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ON INFINITE UNIQUELY PARTITIONABLE GRAPHS AND GRAPH PROPERTIES OF FINITE CHARACTER

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Abstract

A graph property is any nonempty isomorphism-closed class of simple (finite or infinite) graphs. A graph property $\mathcal{P}$ is of *finite character* if a graph G has a property $\mathcal{P}$ if and only if every finite induced subgraph of G has a property $\mathcal{P}$. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties of finite character, a graph G is said to be (uniquely) $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable if there is an (exactly one) partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that $G[V_i] \in \mathcal{P}_i$ for $i = 1, 2, \ldots, n$. Let us denote by $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ the class of all $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs. A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n, n \geq 2$ is said to be reducible. We prove that any reducible additive graph property $\mathcal{R}$ of finite character has a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable countable generating graph. We also prove that for a reducible additive hereditary graph property $\mathcal{R}$ of finite character there exists a weakly universal countable graph if and only if each property $\mathcal{P}_i$ has a weakly universal graph.

Keywords: graph property of finite character, reducibility, uniquely partitionable graphs, weakly universal graph.

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1. Introduction

In this paper we deal with the generators of reducible graph properties of finite character and their relations to uniquely partitionable countable graphs. In general, we follow standard graph terminology (see e.g. [9]). Let us denote by $\mathcal{I}^\omega, \mathcal{I}$ and $\mathcal{I}^{conn}$ the class of simple countable graphs, simple finite graphs and simple finite connected graphs, respectively. All our considerations can be done for arbitrary infinite graphs, however, in order to avoid formal set-theoretical problems, we shall consider only countable infinite graphs. Moreover, we assume that the vertex set $V(G)$ of a graph G is a subset of a given countable set. A graph property $\mathcal{P}$ is any isomorphism-closed nonempty subclass of $\mathcal{I}^\omega$. It means that investigating graph properties, in principle, we restrict our considerations to unlabeled graphs. We also say that a graph G has the property $\mathcal{P}$ if $G \in \mathcal{P}$. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties, a (vertex) $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition (colouring) of a graph $G = (V, E)$ is a partition $\{V_1, V_2, \ldots, V_n\}$ of V such that each partition class V_i induces a subgraph $G[V_i]$ having property $\mathcal{P}_i$. If a graph G possesses a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition, we say that G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable. Let us denote by $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ the class of all $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs. A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n, n \geq 2$ is said to be *reducible*, a property $\mathcal{P}$ which cannot be expressed in the form $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n, n \geq 2$ is said to be *irreducible*. For convenience, we allow empty partition classes in $\{V_1, V_2, \ldots, V_n\}$. An empty partition class induces the null graph $K_0 = (\emptyset, \emptyset) \in \mathcal{I}$. If for each $i = 1, 2, \ldots, n$, the property $\mathcal{P}_i$ is the property $\mathcal{O}$ of being edgeless, we have a proper n-colouring, thus reducibility can be considered as generalization of n-colourability. Many other examples, references and results on generalized colourings of finite graphs may be found e.g. in the survey [1].

In 1951, de Bruijn and Erdős proved that an infinite graph G is k-colourable if and only if every finite subgraph of G is k-colourable. An analogous compactness theorem for generalized colourings was proved in [6]. The key concept for the Vertex Colouring Compactness Theorem VCCT [6] is that properties are of *finite character*. Let $\mathcal{P}$ be a graph property, $\mathcal{P}$ is of *finite character* if a graph in $\mathcal{I}^\omega$ has the property $\mathcal{P}$ if and only if each its finite vertex-induced subgraph has the property $\mathcal{P}$. It is easy to see that if $\mathcal{P}$ is of finite character and a graph has the property $\mathcal{P}$ then so does every induced subgraph. A property $\mathcal{P}$ is said to be *induced-hereditary* if $G \in \mathcal{P}$ and $H \leq G$ imply $H \in \mathcal{P}$, that is $\mathcal{P}$ is closed under taking induced subgraphs. Thus properties of finite character are induced-hereditary. However not all induced-hereditary properties are of finite character; for example the graph property $\mathcal{Q}$ of not containing a vertex of infinite degree is induced-hereditary but not of finite character. Let us also remark that every property which is hereditary with respect to every subgraph (we say simply *hereditary*) is induced-hereditary as well. A property $\mathcal{P}$ is said to be *additive* if it is closed under taking disjoint union of graphs.The properties of being edgeless, of maximum degree at most k, K_n-free, acyclic, complete, perfect, etc. are additive properties of finite character. Let us denote by $\mathbb{M}^{af}$ the class of all additive graph properties of finite character. Throughout this paper all graph properties, which are considered, are additive graph properties of finite character, all such properties can be characterized by finite connected minimal forbidden subgraphs (see [6, 11]. The compactness theorem for $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitions (colourings), where the $\mathcal{P}_i$'s are of finite character, have been proved using Rado's Selection Lemma in [6]:

