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CARDINALITY OF A MINIMAL FORBIDDEN GRAPH FAMILY FOR REDUCIBLE ADDITIVE HEREDITARY GRAPH PROPERTIES

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Abstract

An additive hereditary graph property is any class of simple graphs, which is closed under isomorphisms unions and taking subgraphs. Let $\mathbf{L}^a$ denote a class of all such properties. In the paper, we consider H-reducible over $\mathbf{L}^a$ properties with H being a fixed graph. The finiteness of the sets of all minimal forbidden graphs is analyzed for such properties.

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1. Introduction and Preliminaries

Let us denote by $\mathcal{I}$ the class of all finite simple graphs having at least one vertex. For a given $G \in \mathcal{I}$ and $V' \subseteq V(G)$ by $G[V']$ we denote a subgraph induced in G by the set V'. If $V' \subseteq V(G)$, then $G - V'$ denotes $G[V(G) - V']$. $V' \subseteq V(G)$ is an *independent set* in G if the graph $G[V']$ is edgeless. For G being isomorphic to a subgraph of G' we write $G \subseteq G'$. Symbols $deg_G(v)$, $\delta(G)$ stand for a degree of a vertex v in a graph G and a minimum vertex degree in G, respectively. $G_1 \cup G_2$ denotes a disjoint union of graphs G_1 and G_2. $G_1 + G_2$ stands for a graph obtained from $G_1 \cup G_2$ by adding all possible edges between vertices of G_1 and G_2. *A graph property* $\mathcal{P}$ is a subclass of

$\mathcal{I}$, closed under isomorphisms. The graph properties $\mathcal{I}, \emptyset$ are called *trivial*. A graph property $\mathcal{P}$ is said to be *hereditary* (*additive*) if it is closed with respect to taking subgraphs (disjoint union of graphs). A set of all graph properties being hereditary (additive and hereditary) will be denoted by $\mathbf{L}$ ($\mathbf{L}^a$). It was stated in [3] that $(\mathbf{L}^a, \subseteq)$ is a complete distributive lattice whose join-operation will be denoted by $\vee$.

Let $[n] = \{1, \ldots, n\}$. We call properties $\mathcal{P}_1, \ldots, \mathcal{P}_n \in \mathbf{L}^a$, $n \geq 2$ incomparable in the lattice $(\mathbf{L}^a, \subseteq)$ if for each $i \in [n]$, $\vee_{j\in[n]}\mathcal{P}_j \neq \vee_{j\in[n]\setminus\{i\}}\mathcal{P}_j$.

A graph property $\mathcal{P}$ is $\vee$-*reducible* in $(\mathbf{L}^a, \subseteq)$ if $\mathcal{P}$ is a join of at least two non-trivial properties, which are incomparable in $(\mathbf{L}^a, \subseteq)$. If $\mathcal{P}$ is a hereditary graph property, then the uniquely determined set of all *minimal forbidden graphs* for $\mathcal{P}$ is defined as follows [9]:

$\mathbf{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P}$ but for each proper subgraph H of G, $H \in \mathcal{P}\}$.

A *generating set* for $\mathcal{P} \in \mathbf{L}$ is a set $\mathcal{G}_{\mathcal{P}} \subseteq \mathcal{P}$ such that each $H \in \mathcal{P}$ is a subgraph of some $G \in \mathcal{G}_{\mathcal{P}}$. $\mathcal{G}_{\mathcal{P}}$ unlike $\mathbf{F}(\mathcal{P})$, is not uniquely determined for a given $\mathcal{P}$. We say that a graph property $\mathcal{P}$ is *generated* by a given set of graphs $\mathcal{G}$ if $\mathcal{P} = \{H :\ H \subseteq F$ for some $F \in \mathcal{G}\}$.

The assignment of colours to vertices of a graph such that two adjacent vertices are distinguished by their colours yields the proper graph colouring. Replacing the adjacency condition by some other features leads to generalized colouring notion. This concept was first considered by Cockayne (see [5]) and studied intensively by many researchers [3, 4, 8].

A graph property $\mathcal{P}$ is $\circ$-*reducible* over $\mathbf{L}^a$ if there are non-trivial properties $\mathcal{P}_1, \ldots, \mathcal{P}_n \in \mathbf{L}^a$, $n \geq 2$, satisfying $G \in \mathcal{P}$ if and only if its vertex-set can be partitioned into sets $V_1, \ldots, V_n$ (empty ones are allowed) so that $G[V_i]$ has a property $\mathcal{P}_i$ for each non-empty V_i.

In 2001 Berger proved that every graph property, which is $\circ$-reducible over $\mathbf{L}^a$ has infinitely many minimal forbidden graphs [1]. In this paper, we generalize her result.

