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ON ODD AND SEMI-ODD LINEAR PARTITIONS OF CUBIC GRAPHS

Jean-Luc Fouquet, Henri Thuillier, Jean-Marie Vanherpe

L.I.F.O., Faculté des Sciences, B.P. 6759
Université d'Orléans
45067 Orléans Cedex 2, France

and

Adam P. Wojda

Wydział Matematyki Stosowanej
Zakład Matematyki Dyskretnej
AGH, Al. Mickiewicza 30, 30–059 Kraków, Poland

Abstract

A *linear forest* is a graph whose connected components are chordless paths. A *linear partition* of a graph G is a partition of its edge set into linear forests and $la(G)$ is the minimum number of linear forests in a linear partition.

In this paper we consider linear partitions of cubic simple graphs for which it is well known that $la(G) = 2$. A linear partition $L = (L_B, L_R)$ is said to be *odd* whenever each path of $L_B \cup L_R$ has odd length and *semi-odd* whenever each path of L_B (or each path of L_R) has odd length.

In [2] Aldred and Wormald showed that a cubic graph G is 3-edge colourable if and only if G has an odd linear partition. We give here more precise results and we study moreover relationships between semi-odd linear partitions and perfect matchings.

Keywords: Cubic graph, linear arboricity, strong matching, edge-colouring.

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1. Introduction

As usually, for any undirected graph G, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges and we consider, as usual, that $|V(G)| = n$ and $|E(G)| = m$. If $F \subseteq E(G)$ then $V(F)$ is the set of vertices which are incident with some edges of F. For any path P we shall denote by $l(P)$ the length of P, that is to say the number of its edges. A vertex of a path P distinct from an end-vertex is said to be an *internal* vertex. If u and v are vertices of a path P then $P[u, v]$ denotes the subpath of P whose end-vertices are u and v. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C, or, equivalently, such that C is the edge-set of the subgraph of G induced by the vertex-set $V(C)$. *A 2-factor* of G is a spanning subgraph whose components are cycles. If every cycle of a 2-factor has an even length then we say that this 2-factor is *an even 2-factor*.

A *linear-k-forest* is a forest whose components are paths of length at most k. The *linear-k-arboricity* of an undirected graph G is defined in [5] as the minimum number of linear-k-forests needed to partition the set $E(G)$. The linear-k-arboricity is a natural refinement of the linear-arboricity introduced by Harary [6] (corresponding to linear-$(n-1)$-arboricity). The linear-k-arboricity will be denoted by $la_k(G)$.

Let $\chi^{'}(G)$ be the classical chromatic index (minimum edge colouring) and let $la(G)$ be the linear arboricity of G. We clearly have:

$$la(G) = la_{n-1}(G) \leq la_{n-2}(G) \leq \cdots \leq la_2(G) \leq la_1(G) = \chi^{'}(G).$$

We know by Vizing's Theorem [10] that $la_1(G) \leq \Delta(G) + 1$ (where $\Delta(G)$ is the maximum degree of G). For any $k \geq 2$, we have (lower bound comes from [5] and upper bound from [3]):

$$\max\left(\left\lceil\frac{\Delta(G)}{2}\right\rceil, \left\lceil\frac{m(k+1)}{kn}\right\rceil\right) \leq la_k(G) \leq \Delta(G).$$

In this paper we consider *cubic graphs*, that is to say finite simple 3-regular graphs. Since in a cubic graph G we have $3n = 2m$, by the previous formula we obtain:

$$la_2(G) = 3 \text{ and for any } k \geq 3, \;\; 2 \leq \; la_k(G) \leq 3.$$

The cubic graph obtained from two disjoint cycles of length 3 connected by a perfect matching is denoted by PR_3. It was shown by Akiyama, Exoo and Harary [1] that $la(G) = 2$ when G is cubic. In [3] Bermond *et al.* conjectured that $la_5(G) = 2$. Thomassen [9] proved the conjecture, which is best possible since, in view of $la_4(K_{3,3}) = 3$ and $la_4(PR_3) = 3$, 5 cannot be replaced by 4.

A partition of $E(G)$ into two linear forests L_B and L_R will be called a *linear partition* and we shall denote this linear partition $L = (L_B, L_R)$. An *odd linear forest* is a linear forest in which each path is a path of odd length. A *semi-odd linear partition* is a linear partition $L = L_B \cup L_R$ such that L_B or L_R is an odd linear forest. An *odd linear partition* is a partition of $E(G)$ into two odd linear forests. For $i \in \{B, R\}$ let $\omega(L_i)$ be the number of components (or maximal paths) of L_i. Since every vertex of G is either end-vertex of a maximal path of L_B or end-vertex of a maximal path of L_R, we have

$$\omega(L_B) + \omega(L_R) = \frac{|V(G)|}{2}.$$

2. Jaeger's Graphs

A special class of cubic graphs introduced by F. Jaeger will be considered.

Definition 2.1. Let us call a *Jaeger's matching* a perfect matching which is the union of two strong matchings. A cubic graph G is a *Jaeger's graph* whenever G contains a Jaeger's matching.

In his thesis [8] Jaeger called these cubic graphs *equitable* and pointed out that the improper 2-colouring $\{B, R\}$ of their vertices induced by a perfect matching M union of two disjoint strong matchings M_B and M_R leads to a *balanced colouring* as defined by Bondy [4].