Theorem 1 (VCCT) [6]. *Let G be a graph in $\mathcal{I}^\omega$ and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties of finite character. Then G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable if every finite induced subgraph of G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable.*

This theorem implies that if the graph properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are of finite character, then also the reducible property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n, n \geq 2$ is of finite character. A graph G of order at least n is said to be *uniquely* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partitionable* if there is exactly one (unordered) $\mathcal{P}_1, \mathcal{P}_2, \ldots,$ $\mathcal{P}_n$-partition. The class of all uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs will be denoted by $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n)$. The binary operation "$\circ$" of additive and hereditary properties of finite graphs have been considered in details in [2, 3]. For technical reasons we consider also the property $\Theta = \{K_0\}$ being the smallest graph property in the lattice $(\mathbb{M}^{af}, \subseteq)$ of all additive induced-hereditary properties of finite character partially ordered by set-inclusion (see [1, 14]). More details on the lattices of hereditary properties may be found in [1] and in Section 3. The properties $\mathcal{I}^\omega$ and Θ of finite character are said to be *trivial*, since for every property $\mathcal{P} \in \mathbb{M}^{af}$, $\Theta \circ \mathcal{P} = \mathcal{P} \circ \Theta = \mathcal{P}$ and $\mathcal{I}^\omega \circ \mathcal{P} = \mathcal{P} \circ \mathcal{I}^\omega = \mathcal{I}^\omega$.

Some basic properties of infinite countable uniquely partitionable graphs with respect to additive graph properties of finite character, based on Theorem 1, are presented in Section 2.

As it have been proved in [15], if $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is the unique factorization of the additive property $\mathcal{R}$ of finite character, then there exists a countable uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph G, which "'generates"' $\mathcal{R}$. We will present more details and apply this result to show the existence of weakly universal graphs for reducible graph properties of finite character in Section 3. We conclude this paper with an open problem on universal graphs.

2. Preliminary Results

We will need some more notions and preliminary results.

The following proposition summarises the basic properties of infinite uniquely partitionable graphs. We omit here the simple proofs, which are the same as the proofs for finite uniquely colourable graphs (see [10] and [4, 12]).

Proposition 1. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n \geq 2$, be any nontrivial additive graph properties of finite character, let G be a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph and let $\{V_1, V_2, \ldots, V_n\}$ be the unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of G, $n \geq 2$. Then*

1. *$G \notin \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_{j-1} \circ \mathcal{P}_{j+1} \circ \cdots \circ \mathcal{P}_n$, for every $j = 1, 2, \ldots, n$,*
2. *for $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$ the set $V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}$ induces a uniquely $(\mathcal{P}_{i_1} \circ \mathcal{P}_{i_2} \circ \cdots \circ \mathcal{P}_{i_k})$-partitionable subgraph of G,*
3. *for every $j = 1, 2, \ldots, n$ the graph G^j_w obtained from G by adding a vertex w and edges joining w to vertices of the set $V_i, i \neq j$, such that $G[V_i \cup \{w\}] \notin \mathcal{P}_i$ for $i = 1, 2, \ldots, j-1, j+1, \ldots, n$, is uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable and $\{V_1, \ldots, V_j \cup \{w\}, \ldots, V_n\}$ is its unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition,*
4. *let $H \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$, $V(H) \cap V(G) = \emptyset$ and $\{W_1, W_2, \ldots, W_n\}$ be a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of $V(H)$. Let the graph $G_H = (V(G) \cup V(H), E(G) \cup E(H) \cup E^*)$ be obtained from $G \cup H$ by adding edges so that for every $j = 1, 2, \ldots, n$ and for each $w \in W_j$ $G_H[V(G) \cup \{w\}] = G^j_w$, then G_H is uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable and $\{V_1 \cup W_1, V_2 \cup W_2, \ldots, V_n \cup W_n\}$ is its unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition.*

To characterize the existence of uniquely partitionable graphs the notion of divisibility for the binary operation "$\circ$" on $\mathbb{M}^{af}$ is used in a natural way.