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n \in \mathbf{L}^a$, G, $H = (\{v_1, v_2, \ldots, v_n\}, E) \in \mathcal{I}$. An $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$-*partition* of G is defined as a partition $(V_1, \ldots, V_n)$ of $V(G)$ (empty parts are allowed) such that the existence of $\{x_i, x_j\} \in E(G)$ with $x_i \in V_i$, $x_j \in V_j$, $i \neq j$ implies the existence of $\{v_i, v_j\} \in E$ and for each $i \in [n]$, with non-empty V_i, a graph $G[V_i]$ has a property $\mathcal{P}_i$.

The symbol $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$ denotes a class of all graphs possessing an $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$-partition. For $\mathcal{P} = H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$ with $\mathcal{P}_1, \ldots, \mathcal{P}_n \in \mathbf{L}^a \setminus \{\mathcal{I}, \emptyset\}$, we say that $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$ is an *H-factorization* (*factorization*) of $\mathcal{P}$ over $\mathbf{L}^a$, $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are *H-factors* (*factors*) of this factorization and

$\mathcal{P}$ is H-factorizable over $\mathbf{L}^a$. Of course, each graph property that is H-factorizable over $\mathbf{L}^a$ is in $\mathbf{L}^a$. An H-factorization $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$ of $\mathcal{P}$ over $\mathbf{L}^a$ is called *proper* if there exists a graph $G \in \mathcal{P}$ such that in each $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$-partition $(V_1, \ldots, V_n)$ of G all V_i are non-empty.

It is worth pointing out that $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n \geq 2$, are incomparable elements in the lattice $(\mathbf{L}^a, \subseteq)$ if and only if $\bar{K}_n[\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n]$ is a proper $\bar{K}_n$-factorization of $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2 \vee \cdots \vee \mathcal{P}_n$ over $\mathbf{L}^a$.

For a given $H \in \mathcal{I}$ we say that a graph property $\mathcal{P}$ is *H-reducible over* $\mathbf{L}^a$ if there exists a proper H-factorization of $\mathcal{P}$ over $\mathbf{L}^a$. Otherwise, $\mathcal{P}$ is called *H-irreducible over* $\mathbf{L}^a$. The concept of H-reducibility over $\mathbf{L}^a$, introduced in [6], covers $\vee$-reducibility in $(\mathbf{L}^a, \subseteq)$ and $\circ$-reducibility over $\mathbf{L}^a$. We shall show, in Section 2 (Theorem 3), that if a graph property $\mathcal{P}$ is H-reducible over $\mathbf{L}^a$ and $\delta(H) \geq 1$, then $\mathbf{F}(\mathcal{P})$ is infinite. Moreover, in Theorem 7, infiniteness of $\mathbf{F}(\mathcal{P})$ shall be observed for graph properties $\mathcal{P}$ that are H-reducible over $\mathbf{L}^a$ in the case $\delta(H) = 0$. Surprisingly, the condition $\delta(H) = 0$ forces equivalence between H-reducibility over $\mathbf{L}^a$ and $\vee$-reducibility in the lattice $(\mathbf{L}^a, \subseteq)$. Some results on this subject can be found in [7]. Our contribution to this topic is presented in Theorems 15 and 16.

2. H-Factorizations

Using the definitions introduced earlier we are in a position to formulate a theorem which generalizes the result from [1]. Its proof imitates the proof of the above mentioned result and must be preceded by two lemmas.

A cyclic block of a graph is its block containing a cycle.

Lemma 1 [1]. *Let $r \geq 2$ be an integer. Let $\mathcal{P} \in \mathbf{L}^a$ and suppose that $\mathbf{F}(\mathcal{P})$ is a set of graphs each with at least one cyclic block, such that the set of all the cyclic blocks forming the graphs in $\mathbf{F}(\mathcal{P})$ is finite. Next for each graph $G \in \mathcal{P}$ satisfying that for every $F \in \mathbf{F}(\mathcal{P})$ there exists a cyclic block of F which is not an induced subgraph of G, there exists a graph $H \in \mathcal{P}$ such that in any partition of $V(H)$ into r parts, G is the subgraph of a graph induced by at least one of the parts.*

Lemma 2 [1]. *Let $F_1, F_2, \ldots$ be a finite or infinite sequence of graphs, each with at least one cyclic block, such that the set of all the cyclic blocks making up $F_1, F_2, \ldots$ is finite. Then there exists a graph G of the form $F_i - I$, with*

$I \subseteq V(F_i)$ independent, such that for each $j \geq 1$, there is a cyclic block of F_j not contained in G.

