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ACYCLIC REDUCIBLE BOUNDS FOR OUTERPLANAR GRAPHS

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Abstract

For a given graph G and a sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ of additive hereditary classes of graphs we define an acyclic $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colouring of G as a partition $(V_1, V_2, \ldots, V_n)$ of the set $V(G)$ of vertices which satisfies the following two conditions:

1. $G[V_i] \in \mathcal{P}_i$ for $i = 1, \ldots, n$,
2. for every pair i, j of distinct colours the subgraph induced in G by the set of edges uv such that $u \in V_i$ and $v \in V_j$ is acyclic.

A class $\mathcal{R} = \mathcal{P}_1 \odot \mathcal{P}_2 \odot \cdots \odot \mathcal{P}_n$ is defined as the set of the graphs having an acyclic $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colouring. If $\mathcal{P} \subseteq \mathcal{R}$, then we say that $\mathcal{R}$ is an *acyclic reducible bound* for $\mathcal{P}$.

In this paper we present acyclic reducible bounds for the class of outerplanar graphs.

Keywords: graph, acyclic colouring, additive hereditary class, outerplanar graph.

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1. Introduction

We consider finite undirected graphs without loops or multiple edges. Let $\mathcal{I}$ denote the class of all such graphs. For a graph $G \in \mathcal{I}$ we denote its vertex set by $V(G)$ and its edge set by $E(G)$. For a vertex $v \in V(G)$ its degree is denoted by $d_G(v)$, while the maximum degree of G is denoted by $\Delta(G)$. A *block* of a graph G is defined as a maximal connected subgraph of G without a cut-vertex.

A graph G is called *outerplanar* if it can be embedded in the plane so that no edges intersect and all the vertices belong to one face. An outerplanar graph G is called *maximal*, if for any edge e from the set $E(\overline{G})$, the graph $G + e$ is not outerplanar, $\overline{G}$ stands here for the complement of G.

Following Borowiecki *et al.* [3], we define a *class of graphs* to be any nonempty subset of $\mathcal{I}$ which is closed under isomorphism. A class of graphs $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ then also $H \in \mathcal{P}$, and *additive* if it is closed under disjoint union, i.e., if every component of G belongs to $\mathcal{P}$, then $G \in \mathcal{P}$. We list some additive hereditary classes:

$$\mathcal{O} = \{G \in \mathcal{I} : \ E(G) = \emptyset\},$$

$$\mathcal{S}_k = \{G \in \mathcal{I} : \ \Delta(G) \leq k\},$$

$$\mathcal{T}_2 = \{G \in \mathcal{I} : \ G \text{ is outerplanar}\}.$$

A hereditary class $\mathcal{P}$ can be uniquely determined by the set of *minimal forbidden subgraphs* defined as follows:

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} : \ G \notin \mathcal{P}, \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}.$$

Therefore we can define, for an arbitrary set $\mathcal{F}$ of graphs, a class $\mathcal{P} = Forb(\mathcal{F})$ as the set of all graphs having no subgraph isomorphic to any graph from $\mathcal{F}$. Clearly, $\mathcal{P}$ is a hereditary class of graphs. If $\mathcal{F} = \{H\}$ then we will write $Forb(H)$ instead of $Forb(\{H\})$.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be additive hereditary classes of graphs. A partition $(V_1, V_2, \ldots, V_n)$ of the vertex set V of G is called an *acyclic* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*colouring* of G, if $G[V_i] \in \mathcal{P}_i$ for $i = 1, \ldots, n$, and for every pair i, j $(1 \leq i, j \leq n)$ of distinct colours the subgraph induced in G by the set of edges uv such that $u \in V_i$ and $v \in V_j$ is acyclic. By $\mathcal{P}_1 \odot \mathcal{P}_2 \odot \cdots \odot \mathcal{P}_n$ we denote the set of all graphs having an acyclic $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colouring. If $\mathcal{R} = \mathcal{P}_1 \odot \cdots \odot \mathcal{P}_n$ and $\mathcal{P} \subseteq \mathcal{R}$, then we say that $\mathcal{R}$ is an *acyclic reducible bound* for $\mathcal{P}$.

The other specific terminology will be introduced in the text. The general concepts not defined in the paper can be found in [8, 10].

One can observe that the above presented definition of an acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colouring corresponds to one presented in [1] and is a generalisation of a definition of an acyclic colouring of a graph, given by Grünbaum in [9] (it is enough to put each $\mathcal{P}_i$ equal to $\mathcal{O}$).

After being introduced by Grünbaum in 1973, the acyclic colouring has been widely studied over past thirty years by Burstein, see [7], Borodin [5], Borodin, Kostochka and Woodall [6], and many others. In 1999 Boiron, Sopena and Vignal considered the acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colouring of planar and outerplanar graphs, see [1], and also of graphs with bounded degree, see [2]. In [1] it was proved that $\mathcal{T}_2 \subseteq \mathcal{S}_5 \odot \mathcal{S}_5$. In [4] another four acyclic reducible bounds for the class $\mathcal{T}_2$ were presented. The aim of our paper is to provide new acyclic reducible bounds for the class of outerplanar graphs. To do this we propose a construction of special families of outerplanar graphs. This construction is presented in Section 2, while Section 3 contains main results. In Section 4 we discuss the relationship between acyclic reducible bounds given in [1] and [4], and ours.

2. Construction of $\mathcal{H}_i$ and its Basic Properties

Let $G = (V, E; L, S)$, $L : V \to \{0, 1, \ldots\}$, $S : E \to \{0, 1, \ldots\} \times \{+, 0, -\}$ be a graph with labels assigned to its vertices and edges. If $S(e) = (k, \cdot)$, then we say that the edge e has *level* k ($\cdot$ stands for any of the signs from the set $\{+, 0, -\}$). Similarly, the vertex v has *level* k, if $L(v) = k$. Moreover, we write $S(e) = (\cdot, +)$, if we mean that $S(e) = (i, +)$, but the value i is not important or unknown yet.

Let $G = (V, E; L, S)$ and $G' = (V', E'; L', S')$ be labelled graphs. If there is an isomorphism $f : G \to G'$ such that $L(v) = L'(f(v))$ for each $v \in V$ and $S(uv) = S'(f(u)f(v))$ for each $uv \in E$, then we say that the graphs are *LS-isomorphic*. Moreover, if F is a (labelled) graph and $\mathcal{A}$ is a set of (labelled) graphs, then by $F \in \mathcal{A}$ we mean F is (LS-)isomorphic to a member of $\mathcal{A}$. Let us remark that taking (induced) subgraphs preserves labels L and S.

We have introduced the labels L and S just for the technical purposes of the construction and the simplicity of the proofs, but because the main result of this paper concerns colouring of unlabelled graphs we must define an operator $\widetilde{\ }$ which transforms a given labelled graph into unlabelled one. So, for a labelled graph $G = (V, E; L, S)$ by $\widetilde{G}$ we mean the unlabelled graph (V, E).

Similarly, if $\mathcal{A}$ is a set of labelled graphs, then the set $\widetilde{\mathcal{A}}$ is defined as follows: a graph $G \in \widetilde{\mathcal{A}}$ if and only if $G \simeq \widetilde{H}$ for some $H \in \mathcal{A}$.