When G is a cubic graph having a 2-factor of C_4's, say $\mathcal{F}$, we consider the auxiliary 2-regular graph G' defined as follows: every C_4 of $\mathcal{F}$ is replaced with its complementary graph (which is a $2K_2$).

Theorem 2.2. *Let G be a connected cubic graph having a 2-factor of squares, say $\mathcal{F}$ and let p be the number of cycles of G'. Then there are 2^{p-1} Jaeger's matchings in G which intersect $\mathcal{F}$.*

Proof. We first prove that there are at most two types of Jaeger's matchings in G.

Claim. Let $M = M_B \cup M_R$ be a Jaeger's matching of G. If M intersects $\mathcal{F}$ then every C_4 of $\mathcal{F}$ contains an edge of M_B and an edge of M_R.

Proof of Claim. Recall that M_B and M_R are strong matchings. Without loss of generality we may assume that there is some edge say ab of some C_4 in $\mathcal{F}$, say $abcd$ which belongs to M_B. Since M is a perfect matching and M_B is a strong matching the vertices c and d must be the endpoint of some edge(s) of M_R. Since M_R is a strong matching we have $cd \in M_R$. Let $a'b'c'd'$ be another C_4 of $\mathcal{F}$ which is connected to $abcd$ by some edge say aa'. The edge aa' is not an edge of M (M is a matching) and since a' must be an endpoint of an edge of M_R, M_R intersects $a'b'c'd'$. Consequently, G being connected we have that M_B and M_R intersect all cycles of $\mathcal{F}$. ■

It follows that a Jaeger's matching of G is either contained into $\mathcal{F}$ or disjoint from $\mathcal{F}$.

We now establish a correspondence between the orientations of the cycles of G' and the Jaeger's matchings of G which intersect $\mathcal{F}$.

Let us give an orientation of the cycles of G'. Going back now to G, each C_4 of $\mathcal{F}$ has an edge connected to two out-going edges and an edge connected to two in-going edges. Let M_B be the set of edges connected to two out-going edges over all the C_4's of $\mathcal{F}$ while M_R contains the edges connected to two in-going edges. It's an easy task to check that $M_B \cup M_R$ is a Jaeger's matching of G.

Conversely let us consider a Jaeger's matching $M = M_B \cup M_R$ of G which intersects $\mathcal{F}$. By the above Claim, each C_4 of $\mathcal{F}$ contains an edge of M_B and an edge of M_R. For any C_4 of $\mathcal{F}$ and for any vertex v of this C_4 we denote e_v the edge of $E(G)\backslash E(\mathcal{F})$ that is adjacent to v. We know that v is an endpoint of an edge in M_B or in M_R. We give an orientation to the edge e_v in such a way that e_v is an out-going edge (that is v is the origin) if and only if v is endpoint of an edge of M_B. Since every edge of $E(G)\backslash E(\mathcal{F})$ is connected to two C_4's of $\mathcal{F}$ those edges are oriented twice; more precisely: when aa' is an edge connecting two cycles of $\mathcal{F}$, say $abcd$ and $a'b'c'd'$, if $aa' = e_a$ is an out-going edge for the cycle $abcd$ then $aa' = e_{a'}$ must be an in-going edge for $a'b'c'd'$ for otherwise M_B would not be a strong matching. Consequently the given orientation of all edges e_v ($v \in V(G)$) extends to an orientation of the cycles of G'.

We have 2^p possible orientations of the cycles of G'. A given orientation of each cycle of G' and the opposite orientations of these cycles yield to the same partition of M, consequently, there are 2^{p-1} Jaeger's matchings intersecting the 2-factor $\mathcal{F}$ of G. This finishes the proof. ■

By Theorem 2.2 every cubic graph having a 2-factor of squares has at least one Jaeger's matching. Hence we conclude this subsection with the following corollaries.

Corollary 2.3. *A cubic graph having a 2-factor of squares is a Jaeger's graph.*

Furthermore, we can derive from Theorem 2.2 a simple linear time algorithm for finding a Jaeger's matching in a connected cubic graph which have a 2-factor of squares.

It can be noticed that every cubic graph with a perfect matching M can be transformed into a Jaeger's graph by using the transformation (*square extension*) depicted in Figure 1 on each edge of M. Indeed, the resulting graph has a 2-factor of squares and we can apply Theorem 2.2.

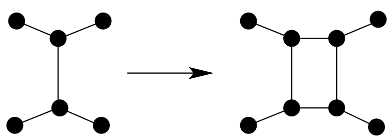

Figure 1. Square extension

Corollary 2.4. *A connected cubic graph is 3-edge colourable if and only if there is a perfect matching N such that the cubic graph obtained in using a square extension on each edge of N leads to a Jaeger's graph having an odd number (at least 3) of Jaeger's matchings.*

Proof. Let G be a cubic graph such that G', obtained from G by square extensions on each edge of N, has an odd number of Jaeger's matchings. Let $\mathcal{F}$ be the 2-factor of C_4's of G' obtained by these square extensions. Since G' has an odd number of Jaeger's matchings, Theorem 2.2 says that there is a Jaeger's matching M of G' which avoids all the edges of $\mathcal{F}$. Clearly, there is a bijection between M and the 2-factor $E(G) \setminus N$. Since M is the union of the strong matchings M_B and M_R, going back to G the edges of $M_B \cup M_R$ give rise to an even 2-factor $E(G) \setminus N$ of G which, together with N, leads to a 3-edge colouring of G.