Theorem 3. *Let H be a graph satisfying $\delta(H) \geq 1$ and let $\mathcal{P}$ be an H-factorizable over $\mathbf{L}^a$ property. Then a set of all cyclic blocks making up the graphs in $\mathbf{F}(\mathcal{P})$ is infinite and hence $\mathbf{F}(\mathcal{P})$ is infinite.*

Proof. Let $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$ for $n \geq 2, \mathcal{P}_1, \ldots, \mathcal{P}_n \in \mathbf{L}^a$ be an H-factorization of $\mathcal{P}$ over $\mathbf{L}^a$. Suppose that the set of all cyclic blocks making up the graphs in the family $\mathbf{F}(\mathcal{P})$ is finite. Because H contains at least one edge and for each $s \in \mathbb{N}$, $\bar{K}_s \in \mathcal{P}_i$, $i \in [n]$, we have that each bipartite graph has a property $\mathcal{P}$. Hence each graph in $\mathbf{F}(\mathcal{P}) = \{F_1, \ldots\}$ (finite or infinite) is non-bipartite and contains at least one cyclic block. Lemma 2 implies that there exist $i^* \in [n]$ and a graph $G = F_{i^*} - I$ with independent $I \subseteq V(F_{i^*})$ such that for each $j \geq 1$ one can find a cyclic block in F_j which is not contained in G. Of course, $G \in \mathcal{P}$. Lemma 1 for $r = n$ guarantees that there exists a graph $H^* \in \mathcal{P}$ such that in each partition of $V(H^*)$ into n parts at least one of them induces in H^* a graph with a subgraph G. Consider an arbitrary $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$-partition $(V_1, \ldots, V_n)$ of H^*. Now, consider a part V_j in which G is contained. Since $\delta(H) \geq 1$, there exists $k \in [n] \setminus \{j\}$ such that $\{v_j, v_k\} \in E(H)$. Therefore the graph $H^*[V_j] + \bar{K}_{|I|}$ has an $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$-partition and it is in $\mathcal{P}$. On the other hand $F_{i^*} \subseteq H^*[V_j] + \bar{K}_{|I|}$. The above shows that the set of cyclic blocks making up all graphs in the family $\mathbf{F}(\mathcal{P})$ is infinite and consequently $\mathbf{F}(\mathcal{P})$ is infinite. ∎

In what follows $\bar{\mathcal{Q}} = \mathcal{I} \setminus \mathcal{Q}$. For $\mathcal{P}_1, \mathcal{P}_2 \in \mathbf{L}^a$ and $\mathcal{P} = \bar{K}_2[\mathcal{P}_1, \mathcal{P}_2]$ we define the following sets:

$A_{\mathcal{P}} = \{G \in \mathbf{F}(\mathcal{P}) : G \in (\mathbf{F}(\mathcal{P}_1) \cap \bar{\mathcal{P}}_2) \cup (\mathbf{F}(\mathcal{P}_2) \cap \bar{\mathcal{P}}_1)\}$,
$B_{\mathcal{P}} = \{G \in \mathbf{F}(\mathcal{P}) \setminus A_{\mathcal{P}} :$ for every edge $e \in E(G)$, $G - e \in \mathcal{P}_1 \cup \mathcal{P}_2\}$,
$C_{\mathcal{P}} = \{G \in \mathbf{F}(\mathcal{P})$: there exists an edge $e \in E(G)$ such that $G - e \in \bar{\mathcal{P}}_1 \cap \bar{\mathcal{P}}_2\}$.
It is not difficult to see that $\mathbf{F}(\mathcal{P}) = A_{\mathcal{P}} \cup B_{\mathcal{P}} \cup C_{\mathcal{P}}$ [7].

Let $G_1 v_1 \overset{k}{\longleftrightarrow} v_2 G_2$ denote a graph obtained from the disjoint graphs G_1, G_2 by joining selected vertices v_1 of G_1 and v_2 of G_2 by a path of length k (with k edges). If the choice of vertices v_1, v_2 has no meaning, we use the notation $G_1 \overset{k}{\longleftrightarrow} G_2$. It was observed in [7] that if $\mathcal{P}_1, \mathcal{P}_2$ are two graph properties, which are incomparable in $(\mathbf{L}^a, \subseteq)$, then graphs in $C_{\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2]}$ are always of the form $G_1 \overset{k}{\longleftrightarrow} G_2$ with $G_1 \in \mathbf{F}(\mathcal{P}_1) \cap \mathcal{P}_2$, $G_2 \in \mathbf{F}(\mathcal{P}_2) \cap \mathcal{P}_1$, $k \in \mathbb{N}$. From the same paper it follows that for each graph G in $B_{\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2]}$ the set of its edges is a union of sets E_1, E_2 such that the graph $G[E_1]$ induced in G by E_1 is an element of $\mathbf{F}(\mathcal{P}_1)$ and similarly $G[E_2] \in \mathbf{F}(\mathcal{P}_2)$.