Let us define a family $\mathcal{H}_0$ of labelled graphs as follows: $\mathcal{H}_0 = \{H_0^1, H_0^2, H_0^3\}$, where

$H_0^1 = (\{u,v\}, \{uv\}; L^1, S^1)$ and $L^1(u) = L^1(v) = 0,\ S^1(uv) = (0,+)$,

$H_0^2 = (\{u,v,w\}, \{uv,uw,vw\}; L^2, S^2)$ and $L^2(u) = L^2(v) = L^2(w) = 0$,

$S^2(uv) = (0,0),\ S^2(uw) = S^2(vw) = (0,+)$,

$H_0^3 = (\{u,v,w\}, \{uv,uw,vw\}; L^3, S^3)$ and $L^3(u) = L^3(v) = L^3(w) = 0$,

$S^3(uv) = S^3(vw) = (0,0),\ S^3(uw) = (0,+)$.

The family $\mathcal{H}_0$ is presented in Figure 1, labels of all the vertices are 0 and they are omitted.

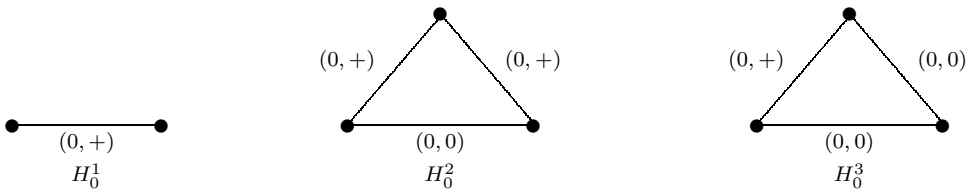

Figure 1. The family $\mathcal{H}_0$ of graphs.

In order to define the family $\mathcal{H}_i$, for $i \geq 1$, we need to introduce the notion of a *child* of a labelled graph G. Namely, if $G = (V, E; L, S)$ is a labelled graph, then we say that a graph $G' = (V', E'; S', L')$ is a *child* of G, if G' is constructed in the following way:

(a) set $V' = V, E' = E, S' = S$ and $L' = L$;

(b) for each edge $uv \in E'$ such that $S'(uv) = (i,+)$

(b.0) set $S'(uv) = (i,-)$;

(b.1) if $deg(u) = deg(v) = 1$, then add to G' vertices u', v', w', edges $u'u, v'u, v'v, w'v$ and set $L'(u') = L'(v') = L'(w') = i+1$, $S'(u'u) = S'(w'v) = (i+1,+)$ and

$([S'(v'u) = (i+1,+), \quad S'(v'v) = (i+1,0)]$ or

$[S'(v'u) = (i+1,0), \quad S'(v'v) = (i+1,+)]$ or

$[S'(v'u) = S'(v'v) = (i+1,+)])$;

(b.2) if $deg(u) = 1$ and $deg(v) > 1$, then add to G' vertices u', v', edges $u'u, v'u, v'v$ and set $L'(u') = L'(v') = i+1$,
$S'(u'u) = (i+1,+)$ and

$([S'(v'u) = (i+1,+), \quad S'(v'v) = (i+1,0)]$ or

$[S'(v'u) = (i+1,0), \quad S'(v'v) = (i+1,+)]$ or

$[S'(v'u) = S'(v'v) = (i+1,+)])$;

(b.3) if $deg(u) > 1$ and $deg(v) > 1$, then add to G' a vertex v', edges $v'u, v'v$ and set $L'(v') = i+1$; let B be the block of G' which contains the edge uv,

(b.3.1) if B has an edge $e, e \neq uv$, such that $S'(e) = (\cdot,+)$, then set

$([S'(v'u) = (i+1,+), \quad S'(v'v) = (i+1,0)]$ or

$[S'(v'u) = (i+1,0), \quad S'(v'v) = (i+1,+)]$ or

$[S'(v'u) = (i+1,0), \quad S'(v'v) = (i+1,0)])$;

(b.3.2) in the other case set

$[S'(v'u) = (i+1,+), \quad S'(v'v) = (i+1,0)]$ or

$[S'(v'u) = (i+1,0), \quad S'(v'v) = (i+1,+)]$;

(c) for each edge $uv \in E'$ such that $S'(uv) = (i-1,0)$ add to G' vertices u', v', edges $u'u, v'v$ and set
$L'(u') = L'(v') = i+1, \quad S'(u'u) = S'(v'v) = (i+1,+)$.

Let us remark that if an edge has label $(\cdot,+)$ in a labelled graph G, then it means that this edge plays a special role in G — it is used in the construction of a child of G.

If G is a labelled graph, then by child(G) we denote the set of all non-LS-isomorphic graphs being the children of G. If a graph $G' \in$ child(G), then we say that the graph G is a *parent* of G'.

The family $\mathcal{H}_i$, for $i \geq 1$, is defined as follows:

$$\mathcal{H}_i = \bigcup_{H \in H_{i-1}} \text{child}(H).$$

The family $\mathcal{H}_1$ is presented in Figure 2, the vertices with label 1 are coloured white, while the vertices with label 0 are coloured black. Let us remark that the first two graphs on the picture are the children of H_0^1, the next five — the children of H_0^2, the last one is the child of H_0^3.

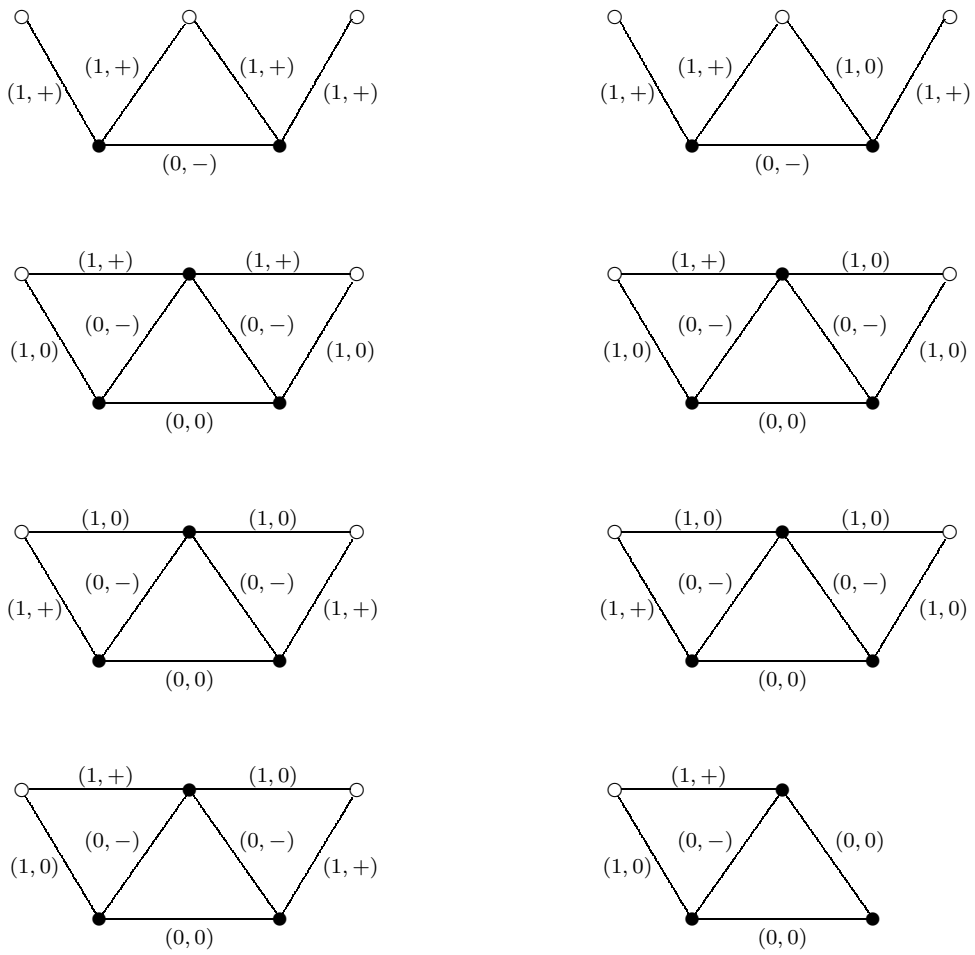

Figure 2. The family $\mathcal{H}_1$ of graphs.