Conversely, assume that G is 3-edge colourable. Then extending each edge of a given colour in a 3-edge colouring of G leads to a graph G' which has a 2-factor of squares. We can choose the square of G' extending an edge of G of the given colour in such a way that any of the two other colours induces a strong matching. Indeed, the edges of the two other colours give rise to a Jaeger's matching in G' avoiding every square so constructed and Theorem 2.2 applies. ∎

3. Semi-Odd Linear Partitions

We are interested by relationships between perfect matchings and semi-odd linear partitions and we generalize a theorem of Aldred and Wormald [2]. We will come again on their result in the next section.

Theorem 3.1. *Let G be a cubic graph having a perfect matching M. Then there exists a set $F \subseteq E(G) - M$ intersecting each cycle of the 2-factor $G - M$ such that $F + M$ is an odd linear forest.*

Proof. Let $\{C_1, C_2, \ldots, C_k\}$ be the cycles of $G - M$ (with $k \geq 1$). Clearly if e is an edge of C_1 then the set $M \cup \{e\}$ induces an odd linear forest of G (made of a path of length 3 and a matching). Let us suppose that $k \geq 2$ and let i such that $1 \leq i < k$. We suppose that for every j with $1 \leq j \leq i$ we have chosen an edge e_j of C_j such that $F_i + M$ is an odd linear forest (with $F_i = \{e_1, e_2, \ldots, e_i\}$). Let xy be an edge of C_{i+1}. If $F_i + M + xy$ contains a cycle then xy belongs to this cycle. Thus, $F_i + M$ contains a path P having x and y as end vertices. Let z be the neighbour of y on C_{i+1} distinct from x. Then, $F_i + M + yz$ contains no cycle (if it contains a cycle, then $F_i + M$ contains a path P' having y and z as end vertices, contradicting the existence of P). So, C_{i+1} contains an edge, say e_{i+1}, such that $F_i + e_{i+1} + M$ is an odd linear forest. Let us denote $F_i + e_{i+1}$ by F_{i+1}. The results follows by induction. ∎

Definition 3.2. For every odd path $P = [a_0, a_1, \ldots, a_{2l+1}]$, with $l \geq 0$, we say that the edges $\{a_0a_1, a_2a_3, \ldots, a_{2l}a_{2l+1}\}$ are at *even distance from the end vertices of P.*

Theorem 3.3. *A cubic graph has a perfect matching if and only if it has a semi-odd linear partition.*

Proof. If M is a perfect matching of a cubic graph G then by Theorem 3.1 the graph G has a set of edges F intersecting every cycle of the 2-factor $G \setminus M$ such that $F + M$ is an odd linear forest. Set $L_B = F + M$ and $L_R = G - F - M$ Then, $L = (L_B, L_R)$ is a semi-odd linear partition.

Conversely, if the graph has a semi-odd linear partition $L = (L_B, L_R)$, we suppose without loss of generality that L_B is an odd linear forest. Let M be the set of edges of L_B at even distance from the end vertices of the maximal paths of L_B. It is a routine matter to check that M is a matching. Since L_B is a spanning forest, M is a perfect matching. ∎

For any cubic graph G having a perfect matching (or, equivalently, a 2-factor) we denote by $\rho(G)$ the minimum number of even maximal paths appearing in a semi-odd linear partition, and we denote by $o(G)$ the minimum number of odd cycles appearing in a 2-factor of G (we note that $o(G)$ is an even number). If $\rho(L)$ denotes the number of even maximal paths of a semi-odd linear partition $L = (L_B, L_R)$, then $\rho(G) = Min\{\rho(L) |\ L$ is a semi-odd linear partition of $G\}$.

Theorem 3.4. *Let G be a cubic graph having a 2-factor (or, equivalently, a perfect matching). Then $\rho(G) = o(G)$.*

Proof. Let us suppose that G has a 2-factor. Let $\{C_1, C_2, \ldots, C_k\}$ be a 2-factor of G having $o(G)$ odd cycles, and let M be the perfect matching associated with this 2-factor. By Theorem 3.1 we can choose a set of edges F (one by cycle) such that $F + M$ is an odd linear forest L_B. The set $E(G) - E(L_B)$ induces a linear forest L_R and we consider the semi-odd linear partition $L = (L_B, L_R)$. The number $\rho(L)$ of even maximal paths of L_R is equal to the number $o(G)$ of odd cycles in $\{C_1, C_2, \ldots, C_k\}$. Thus, $\rho(G) \le o(G)$.