Lemma 4. *Let $\bar{K}_n[\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n]$, $n \in \mathbb{N}$, be a factorization of $\mathcal{P}$ over $\mathbf{L}^a$. If for each $F \in \mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2) \cup \cdots \cup \mathbf{F}(\mathcal{P}_n)$ the condition $\delta(F) \geq 2$ is satisfied, then $\delta(F^*) \geq 2$ for each $F^* \in \mathbf{F}(\bar{K}_n[\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n])$.*

Proof. We can assume that $\bar{K}_n[\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n]$ is the proper factorization of $\mathcal{P}$, which implies that $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are graph properties which are incomparable in $(\mathbf{L}^a, \subseteq)$. Moreover, each $F^* \in \mathbf{F}(\bar{K}_n[\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n])$ is connected because of $\mathcal{P} \in \mathbf{L}^a$. For $n = 1$ the assertion is obvious, for $n = 2$ it follows from the definitions of $A_{\mathcal{P}}$, $B_{\mathcal{P}}$, $C_{\mathcal{P}}$ and the construction of graphs in $C_{\mathcal{P}}$ and $B_{\mathcal{P}}$. In a general case, it can be obtained by the induction on n, using associativity of the operation $\vee$. ∎

Lemma 5. *Let $H = (\{v_1, \ldots, v_n\}, E)$ be a connected graph on at least two vertices and let $\mathcal{P}$ be a property which is H-factorizable over $\mathbf{L}^a$. Then for an arbitrary $F \in \mathbf{F}(\mathcal{P})$ the condition $\delta(F) \geq 2$ is satisfied.*

Proof. Recall that each $F \in \mathbf{F}(\mathcal{P})$ is connected because of $\mathcal{P} \in \mathbf{L}^a$. Let us assume that $F^* \in \mathbf{F}(\mathcal{P})$ and $\delta(F^*) = 1$. Let $v \in V(F^*)$ be a vertex of degree 1 in F^* and w be a neighbour of v in F^*. It is clear that $F^* - v \in \mathcal{P}$. Consider an arbitrary $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$-partition $(V_1, \ldots, V_n)$ of $F^* - v$. Let $l \in [n]$ be an index satisfying $w \in V_l$. Because $deg_H(v_l) \geq 1$, we know that there exists an index $s \in [n] \setminus \{l\}$ such that $\{v_s, v_l\} \in E(H)$. It is evident that $(V_1, \ldots, V_s \cup \{v\}, \ldots, V_n)$ is an $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$-partition of F^*, contrary to the assumption $F^* \in \mathbf{F}(\mathcal{P})$. ∎

To obtain the main result of this section we have to recall the following lemma.

Lemma 6 [7]. *Let $\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2]$, be a proper factorization of $\mathcal{P}$ over $\mathbf{L}^a$. If for each $F \in \mathbf{F}(\mathcal{P}_1) \cup \mathbf{F}(\mathcal{P}_2)$ the condition $\delta(F) \geq 2$ is satisfied, then $\mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])$ is infinite.*

Theorem 7. *Let $H = (\{v_1, \ldots, v_n\}, E)$ be a graph, $n \geq 2$, and let $H[\mathcal{P}_1, \ldots, \mathcal{P}_n]$ be a proper H-factorization of $\mathcal{P}$ over $\mathbf{L}^a$ satisfying that for each $i \in [n]$ with $deg_H(v_i) = 0$ and arbitrary $F \in \mathbf{F}(\mathcal{P}_i)$ the condition $\delta(F) \geq 2$ holds. Then the family $\mathbf{F}(\mathcal{P})$ is infinite.*

Proof. Denote $W_1 = \{v_j : deg_H(v_j) = 0\}$, $W_2 = V(H) \setminus W_1$. The proof will be divided into three parts:

(1) $W_1 = \emptyset$

The assertion follows by Theorem 3.

(2) $W_2 = \emptyset$

In this case we can apply Lemmas 4, 6 and associativity of the operation $\vee$.

