. **156** (1996) 103–112.
- [8] H. Galeana-Sánchez, *Kernels in edge coloured digraphs*, Discrete Math. **184** (1998) 87–99.
- [9] H. Galeana-Sánchez and J.J. García Ruvalcaba, *Kernels in the clousure of coloured digraphs*, Discuss. Math. Graph Theory **20** (2000) 243–354.
- [10] H. Galeana-Sánchez and V. Neumann-Lara, *On kernel and semikernels of digraphs*, Discrete Math. **48** (1984) 67–76.
- [11] H. Galeana-Sánchez and V. Neumann-Lara, *On kernel-perfect critical digraphs*, Discrete Math. **59** (1986) 257–265.
- [12] H. Galeana-Sánchez and R. Rojas-Monroy, *On monochromatic paths and monochromatic 4-cycles in edge coloured bipartite tournaments*, Discrete Math. **285** (2004) 313–318.

- [13] H. Galeana-Sánchez and R. Rojas-Monroy, *Monochromatic paths and most 2-coloured arc sets in edge-coloured tournaments*, Graphs and Combin. **21** (2005) 307–317.
- [14] G. Hahn, P. Ille and R. Woodrow, *Absorbing sets in arc-coloured tournaments*, Discrete Math. **283** (2004) 93–99.
- [15] T.W. Haynes, T. Hedetniemi and P.J. Slater, editors, *Domination in Graphs (Advanced Topics)*, Marcel Dekker Inc., 1998).
- [16] T.W. Haynes, T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker Inc., 1998).
- [17] M. Kucharska, *On  $(k,l)$ -kernels of orientations of special graphs*, Ars Combin. **60** (2001) 137–147.
- [18] M. Kucharska and M. Kwaśnik, *On  $(k,l)$ -kernels of superdigraphs of  $P_m$  and  $C_m$* , Discuss. Math. Graph Theory **21** (2001) 95–109.
- [19] M. Kwaśnik, *The generalization of Richardson's Theorem*, Discuss. Math. Graph Theory **IV** (1981) 11–14.
- [20] M. Kwaśnik, *On  $(k,l)$ -kernels of exclusive disjunction, cartesian sum and normal product of two directed graphs*, Discuss. Math. Graph Theory **V** (1982) 29–34.
- [21] S. Minggang, *On monochromatic paths in  $m$ -coloured tournaments*, J. Combin. Theory (B) **45** (1988) 108–111.
- [22] M.V. Neumann-Lara, *Seminúcleos de una digráfica*, An. Inst. Mat. UNAM **11** (1971) 55–62.
- [23] M. Richardson, *Solutions of irreflexive relations*, Ann. Math. **58** (1953) 573.
- [24] M. Richardson, *Extensions theorems for solutions of irreflexive relations*, Proc. Nat. Acad. Sci. USA **39** (1953) 649.
- [25] B. Sands, N. Sauer and R. Woodrow, *On monochromatic paths in edge-coloured digraphs*, J. Combin. Theory (B) **33** (1982) 271–275.
- [26] A. Włoch and I. Włoch, *On  $(k,l)$ -kernels in generalized products*, Discrete Math. **164** (1997) 295–301.
- [27] I. Włoch, *On imp-sets and kernels by monochromatic paths in duplication*, Ars Combin. **83** (2007) 93–99.
- [28] I. Włoch, *On kernels by monochromatic paths in the corona of digraphs*, Cent. Eur. J. Math. **6** (2008) 537–542.

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