Now let $L = (L_B, L_R)$ be a semi-odd linear partition such that L_B is an odd linear forest. As in Proof of Theorem 3.3, let M be the perfect matching made of the edges of L_B at even distance from the end vertices of the maximal paths of L_B, and let $\{C_1, C_2, \ldots, C_k\}$ be the 2-factor $G - M$. Every path of L_B of length ≥ 3 intersects this 2-factor and we see that $E(L_B) \cap (E(C_1) \cup E(C_2) \cdots \cup E(C_k))$ is a matching. Now consider any cycle C_i of this 2-factor. Clearly, $E(L_B)$ intersects $E(C_i)$. Let $\{e_1, e_2, \ldots, e_r\} = E(L_B) \cap E(C_i)$. We see that $E(C_i) - E(L_B)$ induces a set of elementary paths $\{P_1, P_2, \ldots, P_r\}$ which are precisely maximal paths of L_R. If $P_1, P_2, \ldots, P_r$ have odd lengths then $|E(C_i)| = r + \sum_{j=1}^{j=r} l(P_j)$ is even. Thus, if C_i is an odd cycle then at least one of these paths has an even length. Then, $\rho(L)$ is greater or equal to the number of odd cycle in $\{C_1, C_2, \ldots, C_k\}$. Hence, $\rho(L) \geq o(G)$. By choosing L such that $\rho(L) = \rho(G)$, we obtain $\rho(G) \geq o(G)$. ■

4. Odd Linear Partitions

Let G be a cubic graph. Assume that $L = (L_B, L_R)$ is a linear partition of its edge set. By colouring alternately the edges of the maximal paths in L_B with α and γ and those of L_R with β and δ, we get a 4-edge colouring.

We note that a cubic graph G can be factored into two odd linear forests if and only if $\rho(G) = 0$ and that G has an even 2-factor (or, equivalently, is 3-edge colourable) if and only if $o(G) = 0$. So, the following result of Aldred and Wormald can be obtained as a corollary of Theorem 3.4.

Theorem 4.1 [2]. *Let G be a cubic graph. Then the following properties are equivalent:*

1. *G is 3-edge colourable (that is $\chi'(G) = 3$).*
2. *G can be factored into two odd linear forests.*

Recall here a sketch of their proof. Suppose that $L = (L_B, L_R)$ is an odd linear partition of G. A proper 3-edge colouring of G is obtained by colouring the edges of the paths in L_B alternately with α and γ so that each path in L_B has its first and last edges coloured with α, and by colouring the edges of the paths in L_R alternately with β and γ so that each path in L_R has its first and last edges coloured with β. Conversely, if G is 3-edge colourable, let us consider a proper 3-edge colouring using α, β and γ as colours. Pick an edge from each cycle of the even 2-factor $\Phi(\alpha, \beta)$ induced by the colours α and β, and let F be the set of these picked edges. The subgraph of G formed by F and the perfect matching R induced by colour γ has connected components which are odd paths or even cycles. Each even cycle is broken by choosing an edge coloured with γ (let F' be this set of edges). Then $L_R = R + F - F'$ is a set of odd paths as well as $L_B = \Phi(\alpha, \beta) - F + F'$, leading to an odd linear partition $L = (L_B, L_R)$ of G.

The remarkable point here is that F is a minimal transversal of the cycles of $\Phi(\alpha, \beta)$ where each edge of F has been chosen at random. Their strategy is a greedy strategy with simple accommodations in order to broke some even cycles. In our proof of Theorem 3.4 we choose randomly an edge on a cycle, and if this edge is not acceptable we choose an incident edge. This strategy is intermediate between the greedy strategy of Aldred and Wormald and the strategy that we will develop in the next subsections. We shall see that when suitably choosing edges in F we extend their result.

4.1. Reductions

Assume that G is a cubic 3-edge colourable graph and let Φ be a 3-edge colouring of G. For any edge e, let us denote the colour of e by $\Phi(e)$. Let α and β be any two distinct colours of Φ and let γ be the third colour. The subset of the edges of G coloured with α or with β induces an even 2-factor. In the following the 2-factor induced by any two distinct colours α and β will be denoted by $\Phi(\alpha, \beta)$. Any cycle of $\Phi(\alpha, \beta)$ is said to be an $\alpha\beta$-cycle. Since α and β are arbitrary colours it is clear that the connected components of a 3-edge colourable cubic graph are 2-connected subgraphs. We need some specific definitions for this section.

Definition 4.2. Let α and β be any two distinct colours of Φ. In the following $SM_G(\alpha, \beta)$ will denote a strong matching of G intersecting every $\alpha\beta$-cycle (when such a strong matching exists).

Definition 4.3. Let α and β be any two distinct colours of Φ. Let xy be an edge of G and let x' and x'' (respectively y' and y'') be the (distinct) neighbours of x (of y, respectively) distinct from y (respectively x) such that $x' \neq y''$ or $x'' \neq y'$ and suppose that $x'y'$ and $x''y''$ are not edges of G. Let us suppose that $\Phi(xy) = \alpha$, $\Phi(xx') = \Phi(yy') = \beta$ and $\Phi(xx'') = \Phi(yy'') = \gamma$. If $x' \neq y''$ or $x'' \neq y'$ then the edge xy is said to be an *α-free edge*. Note that edge $x'y''$ (respectively $x''y'$) may exist, and in this case $\Phi(x'y'') = \alpha$ (respectively $\Phi(x''y') = \alpha$). We notice that, without loss of generality, there are two cases:

- Case 1. $x' \neq y''$ and $x'' \neq y'$.
- Case 2. $x' = y''$ and $x'' \neq y'$.