(3) $W_1 \neq \emptyset$ and $W_2 \neq \emptyset$

Without restriction of generality assume that $W_1 = \{v_1, \ldots, v_j\}$, $W_2 = \{v_{j+1}, \ldots, v_n\}$ for some $j \in [n-1]$. Let $H_1 = H[W_1]$, $H_2 = H[W_2]$. The conclusion holds by Lemmas 4, 5, 6. It is so because $\mathcal{P} = H_1[\mathcal{P}_1, \ldots, \mathcal{P}_j] \vee H_2[\mathcal{P}_{j+1}, \ldots, \mathcal{P}_n] = \mathcal{P}_1 \vee \mathcal{P}_2 \vee \cdots \vee \mathcal{P}_j \vee H_2[\mathcal{P}_{j+1}, \ldots, \mathcal{P}_n] = \bar{K}_{j+1}[\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_j, H_2[\mathcal{P}_{j+1}, \ldots, \mathcal{P}_n]]$ and properties $\mathcal{P}_1, \ldots, \mathcal{P}_j, H_2[\mathcal{P}_{j+1}, \ldots, \mathcal{P}_n]$ are incomparable in $(\mathbf{L}^a, \subseteq)$. ■

Let H be a graph on at least two vertices. The only graph property $\mathcal{P}$ that is H-reducible over $\mathbf{L}^a$ which is left to be considered with respect to the finiteness of the family of all minimal forbidden graphs has the following properties:

- $\mathcal{P}$ is $\bar{K}_2$-reducible over $\mathbf{L}^a$ ($\mathcal{P}$ is a join of two incomparable properties $\mathcal{P}_1, \mathcal{P}_2$ in $(\mathbf{L}^a, \subseteq)$),
- at least one out of the properties $\mathcal{P}_1, \mathcal{P}_2$ is H^*-irreducible over $\mathbf{L}^a$ for each connected H^* satisfying $|V(H^*)| \geq 2$ and its minimal forbidden graph family contains a graph F such that $\delta(F) = 1$.

In the next section, we try to give as much information as possible about such properties in terms of families $\mathbf{F}(\mathcal{P}_1)$, $\mathbf{F}(\mathcal{P}_2)$.

3. Quasi-Orders

We are going to show a theorem which will be sharper than the one which was proven in [7]. First, we shall recall the definitions of a ranking number of a graph [4] and the well quasi-ordered set. Then we shall establish the connection between these notions and the finiteness of the family of all minimal forbidden graphs for a property $\mathcal{P} \in \mathbf{L}^a$. For all undefined notions in the section please refer to [10].

Let A be an arbitrary set and ρ be a relation defined on it. A couple (A, ρ) is called a *quasi-ordered set* if ρ is reflexive and transitive on A. Below, we cite a theorem giving equivalent conditions for a quasi-ordered set to be a well-quasi-ordered set.

Theorem 8 [10]**.** *If (A, ρ) is a quasi-ordered set, then the following conditions are equivalent.*

(1) *A has no infinite strictly ρ-decreasing sequence and no infinite ρ-antichain.*

(2) *For each infinite sequence $a_1, a_2, \ldots$ of elements in A there exist i and j such that $i < j$ and $a_i \rho a_j$.*

(3) *Each infinite sequence of elements in A has an infinite strictly ρ-increasing subsequence.*

(4) *For each subset B of A, the set of ρ-minimal elements in B is finite.*

For a given path $P = v_1, \ldots, v_n$ and a disjoint vertex u, by (u, P), $P(v_i, v_j)$, we denote a path $u, v_1, v_2, \ldots, v_n$ and a subpath of P joining v_i and v_j, respectively. If we are interested only in end-vertices of P, we can denote it as (v_1, v_n)-path.

Given a graph $G = (V, E) \in \mathcal{I}$, a mapping $c : V \longrightarrow [k]$ such that for each pair of vertices u, v satisfying $c(u) = c(v)$ and for each (u, v)-path in G there exists an internal vertex z of this path, such that $c(z) > c(u)$ is called *an ordered k-colouring of G*. We define *a ranking number of a graph G*, $\chi_0(G)$, as the smallest integer k such that G possesses an ordered k-colouring.

Let Q_i be a path with an end-vertex x_0^i, $i \in [n]$ and $H = (\{v_1, \ldots, v_n\}, E)$ be a graph such that $H, Q_1, \ldots, Q_n$ are disjoint. Let G be a graph obtained by identifying each x_0^i with a vertex v_i, $i \in [n]$. We say that H is a *core* of G. Putting G instead of H and taking all paths $Q_1, \ldots, Q_n$ having only one vertex, we see that G can be a core of itself. In general, a graph G does not have a unique core. By $\mathcal{B}_k$ we denote the set of all graphs G such that there exists a core H of G satisfying $\chi_0(H) \leq k$. All notions defined above were introduced in [12].

Let G_i stand for a path with three vertices and a marked vertex v_i of degree two, $i = 1, 2$ and let the symbol H_m, $m \in \mathbb{N}$, denote the graph $G_1 v_1 \stackrel{m}{\longleftrightarrow} v_2 G_2$. It was shown in [12] that the graph H_m defined above and the cycle C_m on m vertices are basic in the theorem whose part we formulate below preceding it with a useful lemma.