One can observe that if $H \in \mathcal{H}_i$, then H is outerplanar and each its block is maximal outerplanar.

In the next two lemmas we list another two properties of the graphs from $\mathcal{H}_i$, $i \geq 1$. Both follows from the construction of $\mathcal{H}_i$ and will be used in the proofs in Section 3.

Lemma 1. *Let $H \in \mathcal{H}_i$, $i \geq 1$, and let B be a block of H. Then*

1. *if $\widetilde{B} \simeq K_2$, then $e \in E(B)$ has label $(i,+)$;*
2. *if $\widetilde{B} \simeq K_3$, then the edges of B have labels $(i,+),(i,+),(i-1,-)$ or $(i,+),(i,0),(i-1,-)$;*
3. *if B has more than three edges, then one of the following cases holds:*

(a) *B contains two adjacent edges e_1, e_2 such that $S(e_1) = (i,+)$ and $S(e_2) = (i,0)$ and all the other edges of B have labels neither $(i,\cdot)$ nor $(\cdot,+)$,*

(b) *B contains two adjacent edges e_1, e_2 such that $S(e_1) = (i,+)$ and $S(e_2) = (i,0)$ and another two adjacent edges e_3, e_4 such that $S(e_3) = (i,0)$ and $S(e_4) = (i,0)$ and all the other edges of B have labels neither $(i,\cdot)$ nor $(\cdot,+)$,*

(c) *B contains two adjacent edges e_1, e_2 such that $S(e_1) = (i,+)$ and $S(e_2) = (i,0)$ and another two adjacent edges e_3, e_4 such that $S(e_3) = (i,+)$, $S(e_4) = (i,0)$ and all the other edges of B have labels neither $(i,\cdot)$ nor $(\cdot,+)$.*

Lemma 2. *Let $H = (V,E;L,S) \in \mathcal{H}_i$, $i \geq 1$, and let B be a block of H. Then*

1. *if $\widetilde{B} \simeq K_3$ with $E(B) = \{e_1,e_2,e_3\}$ and $S(e_1) = (i-1,-)$, then there are graphs $H^1 = (V,E;L,S^1) \in \mathcal{H}_i$, $H^2 = (V,E;L,S^2) \in \mathcal{H}_i$ and $H^3 = (V,E;L,S^3) \in \mathcal{H}_i$ such that $S^1(e) = S^2(e) = S^3(e) = S(e)$ for each edge $e \in E$ different from e_2, e_3 and $S^1(e_2) = S^1(e_3) = (i,+)$, $S^2(e_2) = (i,+)$ and $S^2(e_3) = (i,0)$, $S^3(e_2) = (i,0)$ and $S^3(e_3) = (i,+)$; moreover, there is a graph $H' \in \mathcal{H}_{i-1}$ such that $H, H^1, H^2, H^3 \in \mathrm{child}(H')$;*
2. *if B has more than three edges, then*

(a) *if B contains exactly two edges of level i, say e_1, e_2, then there are graphs $H^1 = (V,E;L,S^1) \in \mathcal{H}_i$ and $H^2 = (V,E;L,S^2) \in \mathcal{H}_i$ such that $S^1(e) = S^2(e) = S(e)$ for each edge $e \in E$ different from e_1, e_2 and $S^1(e_1) = (i,+)$ and $S^1(e_2) = (i,0)$, $S^2(e_1) = (i,0)$ and $S^2(e_2) = (i,+)$; moreover, there is a graph $H' \in \mathcal{H}_{i-1}$ such that $H, H^1, H^2 \in \mathrm{child}(H')$;*

(b) *if B contains exactly four edges of level i, say e_1, e_2, e_3, e_4, and if e_1, e_2 are adjacent, and e_3, e_4 are adjacent, then there are graphs $H^j = (V,E;L,S^j) \in \mathcal{H}_i$ for $j = 1,\ldots,8$ such that $S^j(e) = S(e)$ for $j = 1,\ldots,8$ and for each edge $e \in E$ different from e_1, e_2, e_3, e_4, each of the edges*

e_1, e_2, e_3, e_4 has either label $(i,+)$ or $(i,0)$ and at least one of them has label $(i,+)$; moreover, if e_1 has label $(i,+)$, then e_2 has label $(i,0)$, and vice versa; the same holds for e_3 and e_4; furthermore, there is a graph $H' \in \mathcal{H}_{i-1}$ such that $H, H^1, \ldots, H^8 \in \mathrm{child}(H')$.

3. Results

We start this section by reminding a special family of labelled graphs. In [4] it was proved that this family is a generator of the class of outerplanar graphs.

Let $A_0 = (V_0, E_0; L_0, S_0)$ be a labelled graph such that $V_0 = \{v_1, v_2, v_3\}$, $E_0 = \{v_1v_2, v_2v_3, v_1v_3\}$, $L_0(v) = 0$ for all $v \in V_0$ and $S_0(e) = (0,0)$ for all $e \in E_0$. The graph $A_{i+1} = (V_{i+1}, E_{i+1}; L_{i+1}, S_{i+1})$ is obtained from $A_i = (V_i, E_i; L_i, S_i)$ by adding for each edge $e = uv$ of level i one new vertex w_e and joining w_e with both u and v. The vertex w_e has level $i+1$ and both new added edges have labels $(i+1, 0)$.

It is worth to mention that each A_i is maximal outerplanar and it has such an embedding in a plane in which all vertices belong to the outer face and only the edges of level i belong to the outer face. Therefore, when we talk about the graph A_i or its subgraphs, we always deal with this embedding.

The following lemma states that the family $\{\widetilde{A_0}, \widetilde{A_1}, \ldots\}$ of graphs is a generator of the class $\mathcal{T}_2$ and render us possible to consider in our proofs only the graphs A_i instead of the whole class $\mathcal{T}_2$.

Lemma 3. [4] *For every $G \in \mathcal{T}_2$ there exists an index $i \geq 0$ such that $G \subseteq \widetilde{A_i}$.*

In [4] it was proved that each outerplanar graph G has an acyclic colouring (V_1, V_2) such that $G[V_1] \in \mathcal{O}$ and $G[V_2] \in Forb(\widetilde{A_1})$.

Lemma 4. [4] $\mathcal{T}_2 \subseteq \mathcal{O} \odot Forb(\widetilde{A_1})$.

In the following lemmas, which easily follows from the construction of the graphs A_i, we present more properties of such graphs, which will be used later.