Given any two graph properties $\mathcal{R}, \mathcal{P} \in \mathbb{M}^{af}$, we say that $\mathcal{P}$ is a *divisor* of $\mathcal{R}$, if $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ for some property $\mathcal{Q} \in \mathbb{M}^{af}$, we can also say that $\mathcal{P}$ *divides* $\mathcal{R}$ and that $\mathcal{R}$ is divisible by $\mathcal{P}$.

Let $\mathcal{P}, \mathcal{Q} \in \mathbb{M}^{af}$. We say that the additive induced-hereditary property $\mathcal{D}$ of finite character is a *greatest common divisor* of $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{D} = gcd(\mathcal{P}, \mathcal{Q})$ if

(1) $\mathcal{D}$ divides $\mathcal{P}$ and $\mathcal{D}$ divides $\mathcal{Q}$;

(2) if $\mathcal{D}' \in \mathbb{M}^{af}$ divides $\mathcal{P}$ and $\mathcal{D}'$ divides $\mathcal{Q}$, then $\mathcal{D}'$ divides $\mathcal{D}$.

Obviousely, a non-trivial additive induced-hereditary property $\mathcal{P}$ of finite character is irreducible, if the only additive induced-hereditary properties which divide $\mathcal{P}$ are Θ and $\mathcal{P}$ itself and *reducible* otherwise.

The introduced notions are well-defined since any additive graph property of finite character can be expressed as a product of irreducible additive induced-hereditary properties of finite character in a unique way.

Theorem 2 [11]. *Every nontrivial additive property of finite character is uniquely* (*up to the order of factors*) *factorizable into finite number of irreducible graph properties belonging to* $\mathbb{M}^{af}$.

Hence any reducible property $\mathcal{R} \in \mathbb{M}^{af}$ can be written as $\mathcal{R} = {\mathcal{P}_1}^{e_1} \circ {\mathcal{P}_2}^{e_2} \circ \cdots \circ {\mathcal{P}_n}^{e_n}$, where $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are distinct irreducible properties from $\mathbb{M}^{af}$ and $e_1, e_2, \ldots, e_r$ are positive integers. Using the symbol $\mathcal{P}^0$ to denote the property Θ, one can clearly use this type of factorization to describe the greatest common divisor of any two properties similar to the way it is done in Number Theory.

The following result is a classical corollary of the proof of Unique Factorization Theorem for finite graphs. It have been presented in [11].

Theorem 3 [11]. *Let* $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$, $n \geq 2$ *be a factorization of a reducible property* $\mathcal{R} \in \mathbb{M}^{af}$ *into irreducible factors. Then* $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n) \neq \emptyset$ *and moreover if* $H \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n \cap \mathcal{I}$, *then* H *is an induced subgraph of some uniquely* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partitionable graph* G.

Based on the results presented in [3] and [11] the following theorem holds.

Theorem 4. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, $n \geq 2$, be any non-trivial additive graph properties of finite character. Then there exists a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph if and only if for each $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ we have that $gcd(\mathcal{P}_i, \mathcal{P}_j) = \Theta$ or $\mathcal{P}_i = \mathcal{P}_j$ is an irreducible property.*

Proof. The proof is going in the same way as in [3]. We recall here the main parts. It is enough to consider $n = 2$ only, since the presented arguments can be repeated in the case $n \geq 3$, analogously.

Suppose that G is a uniquely $(\mathcal{P}_1, \mathcal{P}_2)$-partitionable graph and there exists an irreducible property $\mathcal{Q}$ such that $\mathcal{P}_1 = \mathcal{Q} \circ \mathcal{P}_1'$ and $\mathcal{P}_2 = \mathcal{Q} \circ \mathcal{P}_2'$ and at least $\mathcal{P}_1' \neq \Theta$. Let $\{V_1, V_2\}$ be any $(\mathcal{P}_1, \mathcal{P}_2)$-partition of the graph G. Since $\mathcal{P}_1 = \mathcal{Q} \circ \mathcal{P}_1'$ and $\mathcal{P}_2 = \mathcal{Q} \circ \mathcal{P}_2'$ let $\{V_{11}, V_{12}\}$ ($\{V_{21}, V_{22}\}$) be the $(\mathcal{Q}, \mathcal{P}_1')$-partition ($(\mathcal{Q}, \mathcal{P}_2')$-partition) of $G[V_1]$ ($G[V_2]$). Then we have at least the following two different $(\mathcal{P}_1, \mathcal{P}_2)$-partitions $\{V_{11} \cup V_{12}, V_{21} \cup V_{22}\}$ and $\{V_{21} \cup V_{12}, V_{11} \cup V_{22}\}$ of G because we can assume that V_{11}, V_{21} and V_{12} are not empty.