Lemma 9 [13]**.** *If every path of a graph G has the length at most $k - 1$, then $\chi_0(G) \leq k$.*

Theorem 10 [12]. *Let $k \in \mathbb{N}$ and $G \notin \mathcal{B}_k$. Then G contains a subgraph H_m or C_m where m is large relatively to k.*

Proof. Without loss of generality we can assume that G is connected (otherwise we can consider an arbitrary component of G). Choose a core H of G that is minimal with respect to the subgraph relation. H does not have an ordered k-colouring. Let $P = u_1, \ldots, u_l$ be the longest path of H. By Lemma 9, $k \leq l$. Let y be a neighbour of u_1 in G. If y is in $V(H) \setminus \{u_1, \ldots, u_l\}$ then (y, P) is a longer path of H than P. If y is in $V(\mathcal{P})$ and the length of $P(u_1, y)$ is large relatively to k then the length of the cycle $C_m = P(u_1, y), u_1$ is large relatively to k. Hence $y \in V(G) \setminus V(H)$ or y is in $V(P)$ and the length of $P(u_1, y)$ is small relatively to k. If u_2 is the only vertex of H adjacent to u_1 and u_1 is adjacent to at most one vertex in $V(G) \setminus V(H)$, then $H - u_1$ is also a core of G, contrary to minimality of H. Therefore u_1 is adjacent to a vertex u_i such that $i \geq 3$ and i is small relatively to k or u_1 is adjacent to at least two vertices in $V(G) \setminus V(H)$. Similarly, u_l is adjacent to at least two vertices in $V(G) \setminus V(H)$. It implies that G has an H_m subgraph such that m is large relatively to k. ∎

Corollary 11. *Let $\mathcal{G}$ be a set of graphs such that $\mathcal{G} \not\subseteq \mathcal{B}_k$ for each $k \in \mathbb{N}$. Then there exists either an infinite family $\{C_{n_i} : i \in \mathbb{N}\}$ or an infinite family $\{H_{n_i} : i \in \mathbb{N}\}$ of graphs, whose elements have the property generated by $\mathcal{G}$.*

The next theorem characterizes hereditary graph properties with infinite families of all minimal forbidden graphs. It is based on the following result.

Lemma 12 [12]. *$(\mathcal{B}_k, \subseteq)$ is a well-quasi-ordered set.*

Theorem 13. *Let $\mathcal{P} \in \mathbf{L}$. The set $\mathbf{F}(\mathcal{P})$ is infinite if and only if the graphs in $\mathbf{F}(\mathcal{P})$ contain an infinite family $\{H_{n_i};\ i \in \mathbb{N}\}$ or an infinite family $\{C_{n_i};\ i \in \mathbb{N}\}$ as subgraphs.*

Proof. Suppose that $\mathbf{F}(\mathcal{P})$ is infinite and its elements do not contain neither an infinite family of cycles nor an infinite family of graphs H_{n_i}, $i \in \mathbb{N}$, as subgraphs. Hence, in accordance to Corollary 11, the property generated by $\mathbf{F}(\mathcal{P})$ is contained in $\mathcal{B}_k$ for some $k \in \mathbb{N}$. By Theorem 8 and Lemma 12 any subset of $\mathcal{B}_k$ contains only finite antichains. But $\mathbf{F}(\mathcal{P})$ creates an antichain in $\mathcal{B}_k$, a contradiction.

If $\mathbf{F}(\mathcal{P})$ is a finite set, then it contains no infinite antichains from $(\mathcal{I}, \subseteq)$, in particular, neither $\{H_{n_i};\ i \in \mathbb{N}\}$ nor $\{C_{n_i};\ i \in \mathbb{N}\}$. ∎

Lemma 14. *Let $\mathcal{P} \in \mathbf{L}$ be a property for which there exists $k \in \mathbb{N}$ such that a path on k vertices is a minimal forbidden graph for $\mathcal{P}$. Then the set $\mathbf{F}(\mathcal{P})$ is finite.*

Proof. Let us suppose that $\mathbf{F}(\mathcal{P})$ is infinite. Thus, by Theorem 13 graphs of the family $\mathbf{F}(\mathcal{P})$ contain an infinite antichain $\{H_{n_i} : i \in \mathbb{N}\}$ or $\{C_{n_i} : i \in \mathbb{N}\}$, as subgraphs. It is not possible because there exists $i \in \mathbb{N}$ such that for each $j > i$, H_{n_j} and C_{n_j} contain a path on k vertices as a subgraph, contrary to the fact that elements of $\mathbf{F}(\mathcal{P})$ are incomparable in the sense of the subgraph-relation. ■

Let $\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of G has at most $k+1$ vertices$\}$.