Lemma 5. *Let B be a 2-connected induced subgraph of A_i, $i \geq 0$. Then B cannot contain any induced cycle of length greater than 3.*

Lemma 6. *An edge $e \in E(A_i)$ belongs to the outer cycle of A_i if and only if $S_i(e) = (i, 0)$. Moreover, every vertex $v \in V(A_i)$ such that $L_i(v) = i$ is of degree 2.*

Lemma 7. *If $H \in \mathcal{H}_i$, $i \geq 0$, then $\widetilde{H}$ cannot contain a subgraph isomorphic to $\widetilde{A_1}$.*

Next we prove that if $\mathcal{P}$ is an additive hereditary class such that $\widetilde{A_1} \notin \mathcal{P}$ and (V_1, V_2) is an acyclic $(\mathcal{O}, \mathcal{P})$-colouring of A_i, $i \geq 0$, then there is a graph $H_i \in \mathcal{H}_i$ such that $\widetilde{A_i}[V_2] = \widetilde{H_i}$.

Now we present some properties of such a colouring.

Lemma 8. *Let $\mathcal{P}$ be an additive hereditary class and assume $\widetilde{A_1} \notin \mathcal{P}$. If (V_1, V_2) is an acyclic $(\mathcal{O}, \mathcal{P})$-colouring of A_i, C is the outer cycle of A_i and B is a block of $A_i[V_2]$, then one of the following situations occurs:*

1. *$E(B) \cap E(C)$ contains exactly four edges: $uv, vw, u'v'$ and $v'w'$; moreover, $L_i(v) = L_i(v') = i$ and $uw, u'w' \in E(A_i)$,*
2. *$E(B) \cap E(C)$ contains exactly two edges: uv, vw; moreover, $L_i(v) = i$ and $uw \in E(A_i)$*
3. *$E(B) \cap E(C)$ contains exactly one edge.*

Proof. First observe that if B is a trivial block of $A_i[V_2]$, it means, if $\widetilde{B} \simeq K_2$, then from the construction of A_i and the fact that the colouring is acyclic it clearly follows that $|E(B) \cap E(C)| = 1$.

On the other hand, if B is a non-trivial block of $A_i[V_2]$, u, v are two consecutive vertices of the outer cycle of this block and the edge uv does not belong to the cycle C, then for each such a pair of vertices there is one vertex $w \in V(A_i) - V(B)$, adjacent to both u and v, which follows from the construction of A_i. Therefore, if $E(B) \cap E(C) = \emptyset$, then the colouring (V_1, V_2) cannot be acyclic.

Hence consider an edge $e = uv \in E(B) \cap E(C)$. Without loss of generality we can assume that $L_i(v) = i$, therefore, by Lemma 6, $d(v) = 2$. Let w be the second neighbour of v in A_i. If the edge $e_1 = vw$ does not belong to B, then $\widetilde{B}$ is isomorphic to K_2. Hence $e_1 \in E(B)$. If $E(B) \cap E(C) = \{e, e_1\}$, then the second situation occurs.

Therefore, we can assume that there is an edge $e' = u'v' \in E(B)$, $e' \neq e$, $e' \neq e_1$, which belongs to the cycle C. Similarly as above, we can assume that v' has level i in A_i. Hence, by Lemma 6, $d(v) = 2$. Moreover, if w' is the remaining neighbour of v', then the edge $e'_1 = w'v' \in E(C) \cap E(B)$, which is clear because B is a block. Moreover, from the construction of A_i

it follows that $e_1' \neq e$ and $e_1' \neq e_1$. If $E(B) \cap E(C) = \{e, e_1, e', e_1'\}$, then we have the first situation.

Observe that if there is another edge $e'' = v''u'' \in E(B) \cap E(C)$, $e'' \notin \{e, e_1, e', e_1'\}$, then assuming that $L(v'') = i$, we will have that there is exactly one vertex $w'' \neq u''$ adjacent to v'' in A_i. Moreover, $w'' \in V(B)$ and $u''w'' \in E(B)$. But it is quite easy to see that in this case $\widetilde{B}$ must contain a subgraph isomorphic to $\widetilde{A_1}$, which follows from the construction of A_i, a contradiction. ■

From the proof of Lemma 8 and the construction of A_i we can conclude the following.

Remark 1. Let $\mathcal{P}$ be an additive hereditary class such that $\widetilde{A_1} \notin \mathcal{P}$. Consider an acyclic $(\mathcal{O}, \mathcal{P})$-colouring (V_1, V_2) of A_i. Let C be the outer cycle of A_i. Furthermore, assume that v is a vertex such that $v \in V_2$ and $L_i(v) = i$. Clearly, $d(v) = 2$. Hence, let u and w be the neighbours of v. Obviously, $vu, vw \in E(C)$. Moreover, if B is the block of $A_i[V_2]$ containing v, then either $\widetilde{B} \simeq K_2$ or $vu, vw \in E(B)$.

Theorem 1. *Let $\mathcal{P}$ be an additive hereditary class and assume that $\widetilde{A_1} \notin \mathcal{P}$. If (V_1, V_2) is an acyclic $(\mathcal{O}, \mathcal{P})$-colouring of A_i, then there is a graph $H_i \in \mathcal{H}_i$ such that $\widetilde{A_i}[V_2] = \widetilde{H_i}$.*

Proof. We prove a stronger statement, namely we prove that if (V_1, V_2) is any acyclic $(\mathcal{O}, \mathcal{P})$-colouring of $A_i = (V_i, E_i; L_i, S_i)$, then there is a graph $H_i = (V, E; L, S) \in \mathcal{H}_i$ such that $\widetilde{A_i}[V_2] = \widetilde{H_i}$ and for each vertex $t \in V_2$ we have $L_i(v) = L(v)$.

We use induction on i. It is easy to check that the theorem is true for A_0 and A_1.
Let $i \geq 1$. Assume the theorem holds for every A_j $(0 \leq j \leq i)$, we will prove it for A_{i+1}.

Consider the graph $A_{i+1} = (V_{i+1}, E_{i+1}; L_{i+1}, S_{i+1})$ and the subgraph $A_i = (V_i, E_i; L_i, S_i)$ induced in A_{i+1} by the vertices of levels $0, \ldots, i$. Let (V_1, V_2) be any acyclic $(\mathcal{O}, \mathcal{P})$-colouring of A_i. By the induction hypothesis it follows that there is the graph $H_i = (V, E; L, S) \in \mathcal{H}_i$ such that $\widetilde{A_i}[V_2] = \widetilde{H_i}$ and moreover for each vertex $t \in V_2$ we have $L_i(t) = L(t)$. It is sufficient to prove that for any acyclic $(\mathcal{O}, \mathcal{P})$-colouring (V_1', V_2') of A_{i+1} such that $V_1 \subseteq V_1'$, $V_2 \subseteq V_2'$, there is also a graph $H_{i+1} = (V', E'; L', S') \in \mathcal{H}_{i+1}$ such that $\widetilde{A_{i+1}[V_2']} = \widetilde{H_{i+1}}$ and for each vertex $t \in V_2'$ we have $L_{i+1}(t) = L'(t)$.

At the beginning observe that for any $s \geq 0$, level of an edge uv of a graph $F \in \mathcal{H}_s$ is the maximum of levels of the vertices u and v. Therefore, from the induction hypothesis and the fact that for any $l \geq 0$ all edges of the outer cycle of A_l have level l, it clearly follows that:

Observation 1. *If $e \in E$, then $S(uv) = (i, \cdot)$ if and only if uv belongs to the outer cycle of A_i.*

Furthermore, from the construction of A_i and the facts that the colouring (V_1, V_2) is acyclic and the set V_1 is independent it follows that $A_i[V_2]$ is connected.