To prove the converse, Theorem 3 can be applied if $\mathcal{P}_1 = \mathcal{P}_2$ is an irreducible property. Suppose that $gcd(\mathcal{P}_1, \mathcal{P}_2) = \Theta$. By Theorem 2 let $\mathcal{P}_1 = \mathcal{P}_{11} \circ \mathcal{P}_{12} \circ \cdots \circ \mathcal{P}_{1n}$ and $\mathcal{P}_2 = \mathcal{P}_{21} \circ \mathcal{P}_{22} \circ \cdots \circ \mathcal{P}_{2m}$ be the unique factorizations of $\mathcal{P}_1$ and $\mathcal{P}_2$ into irreducible factors. From our assumption that $gcd(\mathcal{P}_1, \mathcal{P}_2) = \Theta$ it follows that for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ $\mathcal{P}_{1i} \neq \mathcal{P}_{2j}$.

Let us take a uniquely $(\mathcal{P}_{11}, \ldots, \mathcal{P}_{1n}, \mathcal{P}_{21}, \ldots, \mathcal{P}_{2m})$-partitionable graph G, which exists by Theorem 3. Let $\{V_{11}, \ldots, V_{1n}, V_{22}, \ldots, V_{2m}\}$ be the unique vertex $(\mathcal{P}_{11}, \ldots, \mathcal{P}_{1n}, \mathcal{P}_{21}, \ldots, \mathcal{P}_{2m})$-partition of G. Following [3] we can construct a $(\mathcal{P}_{11}, \ldots, \mathcal{P}_{1n}, \mathcal{P}_{21}, \ldots, \mathcal{P}_{2m})$-partitionable graph H with an appropriate vertex partition $\{W_{11}, W_{12} \ldots, W_{1n}, W_{21}, W_{22}, \ldots, W_{2m}\}$ with $H[W_{ki}] \in \mathcal{P}_{ki}$ such that the graph G_H constructed by 4 of Proposition 1 is uniquely $(\mathcal{P}_1, \mathcal{P}_2)$-partitionable.

3. Generators and Universal Graphs of Reducible Properties

Given a graph property $\mathcal{P}$, a graph $U \in \mathcal{P}$ is called universal in $\mathcal{P}$ if each member of $\mathcal{P}$ is isomorphic to an induced subgraph of U. R. Rado first remarked that among the countable graphs there exists a universal one, often called "‘the Rado graph"’ R. However there are properties of finite character which do not possess a universal graph. For example the class of graphs which do not contain a subgraph isomorphic to C_4 do not contain any universal graph. For more details see e.g. [5].

For hereditary graph properties of finite character, a graph $W \in \mathcal{P}$ is called weakly universal in $\mathcal{P}$ if each member of $\mathcal{P}$ is isomorphic to a subgraph of W. In practice the two notions of universality for hereditary properties behave similarly. A universal graph is evidently weakly universal, and in practice proofs of the nonexistence of a universal graph can often be made by excluding weakly universal graphs (see [5]).

To prove the Unique Factorization Theorem in [15] the Formal Concept Analysis (briefly FCA) was used. FCA was introduced by R. Wille in 1982 and since then has grown rapidly (for a comprehensive overview see [8]). It is quite easy to prove that the sets $\mathbb{M}^{af}$ of all additive graph properties of finite character) partially ordered by set inclusion, forms a complete distributive lattice. The lattices of hereditary graph properties have been studied intensively, references may be found in [1, 14]. In order to proceed we need to recall some formal concepts of FCA according to a fundamental book of B. Ganter and R. Wille [8].

Definition 1. A *formal context* $\mathbb{K} := (O, M, I)$ consists of two sets O and M and a relation I on the product $O \times M$. The elements of O are called the *objects* and the elements of M are called the *attributes* of the context.

For a set $A \subseteq O$ we define

$$A' := \{m \in M : gIm \text{ for all } g \in A\}.$$

Analogously, for a set $B \subseteq M$ we define

$$B' := \{g \in O : gIm \text{ for all } m \in B\}.$$

A *formal concept* of the context (O, M, I) is a pair (A, B) with $A \subseteq O, B \subseteq M, A' = B$ and $B' = A$.