The 3-edge coloured cubic graph G' on $(n-2)$ vertices obtained from G by deleting vertices x and y and their incident edges and adding the edges $x'y'$ and $x''y''$, coloured respectively by β and γ, is said to be obtained from G by *reduction of an α-free edge*. Situations are depicted on Figures 2 and 3. Clearly, if G contains a triangle (a cycle of length 3) T such that the three edges connecting T to $G - T$ are independent then every edge of T is a free edge (i.e., α-free edge if its colour is α).

Remark 4.4. Following the notations of Definition 4.3, if xy is an α-free edge of G, the $\alpha\beta$-cycle of G containing xy gives the $\alpha\beta$-cycle of G' containing the β-coloured edge $x'y'$ of the graph G' obtained from G by reduction of the α-free edge xy. The others $\alpha\beta$-cycles, if they exist, are identical in G and in G'.

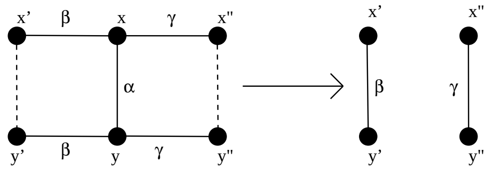

Figure 2. α-free edge and reduction — Case 1.

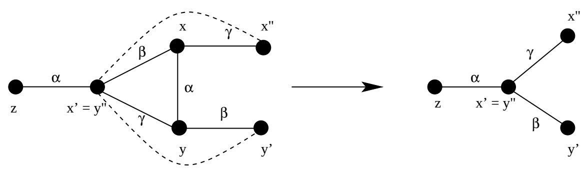

Figure 3. α-free edge and reduction — Case 2.

Definition 4.5. Following the notations of Definition 4.3, let us suppose that $x' = y''$ and $x'' = y'$ (that is xy is a chord of the subgraph induced on $\{x, x', y, y'\}$) and suppose that the component of G containing $\{x, x', y, y'\}$ is distinct from K_4. Let z (respectively z') be the neighbour of x' (respectively y') distinct from x and y. We note that $z \neq y'$ and $z' \neq x'$. Since any component of G is 2-connected, z and z' are distinct vertices. The subgraph D induced on $\{x, x', y, y'\}$ is usually called a *diamond.* The edge xy is called *the central edge* of D. Clearly, the central edge of D and the two edges of the 2-cut connecting D to the rest of G have the same colour. A diamond whose central edge have colour α is said to be an α*-diamond.* There are two cases according to $zz' \notin E(G)$ (Case 1) or $zz' \in E(G)$ (Case 2). In Case 1, an α-diamond is said to be an α*-free diamond.* The 3-edge coloured cubic graph G' on $(n-4)$ vertices obtained from G by deleting D and its incident edges and adding the edge zz' coloured with α is said to be obtained from G by *reduction of an* α*-free diamond.* See Figure 4.

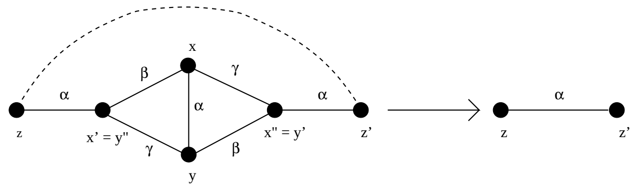

Figure 4. α-free diamond and reduction — Case 1.

In Case 2 we denote by u (respectively u') the neighbour of z (respectively z') distinct from x' and z' (respectively y' and z). We note that u and u' are distinct vertices (recall that every component of G is 2-connected). According to the colour β or γ of the edge zz', there are two sub-cases, Case 2.1 and Case 2.2. We consider the cubic graph G' on $(n-2)$ vertices obtained from G by deleting the edge zz' and replacing the paths uzx' and $u'z'y'$ by ux' and $u'y'$ respectively (we shall say that G' is obtained from G by *edge suppression* of zz'). In Case 2.1 we consider the 3-edge colouring Φ_1 of G' such that $\Phi_1(xy) = \Phi_1(x'u) = \Phi_1(y'u') = \gamma$, $\Phi_1(xx') = \Phi_1(yy') = \alpha$, $\Phi_1(x'y) = \Phi_1(xy') = \beta$ and $\Phi_1(e) = \Phi(e)$ for any other edge. See Figure 5.

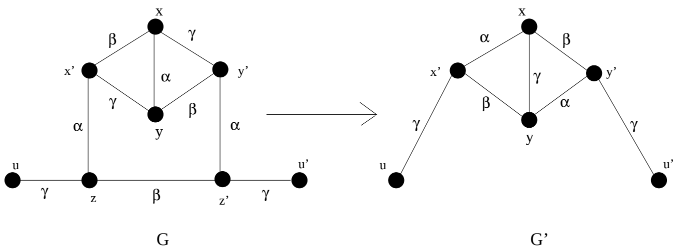

Figure 5. α-diamond and edge suppression — Case 2.1.

In Case 2.2, we have the 3-edge colouring Φ_2 of G' such that $\Phi_2(xy) = \Phi_2(x'u) = \Phi_2(y'u') = \beta$, $\Phi_2(xx') = \Phi_2(yy') = \alpha$, $\Phi_2(x'y) = \Phi_2(xy') = \Phi$ and $\Phi_2(e) = \Phi(e)$ for any other edge. See Figure 6.