The next theorem depends strongly on the fact which has been just proven. It generalizes the result from [7] stating the finiteness of $\mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])$ in the language of $\mathbf{F}(\mathcal{P}_2)$ provided $\mathcal{P}_1 \subseteq \mathcal{O}_k$ for some $k \in \mathbb{N}$ and $\mathcal{P}_1, \mathcal{P}_2$ are incomparable in $(\mathbf{L}^a, \subseteq)$ properties.

Theorem 15. *Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathbf{L}^a$ be graph properties, which are incomparable in $(\mathbf{L}^a, \subseteq)$. If there exists $k \in \mathbb{N}$ such that a path on k vertices is an element of $\mathbf{F}(\mathcal{P}_1)$, then $\mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])$ is finite if and only if $\mathbf{F}(\mathcal{P}_2)$ is finite.*

Proof. Suppose that $\mathbf{F}(\mathcal{P}_2)$ is a finite set. Because of Lemma 14, $\mathbf{F}(\mathcal{P}_1)$ is a finite set, too. Hence we have the finiteness of $A_{\bar{K}_2[\mathcal{P}_1,\mathcal{P}_2]}$ and $B_{\bar{K}_2[\mathcal{P}_1,\mathcal{P}_2]}$ in accordance with their constructions. Moreover, the parameter s in graphs $F_1 v_1 \overset{s}{\longleftrightarrow} v_2 F_2$ of the family $C_{\bar{K}_2[\mathcal{P}_1,\mathcal{P}_2]}$ is bounded above, which implies the finiteness of the set $C_{\bar{K}_2[\mathcal{P}_1,\mathcal{P}_2]}$ and consequently the finiteness of $\mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])$.

Let $\mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])$ be finite. If $\mathbf{F}(\mathcal{P}_2)$ is an infinite set then, by Theorem 13, graphs of the family $\mathbf{F}(\mathcal{P}_2)$ contain an infinite family $\mathcal{C} = \{C_{n_i} : i \in \mathbb{N}\}$ or $\mathcal{H} = \{H_{n_i} : i \in \mathbb{N}\}$, as subgraphs. Next it is possible to choose an infinite family of graphs in $\mathbf{F}(\mathcal{P}_2)$, whose subgraphs are elements of $\mathcal{C}$ or $\mathcal{H}$. These graphs, beginning with some number of vertices are the graphs of the family $A_{\bar{K}_2[\mathcal{P}_1,\mathcal{P}_2]}$ simultaneously, because each of them contains a path on k vertices as a subgraph, a contradiction. ■

Lemma 6 and Theorem 15 give us the possibility of constructing properties that are $\bar{K}_2$-reducible over $\mathbf{L}^a$ with infinitely many minimal forbidden graphs irrespective of the finiteness of minimal forbidden graph families for factors. Using Theorem 15 we can construct graph properties having finite families of minimal forbidden graphs that are $\bar{K}_2$-reducible over $\mathbf{L}^a$.

In this case the factors constructed properties have finite families of minimal forbidden graphs too. Unfortunately, all other possibilities are permissible. For comprehensibility, we illustrate this fact by the following examples. Let C_5^* be a cycle on five vertices with one chord and $K_{1,3}$ be a star on four vertices. Consider properties $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2 \in \mathbf{L}^a$ such that $\mathbf{F}(\mathcal{P}_1) = \{C_5\}$, $\mathbf{F}(\mathcal{P}_2) = \{K_{1,3}\}$ $\mathbf{F}(\mathcal{Q}_1) = \{C_n : n \text{ is odd}\} \cup \{K_{1,3}\}$, $\mathbf{F}(\mathcal{Q}_2) = \{C_n : n$ is even$\} \cup \{K_{1,3}\}$. It is easy to verify that $\mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2]) = \{C_5^*, C_5 \stackrel{1}{\longleftrightarrow} K_1\}$ and $\mathbf{F}(\bar{K}_2[\mathcal{Q}_1, \mathcal{Q}_2]) = \{K_{1,3}\}$.

The last example corresponds with a new result, presented below, which seems to be helpful for obtaining a full characterization of properties which are H-reducible over $\mathbf{L}^a$ in the aspect under consideration.