The rest of the proof is dived into two steps. In the first step we use Lemma 8 to prove that there is a graph $H_{i+1} = (V', E'; L', S') \in \mathcal{H}_{i+1}$ which satisfies $\widetilde{A_{i+1}}[V_2'] \subseteq \widetilde{H_{i+1}}$ and for each vertex $t \in V_2'$ we have $L_{i+1}(t) = L'(t)$.

First, we choose an arbitrary graph $H_{i+1} = (V', E'; L', S') \in \mathrm{child}(H_i)$. Clearly, from the induction hypothesis it follows that it is sufficient to prove that for each vertex $x \in V_2'$ which has level $i+1$ in A_{i+1} there is a suitable vertex of level $i+1$ in H_{i+1}.

From the construction of the graph A_{i+1} it follows that each such x has a unique neighbour v of level i in A_{i+1} and there is also a unique vertex y, $y \neq x$, of level $i+1$ in A_{i+1} which is adjacent to v. We will consider each such pair of vertices x and y. Obviously, $d_{A_{i+1}}(x) = d_{A_{i+1}}(y) = 2$.

Assume at the beginning that $v \in V_2$. Let B be the block of $A_i[V_2]$ containing v. Furthermore, let C be the outer cycle of A_i. According to Lemma 8 we have to distinguish three cases.

Case 1. If $E(B) \cap E(C) = \{uv, vw, u'v', v'w'\}$, then notice that $S_i(uv) = S_i(vw) = S_i(u'v') = S_i(v'w') = (i, 0)$ and moreover $L_i(v) = L_i(v') = i$. Hence $L(v) = L(v') = i$. Therefore v, v' are of degree two in A_i and in H_i. Moreover, $uw \in E(A_i)$ and $u'w' \in E(A_i)$. Let x', y' be the neighbours of v' in A_{i+1} which have level $i+1$. Without loss of generality we can assume that $xu, yw, x'w', y'w' \in E(A_{i+1})$. It is evident that the block B has at least 5 vertices. Furthermore, from the definition of A_i it follows that there is a vertex $z \in V(A_i)$, different from u, v, w, such that $uz, wz \in E(A_i)$. The vertex z is unique, for otherwise we will have a subgraph isomorphic to $K_{2,3}$ in $\widetilde{A_i}$, which is impossible. Similarly, there is exactly one vertex $z' \in V(A_i)$, different from u', v', w', such that $u'z', w'z' \in E(A_i)$. Since B is a block, we have $z, z' \in V(B)$.

If $x, y, x', y' \in V_1'$, then (V_1', V_2') cannot be acyclic, which follows from the proof of Lemma 8.

If we assume that $x, y \in V_2'$ or $x', y' \in V_2'$, then $\widetilde{A_{i+1}}[V_2']$ contains a subgraph isomorphic to $\widetilde{A_1}$, namely a subgraph induced by the vertices $\{x, y, u, v, w, z\}$ or $\{x', y', u', v', w', z'\}$.

If exactly one of the vertices x, y, x', y', say x', belongs to V_2', and the rest of them belong to V_1', then from the fact that $L_i(v) = i$ we have $L(v) = i$. Hence $S(uw) = (i-1, -)$. Likewise, $S(u'w') = (i-1, -)$. Therefore by Lemma 2 there is a graph $H_i' = (V, E; L, S'') \in \mathcal{H}_i$ such that $S''(e) = S(e)$ for every edge $e \in E$, $e \notin \{u'v', vw, uv, v'w'\}$ and $S''(u'v') = (i, +)$, $S(vw) = S(uv) = S(v'w') = (i, 0)$. We set $H_i = (V, E; L, S'')$. In the step (b.3) of the construction of a graph H_{i+1}, we add a vertex a adjacent to both u' and v' with $L'(a) = i+1$. We set $x' = a$. Furthermore, from Observation 1 we conclude that in $B' = H_i[V(B)]$ only the edge $u'v'$ has label $(\cdot, +)$.

In the remaining case, if exactly two of the vertices x, y, x', y' are in V_2' and the two others are in V_1', then without loss of generality we can assume that $x, x' \in V_2'$. Similarly as above, from the fact that $L_i(v) = i$ we have $L(v) = i$ and $S(uw) = (i-1, -)$. Likewise, $S(u'w') = (i-1, -)$. Therefore, by Lemma 2 there is a graph $H_i' = (V, E; L, S'') \in \mathcal{H}_i$ such that $S''(e) = S(e)$ for every edge $e \in E$, $e \notin \{u'v', vw, uv, v'w'\}$ and $S''(uv) = S''(u'v') = (i, +)$ and $S''(vw) = S''(v'w') = (i, 0)$. We set $H_i = (V, E; L, S'')$. In the step (b.3) of the construction of a graph H_{i+1}, we add two new vertices a and b (one for the edge uv and another for $u'v'$), such that a is adjacent to both u and v, and b is adjacent to both u' and v' with $L'(a) = L'(b) = i+1$. We set $x = a$, $x' = b$. Observation 1 implies that in $B' = H_i[V(B)]$ there are no edges with labels $(\cdot, +)$, except of uv and $u'v'$.

Case 2. Assume that $E(B) \cap E(C) = \{uv, vw\}$. It is easy to observe that the edges uv and vw both have level i in A_i, since $L_i(v) = i$. Hence $L(v) = i$. Therefore, $d_{A_i}(v) = 2$ and also $d_{H_i}(v) = 2$. Besides, $uw \in E_i$. Without loss of generality we can assume that $xu \in E_{i+1}$ and $yw \in E_{i+1}$.

If $x, y \in V_1'$, then the colouring (V_1', V_2') cannot be acyclic, which follows from the proof of Lemma 8.

On the other hand, if both x, y are in V_2', but B has a vertex z adjacent to both w and u, $z \neq v$, then the graph $\widetilde{A_{i+1}}[\{x, y, u, v, w, z\}]$ is isomorphic to $\widetilde{A_1}$, which is impossible.

If we assume that $x, y \in V_2'$ and B does not have such a vertex z, then $\widetilde{B} \simeq K_3$ and, by Lemma 1, the edge uw has label $(i-1, -)$ in H_i. Therefore by Lemma 2 there is a graph $H_i' = (V, E; L, S'') \in \mathcal{H}_i$ such that $S''(e) = S(e)$ for every edge $e \in E$, $e \neq uv$, $e \neq vw$ and $S''(uv) = S''(vw) = (i, +)$.

We set $H_i = (V, E; L, S'')$. Notice that in the step (b.3) of the construction of a graph H_{i+1} we add two new vertices a and b (one for the edge uv and another for vw), such that a is adjacent to both u and v, and b is adjacent to both v and w with $L'(a) = L'(b) = i + 1$. We set $x = a$, $y = b$. From Observation 1 we conclude that in $B' = H_i[V(B)]$ there are no edges with labels $(\cdot, +)$, except of uv and vw.