Remark 4.6. Following notations of Definition 4.5, if xy is the central edge of an α-free diamond D (Case 1) then an $\alpha\beta$-cycle containing xy gives an $\alpha\beta$-cycle of G' containing the α-coloured edge zz' of the graph G' obtained from G by reduction of the α-free diamond D. If D is an α-diamond that is not α-free (Case 2), then in Case 2.1 the $\alpha\beta$-cycle of G containing xy gives the $\alpha\beta$-cycle $\{x, x', y, y'\}$ of G' and in Case 2.2 an $\alpha\beta$-cycle containing xy

(α-coloured in G) gives an $\alpha\beta$-cycle of G' containing xy (β-coloured in G'). The others $\alpha\beta$-cycles, if there exist, are identical in G and in G'.

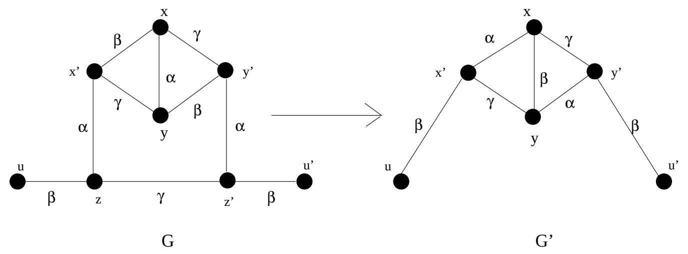

Figure 6. α-diamond and edge suppression — Case 2.2.

4.2. Choosing a strong matching as a transversal

As pointed out before, we are interested in finding a particular transversal of $\Phi(\alpha, \beta)$ when α and β are any two distinct colours of a 3-edge colouring.

Theorem 4.7. *Let G be a 3-edge coloured cubic graph and let Φ be a 3-edge colouring of G. Let α and β be any two distinct colours of Φ. Then there exists a strong matching $SM_G(\alpha, \beta)$ intersecting every cycle belonging to the 2-factor $\Phi(\alpha, \beta)$.*

Proof. It is easily seen that the theorem is true for graphs with at most 8 vertices. Let us suppose that Theorem 4.7 is false and let G be a counterexample having the smallest number of vertices. Without loss of generality we can suppose that G is connected. Let α and β be two colours such that there is no strong matching of G intersecting every $\alpha\beta$-cycle of G.

Claim 1. G has neither an α-free edge nor a β-free edge.

Proof. By symmetry between α and β it suffices to prove that G has no α-free edge. Suppose, for contradiction, that xy is an α-free edge of G. By minimality of G, the graph G' obtained from G by reduction of the α-free edge xy has a strong matching $SM_{G'}(\alpha, \beta)$ intersecting every $\alpha\beta$-cycle of G'. By Remark 4.4, every $\alpha\beta$-cycle C of G' is either an $\alpha\beta$-cycle of G or is obtained by reduction from an $\alpha\beta$-cycle of G containing xy. In the last case, let $\{e\} = SM_{G'}(\alpha, \beta) \cap E(C)$. If $e \neq x'y'$ then $SM_{G'}(\alpha, \beta)$ is a strong matching $SM_G(\alpha, \beta)$ of G. If $e = x'y'$ (coloured with β) then either x'' and

y'' are not incident to $SM_{G'}(\alpha,\beta)$, and we put

$$SM_G(\alpha,\beta) = SM_{G'}(\alpha,\beta) - x'y' + xy$$

or else

- in Case 1, according to x'' or y'' is incident to $SM_{G'}(\alpha,\beta)$ we put

$$SM_G(\alpha,\beta) = SM_{G'}(\alpha,\beta) - x'y' + yy'$$

  or we put

$$SM_G(\alpha,\beta) = SM_{G'}(\alpha,\beta) - x'y' + xx'$$

- in Case 2 we put $SM_G(\alpha,\beta) = SM_{G'}(\alpha,\beta) - x'y' + yy'$.

In any case, it is a routine matter to check that $SM_G(\alpha,\beta)$ so obtained is a strong matching intersecting every $\alpha\beta$-cycle of G, a contradiction. Thus, G has no α-free edge. ■

Claim 2. G has neither an α-diamond nor a β-diamond.

Proof. By symmetry between α and β it suffices to prove that G has no α-diamond. By minimality of G, the graph G' obtained from G by reduction of an α-free diamond D (Case 1, see Figure 4) or by reduction of the edge zz' (Cases 2.1 and 2.2, see Figures 5 and 6 has a strong matching $SM_{G'}(\alpha,\beta)$ intersecting every $\alpha\beta$-cycle C of G'.

- In Case 1, if $zz' \notin SM_{G'}(\alpha,\beta)$ then set $SM_G(\alpha,\beta) = SM_{G'}(\alpha,\beta)$ else set $SM_G(\alpha,\beta) = SM_{G'}(\alpha,\beta) - zz' + xy$.
- In Cases 2.1 and 2.2, let uv be the edge of $SM_{G'}(\alpha,\beta)$ contained in the $\alpha\beta$-cycle of G' using $\{x,x',y,y'\}$ set $SM_G(\alpha,\beta) = SM_{G'}(\alpha,\beta) - uv + xy$.

By Remark 4.6 $SM_G(\alpha,\beta)$ is a strong matching of G intersecting every $\alpha\beta$-cycle of G, a contradiction. Thus, G has no α-diamond. ■

Claim 3. Every $\alpha\beta$-cycle C of G of length ≥ 6 has no chord.