We call A the *extent* and B the *intent* of a formal concept (A, B). $\mathbf{L}(O, M, I)$ denotes the set of all formal concepts of the context (O, M, I).

If (A_1, B_1) and (A_2, B_2) are formal concepts of a given context and $A_1 \subseteq A_2$ (which is equivalent to $B_2 \subseteq B_1$), we write $(A_1, B_1) \leq (A_2, B_2)$.

For an object $g \in O$ we write $g' = \{m \in M : gIm\}$ and γg for the *object concept* (g'', g'), where $g'' = \{\{g\}'\}'$.

Let us mention that by the Basic Theorem on Concept Lattices [8] the set $\mathbf{L}(O, M, I)$ of all formal concepts of the context $\mathbb{K} = (O, M, I)$ partially ordered by the relation $\leq$ (see Definition 1) is a complete lattice, called the *concept lattice* of the context $\mathbb{K}$.

Let us present additive graph properties of finite character as formal concepts in a given formal context. Using FCA we can proceed in the following way. Let us define a formal context $\mathbb{K} = (\mathcal{I}^{\omega}, \mathcal{I}^{conn}, I)$ by setting objects to be the class of countable simple graphs and for each connected finite simple graph $F \in \mathcal{I}^{conn}$ let GIF if nad only if the graph $G \in \mathcal{I}^{\omega}$ does not contain any induced subgraph isomorphic to F. We can immediately observe the following:

The formal concepts of the formal context $\mathbb{K} = (O = \mathcal{I}^{\omega}, M = \mathcal{I}^{conn}, I)$ are additive graph properties of finite character and the concept lattice $(\mathbb{L}(O, M, I), \leq)$ is isomorphic to the lattice $(\mathbb{M}^{af}, \subseteq)$. Moreover, for each formal concept $\mathcal{P} = (A, B)$ there is an object - a countable graph $G \in \mathcal{I}^{\omega}$ such that $\mathcal{P} = \gamma G = (G'', G')$.

For example: let us denote by $\mathcal{D}_1$ the property "'to be a forest"' and by T_{ω} the infinite ω-regular tree, then $\mathcal{D}_1 = \gamma T_{\omega}$, obviously if U is a universal graph in $\mathcal{P}$, then $\mathcal{P} = \gamma U$.

It is easy to verify that the extent of any formal concept (A, B) of $\mathbb{K} = (\mathcal{I}^{\omega}, \mathcal{I}^{conn}, I)$ forms an additive graph property $\mathcal{P} = A$ of finite character. Obviously, each countable graph $G = (V, E)$ in the context $\mathbb{K}$ leads to an "object concept" $\gamma G = (G'', G')$. On the other hand, because of additivity, the disjoint union of all finite graphs having a given additive property $\mathcal{P} \in \mathbb{M}^{af}$ is a countable infinite graph K satisfying $\gamma K = (\mathcal{P}, \mathcal{I}^{conn} - \mathcal{P})$.

In order to describe additive induced-hereditary properties contained in $\mathcal{I}$, mainly two different approaches were used: a characterization by generating sets and/or by minimal forbidden subgraphs (see [1] and [7]). While the extent A of a formal concept $(A, B) \in \mathbb{L}(O, M, I)$ is related to a graph property P, the intent B consists of forbidden connected subgraphs of P. For a given countable graph $G \in \mathcal{I}^{\omega}$ let us denote by $age(G)$ the class of all finite graphs isomorphic to finite induced-subgraph of G (see e.g. [16]). The following result was proved in [15]:

Theorem 5. *Let $\mathcal{R} \in \mathbb{M}^{af}$ be a nontrivial reducible graph property of finite character and $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ be its unique factorization into irreducible properties. Then there exists a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable infinite countable graph G such that $\gamma G = (\mathcal{R}, \mathcal{I}^{conn} - \mathcal{R})$ and $age(G) = \mathcal{R} \cap \mathcal{I}$.*