It remains to consider the case $x \in V_2'$ and $y \in V_1'$. Clearly, from the fact that $L_i(v) = i$ and v is adjacent to both u and w, we conclude that $S(uw) = (i - 1, -)$. Therefore, by Lemma 2 there is a graph $H_i' = (V, E; L, S'') \in \mathcal{H}_i$ such that $S''(e) = S(e)$ for every edge $e \in E$, $e \neq uv$, $e \neq vw$ and $S''(uv) = (i, +)$, $S''(vw) = (i, 0)$. We set $H_i = (V, E; L, S'')$. Notice that in the step (b.3) of the construction of a graph H_{i+1}, we add a new vertex a adjacent to both u and v with $L'(a) = i + 1$. We set $x = a$. From Observation 1 we conclude that in $B' = H_i[V(B)]$ only the edge uv has label $(\cdot, +)$. Similar considerations can be applied to the case when $x \in V_1'$ and $y \in V_2'$.

Case 3. If $E(B) \cap E(C) = \{uv\}$, then $\widetilde{B} \simeq K_2$. It is easy to observe that in A_i there is a vertex w such that $uw, vw \in E(A_i)$. As $\widetilde{B} \simeq K_2$ we have $w \in V_1$. Moreover, there is no loss of generality in assuming that $xu, yw \in E(A_{i+1})$, which follows from the definition of A_{i+1}.

If either x or y belongs to V_1', then either the colouring (V_1', V_2') cannot be acyclic or the set V_1' cannot be independent.

On the other hand, if $x, y \in V_2'$, then from the fact that $L_i(v) = i$ it follows that $L(v) = i$. Moreover, v is of degree 1 in B and cannot be adjacent in A_i to any vertex from V_2, except u. Thus v is of degree 1 in H_i. Therefore, $S(uv) = (i, +)$. In the step (b.2) of the construction of a graph H_{i+1}, we add two new vertices a and b such that a is adjacent to both u and v, b is adjacent to v. We set $x = a$, $y = b$. Moreover, from Observation 1 we can conclude that in $B' = H_i[V(B)]$ only the edge uv has label $(\cdot, +)$.

Let us assume now that $v \notin V_2$. Hence $v \notin V(H_i)$. By the definition of A_i we have that there are vertices $u, w \in V(A_i)$ such that $uv, vw, wu \in E(A_i)$ and $ux, wy \in E(A_{i+1})$. Obviously, $u, w \in V_2$. Let $z \in V(A_i)$ be the unique vertex different from v which is adjacent to both u and w in A_i. The uniqueness of the vertex z follows from the definition of A_i. The colouring (V_1, V_2) is acyclic so $z \in V_2$. If either $x \in V_1'$ or $y \in V_1'$, then V_1' cannot be independent. Therefore, we can assume $x, y \in V_2'$. Moreover, one

of the vertices u, w, say u, has level $i-1$ in A_i, and another one, it means w, has level lower than or equal to $i-1$ in A_i. Hence $L(u) = i-1$ and $L(w) \leq i-1$. Thus $S(uw) = (i-1, \cdot)$. Since $z \in V_2$, we have $z \in V(H_i)$. Moreover, z has level less than or equal to $i-1$ in A_i. Furthermore, the edge uw cannot have label $(i-1, -)$ in H_i, because in this case we would have a vertex a in H_i, adjacent to both u and w, and therefore, since $\widetilde{H_i} = \widetilde{A_i}[V_2]$, we would have $a \in V_2$ and $L_i(a) = i$. This would contradict the fact that the vertex $v \in V_1$ is the only neighbour of both u and w which has L_i equal to i. So we must have $S(uw) = (i-1, 0)$. Therefore, in the step (c) of the construction of H_{i+1} we add vertices b and c and edges ub and uc. We set $x = b$, $y = c$.

Notice that we have just obtained certain graphs $H_i = (V, E; L, S)$ and $H_{i+1} = (V', E'; L', S')$. We use this particular graphs to finish the proof by showing that $\widetilde{A_{i+1}}[V_2'] = \widetilde{H_{i+1}}$.

Clearly, it is sufficient to prove that

- if a vertex $t \in V'$ and $L'(t) = i+1$, then $t \in V_2'$ and
- if an edge $e \in E'$ and $S'(e) = (i+1, \cdot)$, then $e \in E(A_{i+1}[V_2'])$.

Let $a \in V(H_{i+1})$ be a vertex satisfying $L'(a) = i+1$. Obviously, a can be either of degree 2 or 1 in H_{i+1}.

Assume that a is of degree 2 in H_{i+1} and let b, c be its neighbours in H_{i+1}. Clearly, $S'(bc) = (i, -)$. Let B' be the block of H_{i+1} containing the edge bc. Moreover, let $B = H_{i+1}[\{t \in V(B') : L'(t) < i+1\}]$. Obviously, B is a block of the graph H_i and $S(bc) = (i, +)$. Hence by Observation 1, bc belongs to the outer cycle of A_i. Therefore, according to the procedure described in *Case* 1 and *Case* 2, we have $a \in V_2'$ and $ab, ac \in E(A_{i+1}[V_2'])$. Now we assume that a is of degree 1 in H_{i+1} and let $b \in V(H_{i+1})$ be its neighbour. It is clear that $L'(b) < i+1$. Hence $b \in V(H_i)$.

If $L(b) = i$, then b is of degree 1 in H_i. Moreover, if c is the neighbour of b in H_i, then $S(bc) = (i, +)$. Therefore, we have the situation described in *Case* 3 and clearly $a \in V_2'$ and $ab \in E(A_{i+1}[V_2'])$.

In the opposite case, if $L(b) = j < i$, then according to the construction of H_{i+1} we see that there is an edge bc in H_i such that $S(bc) = (i-1, 0)$. Hence bc has level $i-1$ in A_i. Therefore, in A_i there is a vertex v of level i, adjacent to both b and c. The vertex v must be in V_1 since $\widetilde{A_i}[V_2] = \widetilde{H_i}$ and in H_i we do not have any vertex of level i adjacent to both b and c, because

$S(bc) = (i-1, 0)$. Hence in A_{i+1} there is a vertex x, adjacent to both v and b. We have $x \in V_2'$ since V_1' is independent and we can set $a = x$. Clearly, $ab \in E(A_{i+1}[V_2'])$. ■

If G is a graph and (V_1, V_2) is a colouring of G, then a cycle $C = (c_1, \ldots, c_n)$ of a graph G is called *alternating* (with respect to the colouring (V_1, V_2)), if n is even and $c_1, c_3, \ldots, c_{n-1} \in V_2$ and $c_2, c_4, \ldots, c_n \in V_1$. Such a cycle C is called *minimal alternating*, if there is no other alternating cycle C' (with respect to the colouring (V_1, V_2)) satisfying $V(C') \subset V(C)$.

In the proof of the next theorem we will use the following lemma, which presents a certain property of minimal alternating cycles with respect to an $(\mathcal{O}, \mathcal{P})$-colouring of A_i, where $\mathcal{P}$ is a given additive hereditary class.