Proof. Suppose that xy is a chord of C. Let x' and x'' be the neighbours of x distinct from y, and let y' and y'' be the neighbours of y distinct from x. We suppose that the vertices x', x, x'', y', y, y'' appear in that order on C. Let x'^{-} and x''^{+} be respectively the neighbours of x' and x'' on C distinct from x. We wish to prove that $x'x''$ and $y'y''$ are not edges. Suppose, for contradiction, that $x'x''$ is an edge of G. By Claim 1 the vertices $x^{'-}$, $x^{''+}$ and y are not three distinct vertices (otherwise $x'x$ and $x''x$ will be α-free or β-free edges). Since C has length at least 6, vertices $x^{'-}$ and $x^{''+}$ are distinct. Without loss of generality we can suppose that $x^{''+} = y$, that is $y' = x''$, and that $\Phi(x'x) = \beta$. Since $x^{'-} \neq y$, the set $\{x^{'-}, x', x, x'', y, y''\}$ induces an α-diamond, contrary to Claim 2. Thus, $x'x''$ is not an edge, and, by symmetry, $y'y''$ is not an edge. Let G' be the cubic graph obtained from G by deleting x and y and their incident edges and by adding the edges $x'x''$ and $y'y''$. The cycle C gives a cycle C' in G' of length $|C| - 2$. By colouring the edges of C' by the colours α and β, and no change for the other edges (which are edges of G), we obtain a 3-edge colouring of G'. Let $SM_{G'}(\alpha, \beta)$ be a strong matching intersecting every $\alpha\beta$-cycle of G'. Let us assume that $SM_{G'}(\alpha, \beta)$ intersects each $\alpha\beta$-cycle of G' exactly once. Whenever neither $x'x''$ nor $y'y''$ are contained in $SM_{G'}(\alpha, \beta) \cap C$ then we set $SM_G(\alpha, \beta) = SM_{G'}(\alpha, \beta)$. Otherwise, let uv be the edge of $SM_{G'}(\alpha, \beta) \cap C$, then we set $SM_G(\alpha, \beta) = SM_{G'}(\alpha, \beta) - x'x'' + x'x$ when $uv = x'x''$ or we set $SM_G(\alpha, \beta) = SM_{G'}(\alpha, \beta) - y'y'' + y'y$ when $uv = y'y''$. Then $SM_G(\alpha, \beta)$ intersects every $\alpha\beta$-cycle of G, a contradiction. Hence, xy is not a chord of C. ∎

Claim 4. Every $\alpha\beta$-cycle C of G is a cycle of length 4.

Proof. Let $C = (a_0, a_1, a_2, \ldots, a_{2k-1})$ be an $\alpha\beta$-cycle of length $2k \geq 6$. Let us consider respectively $a'_0, a'_1, a'_2, \ldots, a'_{2k-1}$ the neighbours of $a_0, a_1, a_2, \ldots,$ a_{2k-1} not belonging to C. For every $i \in \{0, \ldots, 2k-1\}$ the edge $a_i a'_i$ is coloured with the third colour γ and hence $a'_0, a'_1, a'_2, \ldots, a'_{2k-1}$ are distinct vertices. By Claim 3, $a_{i-1}a_{i+2}$ is not an edge. Since $a_i a_{i+1}$ is neither an α-free nor a β-free edge, $a'_i a'_{i+1} \in E(G)$. Thus, $\{a'_0, a'_1, a'_2, \ldots, a'_{2k-1}\}$ induces an $\alpha\beta$-cycle. Hence, G is the union of two chordless $\alpha\beta$-cycles $C = (a_0, a_1, a_2, \ldots, a_{2k-1})$ and $C' = (a'_0, a'_1, a'_2, \ldots, a'_{2k-1})$ connected by the matching $\{a_0a'_0, a_1a'_1, a_2a'_2, \ldots, a_{2k-1}a'_{2k-1}\}$. Since $k \geq 3$, it is clear that we can choose an edge e on C and an edge e' on C' such that $\{e, e'\}$ is a strong matching, a contradiction. Thus, $k = 2$ and C is a cycle of length 4. ∎

Hence the 2-factor $\Phi(\alpha, \beta)$ is reduced to a set of squares. By Theorem 2.2 G would have a Jaeger's matching $M = M_B + M_R$ such that the strong matching M_B (or indifferently M_R) intersects every square, a contradiction. Thus, G does not exist and Theorem 4.7 is proved. ∎

Corollary 4.8. *Let G be a cubic graph. Then G can be factored into two odd linear forests $L = (L_B, L_R)$ such that*

(i) *Each path in L_B has odd length at most* 3,

(ii) *Each path in L_R has odd length at least* 3

if and only if $\chi'(G) = 3$.

Proof. Assume that G has an odd linear partition $L = (L_B, L_R)$ with these properties. As in Theorem 4.1 we get immediately a 3-edge colouring.