Theorem 16. *Let $\mathcal{P}_1, \mathcal{P}_2$ be graph properties which are incomparable in $(\mathbf{L}^a, \subseteq)$ and let sets $\mathbf{F}(\mathcal{P}_1)$, $\mathbf{F}(\mathcal{P}_2)$ be infinite. If $\mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])$ is finite, then there exist graphs $F^{**} \in \mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2]) \cap \mathbf{F}(\mathcal{P}_i)$ and $F_2 \in \mathbf{F}(\mathcal{P}_j)$ such that $\delta(F^{**}) = \delta(F_2) = 1$ where i, j are different indices.*

Proof. Suppose $l = \max\{|V(F)| : F \in \mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])\}$, $\mathcal{A}_i = \{F \in \mathbf{F}(\mathcal{P}_i) : |V(F)| > l\}$, $i = 1, 2$. It is obvious that $\mathcal{A}_i$ are infinite sets for $i = 1, 2$. Let $F_i \in \mathcal{A}_i$ be a fixed graph, $i = 1, 2$. It is clear that $F_i \in \mathbf{F}(\mathcal{P}_i) \cap \mathcal{P}_j$, $i \neq j$, $i, j = 1, 2$. Otherwise, F_i should be a minimal forbidden graph for $\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2]$, which is impossible by the assumption $|V(F_i)| > l$. Let $F^* = F_1 \stackrel{l+1}{\longleftrightarrow} F_2$. Thus $F^* \in \bar{\mathcal{P}}_1 \cap \bar{\mathcal{P}}_2$ and consequently it contains a graph $F^{**} \in \mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])$. Obviously, F^{**} is neither a proper subgraph of F_1 nor F_2. Moreover, because the path joining F_1 and F_2 in F^{**} is of length $l+1$, it is not possible for $V(F^{**})$ to overlap $V(F_1)$ and $V(F_2)$. Hence either $F^{**} \subseteq F_1 \stackrel{l+1}{\longleftrightarrow} K_1$ or $F^{**} \subseteq K_1 \stackrel{l+1}{\longleftrightarrow} F_2$ and F^{**} contains at least one vertex of the path of length $l+1$. By additivity of $\mathcal{P}$ it means that $\delta(F^{**}) = 1$.

The construction $F_1 \stackrel{l+1}{\longleftrightarrow} F_2$ can be done infinitely many times taking all possible graphs $F_i \in \mathcal{A}_i$, $i = 1, 2$. Consider two sequences $(F_1^i)_{i=1}^{\infty} \subseteq \mathcal{A}_1$, $(F_2^j)_{j=1}^{\infty} \subseteq \mathcal{A}_2$ such that $l + 1 \leq |V(F_1^1| < |V(F_1^2| < \cdots$ and $l + 1 \leq |V(F_2^1)| < |V(F_2^2)| < \cdots$. The existence of such sequences follows by the assumption $|\mathbf{F}(\mathcal{P}_1)| = |\mathbf{F}(\mathcal{P}_2)| = \infty$. Previously we showed that for each graph $F_1^i \stackrel{l+1}{\longleftrightarrow} F_2^j$ there exists $F^{**} \in \mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])$ which is contained in $F_1^i \stackrel{l+1}{\longleftrightarrow} K_1$ or $F_2^j \stackrel{l+1}{\longleftrightarrow} K_1$. Because the number of such graphs F^{**} is finite, there exist sequences $(F_1^{n_i})_{i=1}^{\infty}$, $(F_2^{n_j})_{j=1}^{\infty}$ satisfying that fixed $F^{**} \in \mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2])$ is a subgraph of each graph $F_1^{n_i} \stackrel{l+1}{\longleftrightarrow} F_2^{n_j}$. The numbers of vertices of the graphs $F_1^{n_i}$, $F_2^{n_j}$ tend to infinity if i and j increase. Thus the fixed F^{**} starting with suitable numbers i, j contains a proper subgraph of $F_1^{n_i}$ or a proper subgraph of $F_2^{n_j}$ as a subgraph. It implies that the minimal forbidden subgraphs $F_1^{'} \in \mathbf{F}(\mathcal{P}_1)$, $F_2^{'} \in \mathbf{F}(\mathcal{P}_2)$, which are included in F^{**} satisfy $\delta(F_1^{'}) = \delta(F_2^{'}) = 1$. If F^{**} is forbidden for $\mathcal{P}_1$ or for $\mathcal{P}_2$, then the last part of the assertion is true. If not, use graphs $F_1^{'}, F_2^{'}$ to construct $F_3 = F_1^{'} \stackrel{l+1}{\longleftrightarrow} F_2^{'}$, which does not possess the property $\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2]$. Take again a graph $F_4 \in \mathbf{F}(\bar{K}_2[\mathcal{P}_1, \mathcal{P}_2]) \subseteq F_3$, but in such a way that it uses the biggest number of edges of $l+1$-path. Such a graph guarantees the assertion.

■

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