The proof is based on the following facts. Following E. Scheinerman [17], a *composition sequence* of a class $\mathcal{P}$ of finite graphs is a sequence of finite graphs $H_1, H_2, \ldots, H_n, \ldots$ such that $H_i \in \mathcal{P}, H_i < H_{i+1}$ for all positive integers i and for all $G \in \mathcal{P}$ there exists a j such that $G \leq G^j$. We can easily find a composition sequence $H_1, H_2, \ldots, H_n, \ldots$ of $\mathcal{R} \cap \mathcal{I}$ consisting of finite uniquely $\mathcal{R}$-decomposable graphs. Without loss of generality, we may assume that if $i < j$, then $V(H_i) \subset V(H_j)$. Let $V(H) = \bigcup_i V(H_i)$ and $\{u, v\} \in E(H)$ if and only if $\{u, v\} \in E(H_j)$ for some j. It is easy to see that $age(H) = \mathcal{R} \cap \mathcal{I}$, implying $\gamma H = (\mathcal{R}, H')$. Let us remark that, according to the Theorem 1, H is uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable since if $\{V_{j_1}, V_{j_2}, \ldots, V_{j_n}\}$, $V_{j_i} \neq \emptyset$ is the unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of H_j, then $\{U_1, U_2, \ldots, U_n\}$, where $U_k = \bigcup_j V_{j_k}$, $k = 1, 2, \ldots, n$, is the unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of H. Indeed, this is because the existence of other $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of H would imply the existence of other $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ partition of some H_i and it provides a contradiction.

Based on Theorem 5 we are ready to prove our main result:

Theorem 6. *Let $\mathcal{R} \in \mathbb{M}^{af}$ be a nontrivial reducible hereditary graph property of finite character and $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ be its unique factorization into irreducible properties. Then there exists a weakly universal uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable infinite countable graph H in $\mathcal{R}$ if and only if for each $i \in \{1, 2, \ldots, n\}$ there is a weakly universal graph H_i in $\mathcal{P}_i$.*

Proof. Let G be a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable infinite countable graph such that $\gamma G = (\mathcal{R}, \mathcal{I}^{conn} - \mathcal{R})$, which exists by Theorem 5 and let $\{V_1, V_2, \ldots, V_n\}$ be the unique $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of $V(G)$, $n \geq 2$. Let H be the weakly universal graph in $\mathcal{R}$. Since $H \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ take $V(G) \cap V(H) = \emptyset$ and let $\{W_1, W_2, \ldots, W_n\}$ be a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of $V(H)$. Then the graph G_H defined in 4 of Proposition 1 is uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable and weakly universal in $\mathcal{R}$. Moreover, it is obvious, that the graphs $H_i = G_H[V_i \cup W_i]$ are weakly universal graphs in $\mathcal{P}_i$, $i = 1, 2, \ldots, n$, since otherwise H would be not universal in $\mathcal{R}$.

On the other hand, let the graphs H_i be weakly universal graphs in $\mathcal{P}_i, i = 1, 2, \ldots, n$ and let $H' = H_1 + H_2 + \cdots + H_n$ be the join (the graph obtained from disjoint union of H_i's adding all possible edges between its different components) of these weakly universal graphs. Let us apply the construction given in 4 of Proposition 1 for $W_i = V(H_i)$. Then the graph $H = G_{H'}$ is a weakly universal uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable infinite countable graph in $\mathcal{R}$, since each graph in $\mathcal{R}$ is a subgraph of H'.

4. Conclusion

It is worth to mention that $\gamma H = (\mathcal{P}, H')$ does not imply, in general, that H is a universal graph for $\mathcal{P}$. Let us define a binary relation $\cong$ on $\mathcal{I}^\omega$ by $G_1 \cong G_2$ whenever $\gamma G_1 = \gamma G_2$ in the context K. Obviously, $\cong$ is an equivalence relation on $\mathcal{I}^\omega$. An additive graph property of finite character $\mathcal{P}$ has a universal graph in $\mathcal{P}$ if the corresponding equivalence class $\{G_i : \gamma G_i = (\mathcal{P}, \mathcal{I}^{conn} - \mathcal{P})$ with respect to the equivalence relation $\cong$ has a maximal element with respect to the partial order $\leq$ - "'to be an induced subgraph"'. As it is known, to answer this question is very difficult (see [5]). Based on Theorem 6 we have the following conjecture:

Conjecture 1. Let $\mathcal{R} \in \mathbb{M}^{af}$ be a reducible graph property of finite character and $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ be its unique factorization into irreducible properties. Then there exists a universal uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable infinite countable graph H in $\mathcal{R}$ if and only if for each $i \in \{1, 2, \ldots, n\}$ there is a universal graph H_i in $\mathcal{P}_i$.

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