Conversely, let α and β be two colours of a 3-edge colouring Φ of G and let $SM_G(\alpha, \beta)$ be a minimal strong matching intersecting each cycle of $\Phi(\alpha, \beta)$. If Γ denotes the set of edges coloured by γ then $L_B = \Gamma + SM_G(\alpha, \beta)$ is a set of odd paths of length at most 3. While $L_R = \Phi(\alpha, \beta) \setminus SM_G(\alpha, \beta)$ is a set of odd paths of length at least 3 (recall that, G being simple, every bicoloured cycle has length at least 4). Hence, (L_B, L_R) is an odd linear partition satisfying conditions (i) and (ii). ∎

4.3. Unicoloured transversal

In this section we derive from Theorem 4.7 a result on unicoloured transversals of the 2-factors induced by any 3 edge-colouring of a cubic graph with chromatic index 3. Let us first state a useful Lemma (folklore).

Lemma 4.9. *Let $G = (V, E)$ be a multi-graph then it is always possible to give an orientation to its edge set in such a way that for any vertex v $|d^+(v) - d^-(v)| \leq 1$ (where $d^+(v)$ denotes as usual the outdegree of v and $d^-(v)$ its indegree).*

Proof. Without loss of generality we consider that G is connected. Add a matching of extra edges between vertices of odd degrees in G (since there is an even number of vertices with odd degree) in order to get an eulerian graph G'. We orient the edges of G' following an eulerian tour. It is a routine matter to check that the orientation induced in G satisfies our requirement. ∎

Theorem 4.10. *Let G be a cubic 3-edge colourable graph and let Φ be a 3-edge colouring of G. Let α and β be any two distinct colours of Φ and let γ be the third colour. Then there exists a set F_α of α-edges intersecting every cycle belonging to the 2-factor $\Phi(\alpha, \beta)$ such that the set F_α together with the γ-edges has no cycle.*

Proof. We know by Theorem 4.7 that there exists a strong matching $SM_G(\alpha,\beta)$ intersecting every cycle of the 2-factor $\Phi(\alpha,\beta)$.

Let A be the set of α-edges of $SM_G(\alpha,\beta)$ while B is the set of remaining β-edges of $SM_G(\alpha,\beta)$. We may assume that B is not empty, for otherwise we set $F_\alpha = A$ and we are done.

Let A' be the set of α-edges of G which are incident to an edge of B. For each edge $e \in A'$, the *attachment vertex* of e will be the vertex incident to the edge of B. B being a strong matching this attachment vertex is well defined. We intend to define F_α as a subset of $A \cup A'$ which contains A and thus we focus on the $\alpha\gamma$-cycles of G whose α-edges belong to $A \cup A'$.

Claim. An $\alpha\gamma$-cycle of G whose all α-edges belong to $A \cup A'$ cannot contain any edge of A.

Proof. Let $\mathcal{C} = x_0y_0x_1y_1 \ldots x_ky_k$ be an $\alpha\gamma$-cycle of G whose all α-edges belong to $A \cup A'$. Assume that x_iy_i are α-edges while y_ix_{i+1} are γ-edges (i being taken modulo $k+1$). Let us suppose that $x_0y_0 \in A$. The edge x_1y_1 is certainly in A', otherwise A should not be a strong matching. The attachement vertex of x_1y_1 cannot be x_1 otherwise $A \cup B$ is not a strong matching. Considering now x_2y_2, we can say that this edge is not in A (otherwise $A \cup B$ is not a strong matching) and its attachment vertex cannot be x_2 (otherwise B is not a strong matching). Running through the set of α-edges x_iy_i we can show in the same way that these edges are in A' and their attachment vertices are certainly the y_i's. We obtain thus a contradiction with x_ky_k since this edge is in A' and its attachment vertex is y_k which is impossible since y_k is adjacent to x_0. ■

Let C be the set of γ-edges which are incident to an edge of A' and H be the subgraph of G whose edge-set is $A' \cup C$, obviously the connected components of H are paths or cycles. By Claim every $\alpha\gamma$-cycle of G whose all α-edges belong to $A \cup A'$ is also a cycle of H.

Every edge of B is incident in G to a connected component of H, thus we define an auxiliary graph, namely H', in the following way: the vertices of H' are the connected components of H while it's edge-set is B. Since every connected component of H contains at least one edge of A' there is no isolated vertex in H'.

Using Lemma 4.9, we can find an orientation of the edges of H' such that every vertex of H' of degree at least 2 has an in-going edge and an out-going edge.

For any edge e of B we denote $o(e)$ the endpoint of e with respect of the previous orientation of H' and we define an injective mapping $f : B \longrightarrow A'$: given an edge e of B, $f(e)$ is the α-edge of A' whose attachment vertex is $o(e)$.

We set $F_\alpha = A \cup \{f(e)|e \in B\}$. Observe that F_α is a set of α-edges. Since $A \cup B$ covers all $\alpha\beta$-cycles of G and since e and $f(e)$ belong to the same $\alpha\beta$-cycle of G, F_α covers all $\alpha\beta$-cycles of G. Moreover, suppose that $\mathcal{C}$ is an $\alpha\gamma$-cycle of G whose α-edges are members of F_α. Then $\mathcal{C}$ is an $\alpha\gamma$-cycle of H and has a vertex of degree at least 2 in H'. But now, the α-edge of $\mathcal{C}$ which is incident to an out-going edge of $\mathcal{C}$ does not belong to F_α, a contradiction. ■

Remark 4.11. It is possible to derive a linear time algorithm for the construction of the unicoloured transversal F_α of Theorem 4.10 once the 3-edge colouring Φ and the strong matching described in Theorem 4.7 are given.

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