Lemma 9. *Let (V_1, V_2) be a colouring of A_i such that $\widetilde{A_i}[V_1] \in \mathcal{O}$ and let $C = (c_1, c_2, \ldots, c_k)$ be any minimal alternating cycle of length $k \geq 6$ with respect to the colouring (V_1, V_2) such that $c_1 \in V_2$. Then $V_2 \cap V(C)$ is a cycle of A_i. Moreover, the graph $A_i[V_2 \cap V(C)]$ is a block in $A_i[V_2]$.*

Proof. Let (V_1, V_2) be a colouring of A_i satisfying $\widetilde{A_i}[V_1] \in \mathcal{O}$ and let $C = (c_1, c_2, \ldots, c_k)$ $(k \geq 6)$ be any minimal alternating cycle with respect to this colouring such that $c_1 \in V_2$. By Lemma 5 we have the set $V(C)$ induces the graph G_C in A_i, which cannot contain any induced cycle of length greater than 3. From the fact that the set V_1 is independent we can conclude that if $u, v \in \{c_2, c_4, \ldots, c_k\}$, then u and v cannot be adjacent in A_i. If the vertex c_{2l}, $1 \leq l \leq k/2$, is adjacent to a certain vertex $c_{2l'-1}$, $1 \leq l' \leq k/2$, then the cycle C cannot be minimal alternating. Therefore the vertex c_{2l-1} is adjacent to c_{2l+1}, for all $l = 1, \ldots, k/2 - 1$, and c_{k-1} is adjacent to c_1, since otherwise in G_C it would be an induced cycle of length greater than 3. The fact that $A_i[V_2 \cap V(C)]$ is a block in $A_i[V_2]$ follows from Lemma 5 and the definition of A_i. ■

Theorem 2. *Let $H_i \in \mathcal{H}_i$. Then there is an acyclic colouring (V_1, V_2) of A_i such that $\widetilde{A_i}[V_1] \in \mathcal{O}$ and $\widetilde{A_i}[V_2] = H_i$.*

Proof. We prove a little stronger statement. Namely, we prove that for any graph $H_i = (V, E; L, S) \in \mathcal{H}_i$ there is an acyclic $(\mathcal{O}, \mathcal{P})$-colouring (V_1, V_2) of $A_i = (V_i, E_i; L_i, S_i)$ such that $\widetilde{A_i}[V_2] = \widetilde{H_i}$ and for each vertex $v \in V$ we have $L(v) = L_i(v)$.

We use induction on i. It is easy to check that the theorem is true for all $H \in \mathcal{H}_0$ and $H \in \mathcal{H}_1$.

Let $i \geq 1$. Assume the theorem is true for all $H_j \in \mathcal{H}_j$ and every $j \leq i$. We will prove it for $i+1$.

Let us consider the graph $H_{i+1} = (V', E'; L', S') \in \mathcal{H}_{i+1}$ and let $H_i = (V, E; L, S)$ be a parent of H_{i+1}. Clearly, $H_i \in \mathcal{H}_i$. Therefore by the induction hypothesis we have that A_i has an acyclic $(\mathcal{O}, \mathcal{P})$-colouring (V_1, V_2), which satisfies the condition $\widetilde{A_i}[V_2] = \widetilde{H_i}$ and such that for each vertex $v \in V$ we have $L(v) = L_i(v)$. We prove that there is an acyclic colouring (V_1', V_2') of A_{i+1}, such that $V_1 \subseteq V_1'$ and $V_2 \subseteq V_2'$, and which satisfies the condition $\widetilde{A_{i+1}}[V_1'] \in \mathcal{O}$, $\widetilde{A_{i+1}}[V_2'] = \widetilde{H_{i+1}}$ and for each vertex $v \in V'$ we have $L'(v) = L_{i+1}(v)$.

Now we show how we construct the colouring (V_1', V_2'). First, we colour every vertex of level at most i in A_{i+1} according to the colouring (V_1, V_2). Next, we will extend this colouring to the whole graph A_{i+1}. Clearly, it is sufficient to consider only the vertices of level $i+1$ in A_{i+1}.

Let x, y be a pair of vertices of level $i+1$ in A_{i+1}, such that there is a vertex v of level i in A_i adjacent to both x and y. Besides let u, w be the vertices such that $uw, vw, vu, xu, yw \in E(A_{i+1})$. Furthermore, let z be a vertex different from v such that $uz, wz \in E(A_{i+1})$. According to the definition of A_{i+1} we see that these vertices exist and moreover, all u, w, z have neither level $i+1$ nor i. Since the colouring (V_1, V_2) is acyclic and the set V_1 is independent, at most one of the vertices u, v, w can belong to V_1. Hence there are four cases to consider.

Case 1. If $u, v, w \in V_2$ and $z \in V_1$, then by the definition of A_i, if B is a block in the graph H_i which contains u, v, w, then $\widetilde{B} \simeq K_3$. Therefore, by Lemma 1, at least one of the edges uv, vw has label $(i, +)$ in H_i. Assume, without loss of generality, that $L(uv) = (i, +)$. We put x into V_2'. If $L(vw) = (i, +)$, then we put y into V_2', otherwise into V_1'.

Case 2. If $u, v, w \in V_2$ and $z \in V_2$ then let B be a block of H_i which contains u, v, w, z. From Lemma 8 it follows that B has either two or four common edges with the outer cycle of A_i.

Subcase 2.1. If there are only two common edges uv and vw, then there are no other edges of level i in B. Therefore, by Lemma 1, exactly one of this two edges, say uv, has label $(i, +)$. Hence in the construction of H_{i+1} we add a vertex a, adjacent to both u and v. Therefore we put x into V_2'. Since the edge vw has label $(i, -)$, we put y into V_1'.

Subcase 2.2. If there are four common edges: uv, vw, $u'v'$ and $w'v'$, then in the block B there are no other edges of level i. Hence, again by

Lemma 1, at least one of these four edges has label $(i,+)$ in H_i. If the edge uv has label $(i,+)$, then we put x into V_2', otherwise into V_1'. If the edge vw has label $(i,+)$, then we put y into V_2', otherwise into V_1'.

Case 3. If $v \in V_1$, then clearly $u, w, z \in V_2$, since the colouring (V_1, V_2) is acyclic and the set V_1 is independent. Moreover, the edge uw has label $(i-1, 0)$ in H_i. Therefore, in the construction of the graph H_{i+1} we add two new vertices adjacent to both u and w, respectively. We put x, y into V_2'.

Case 4. If $w \in V_1$ (or, similarly, $u \in V_1$), then $z \in V_2$, because the set V_1 is independent. Furthermore, the edge uv is a trivial block in H_i. Therefore, $L(uv) = (i,+)$ in H_i and the vertex v is of degree 1 in H_i. Hence in the construction of the graph H_{i+1} we add one new vertex adjacent to both u and v and another one adjacent to v. We put x, y into V_2'.

From the above it clearly follows that $\widetilde{A_{i+1}[V_2']} \subseteq \widetilde{H_{i+1}}$, $\widetilde{A_{i+1}[V_1']} \in \mathcal{O}$ and each vertex from the set V_2', which has level l in H_{i+1}, has level l in A_{i+1}.

In order to finish the proof it remains to show that the colouring (V_1', V_2') is acyclic and that $\widetilde{A_{i+1}[V_2']} = \widetilde{H_{i+1}}$.

Assume, on the contrary, that the colouring (V_1', V_2') is not acyclic. From the fact that $V_1 \subseteq V_1'$ and $V_2 \subseteq V_2'$ it follows that any alternating cycle must contain a vertex of level $i+1$ in A_{i+1}. Notice, that each vertex of level $i+1$ is of degree 2 in A_{i+1}. Next, observe that putting a vertex x into V_2', we cannot create an alternating cycle. Moreover, in all the cases, except of *Subcase* 2.2, a given vertex x is put into V_1' only if the following two conditions hold: $|N_{A_{i+1}}(x) \cap V_2| = 2$ and at least one neighbour of x does not have any neighbour in V_1'. Clearly, we cannot obtain an alternating cycle in this way. Therefore, there is only one situation when an alternating cycle can occur. Namely, in *Subcase* 2.2 when both x and y have level $i+1$ and belong to V_1'. Observe that each such alternating cycle must contain at least 6 vertices. Assume that C is the shortest one. By Lemma 9 it follows that if $W_2 = V_2' \cap V(C)$, then the vertices of W_2 create a cycle in A_{i+1} and moreover, they induce a block in $A_{i+1}[V_2']$. But, as it was described in *Subcase* 2.2, at least one of the vertices from the set W_2 has level $i+1$ in A_{i+1}, so is of degree 2. Hence both its neighbours also belong to W_2. This fact clearly implies that C is not alternating.

Now we prove that $\widetilde{A_{i+1}[V_2']} = \widetilde{H_{i+1}}$. On the contrary, assume that there is a vertex a of level $i+1$ in H_{i+1}, such that $a \notin V_2'$. Observe that

from the above it follows that a cannot be of degree 2. Therefore, we can assume, that a is of degree 1. Let b be the neighbour of a in H_{i+1}. Clearly, b does not have level $i+1$ in H_{i+1}.

If b has level i in H_i, then b is of degree 1 in H_i. Moreover, if c is the neighbour of b in H_i, then $L(bc) = (i, +)$. Thus we have the situation described in *Case* 4 and the vertex $a \in V_2'$, a contradiction.

On the other hand, if b has level $j < i$ in H_i, then according to the construction of H_{i+1} there is an edge bc in H_i such that $L(bc) = (i-1, 0)$. If bc is of level $i-1$ in H_i, then it is also of level $i-1$ in A_i. Hence, as in *Case* 3, the vertex $a \in V_2'$, a contradiction. ■

Let $\mathbf{H} = \{H_0, H_1, H_2, \ldots\}$ be a set of graphs such that each $H_i \in \mathcal{H}_i$ and each H_i, for $i > 0$, is a child of H_{i-1}. We define a class $\mathcal{P}_{\mathbf{H}}$ of graphs as follows: a graph G belongs to $\mathcal{P}_{\mathbf{H}}$ if and only if G is a subgraph of the disjoint union of some graphs from the set $\widetilde{\mathbf{H}}$. Clearly, the class $\mathcal{P}_{\mathbf{H}}$ is additive and hereditary.

Theorem 3. *Let $\mathcal{P}_{\mathbf{H}}$ be the class of graphs defined as above. Then*

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot \mathcal{P}_{\mathbf{H}}.$$

Proof. It clearly follows from Lemma 3 and Theorem 2. ■

4. Concluding Remarks

Boiron, Sopena and Vignal proved in [1], that $\mathcal{T}_2 \subseteq \mathcal{S}_5 \odot \mathcal{S}_5$, where $\mathcal{S}_5$ is the class of all graphs of maximum degree at most 5. We prove that this bound and our bounds are incomparable.

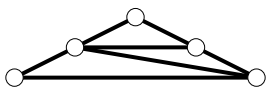

Figure 3. Graph B_0

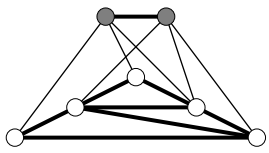

Figure 4. Graph B_1

Let us consider at the beginning the graph B_0 presented in Figure 3. We show that in every acyclic colouring (V_1, V_2) of the graph A_3 either $B_0 \subseteq \widetilde{A_3}[V_1]$ or $B_0 \subseteq \widetilde{A_3}[V_2]$.

Assume on the contrary, that there is an acyclic colouring (V_1, V_2) of A_3, such that $B_0 \not\subseteq \widetilde{A_3}[V_1]$ and $B_0 \not\subseteq \widetilde{A_3}[V_2]$. Assume at the beginning, that all the vertices of level 0 belong to V_1. Clearly, at least two of the vertices of level 1 have to be in the set V_2, since otherwise $B_0 \subseteq \widetilde{A_3}[V_1]$. Moreover, at least one vertex of level 1 has to be in V_1, because the colouring is acyclic. Let x and y be the vertices of level 2 such that each of them is adjacent to two vertices from V_1.

The colouring is acyclic, hence both x and y cannot be in V_2. Therefore, either $x \in V_1$ or $y \in V_1$, but in this case $B_0 \subseteq \widetilde{A_3}[V_1]$, a contradiction.

On the other hand, if we assume that two vertices of level 0, say x and y, are in V_1, and the remaining one in V_2, then let z be the vertex of level 1 adjacent to both x and y. Clearly, $z \in V_1$, because the colouring is acyclic. Now we can proceed as in the previous case, starting from the triangle x, y, z.

Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary classes of graphs such that $\mathcal{T}_2 \subseteq \mathcal{P} \odot \mathcal{Q}$. Clearly, from the above it follows that either $B_0 \in \mathcal{P}$ or $B_0 \in \mathcal{Q}$. Moreover, if we assume additionally that the classes $\mathcal{P}$ and $\mathcal{Q}$ are both different from $\mathcal{O}$, then the graph B_1, presented in Figure 4, admits an acyclic $(\mathcal{P}, \mathcal{Q})$-colouring. Now we show that B_1 does not have any acyclic $(\mathcal{O}, \mathcal{T}_2)$-colouring. Assume on the contrary, that (V_1, V_2) is an acyclic $(\mathcal{O}, \mathcal{T}_2)$-colouring of B_1. Observe at the beginning, that if we remove any vertex from B_1, then the remaining graph is not outerplanar. Hence we have to put at least two vertices to V_1, but in this case the colouring is not acyclic.

Remark 2. If $\mathcal{P}$ and $\mathcal{Q}$ are additive hereditary classes of graphs, both different from $\mathcal{O}$, and such that $\mathcal{T}_2 \subseteq \mathcal{P} \odot \mathcal{Q}$, then $\mathcal{P} \odot \mathcal{Q} \not\subseteq \mathcal{O} \odot \mathcal{T}_2$.

In particular, we can put both $\mathcal{P}$ and $\mathcal{Q}$ equal to $\mathcal{S}_5$ in the above remark. Moreover, instead of the class $\mathcal{T}_2$ we can take the class $\mathcal{P}_{\mathbf{H}}$, with $\mathcal{P}_{\mathbf{H}}$ defined as at the end of Section 3. Hence, $\mathcal{S}_5 \odot \mathcal{S}_5 \not\subseteq \mathcal{O} \odot \mathcal{P}_{\mathbf{H}}$.

Next we show that there is a graph which belongs to $\mathcal{O} \odot \mathcal{P}_{\mathbf{H}} \setminus \mathcal{S}_5 \odot \mathcal{S}_5$, where the class $\mathcal{P}_{\mathbf{H}}$ is defined as at the end of Section 3. Consider the graph $B_3 = K_1 + 6K_{1,6}$. It is obvious that B_3 has an acyclic $(\mathcal{O}, \mathcal{P}_{\mathbf{H}})$-colouring (V_1, V_2), because we can put the vertex of maximal degree to V_1 and the other vertices to V_2. On the other hand, B_3 does not admit any acyclic $(\mathcal{S}_5, \mathcal{S}_5)$-colouring, which follows from that fact that any copy of the graph $K_{1,6}$ cannot be monochromatic, hence the vertex v of maximum degree is adjacent to at least six vertices of the same colour, as colour of v.

Remark 3. If $\mathcal{P}_{\mathbf{H}}$ is the class of graphs defined as above, then $\mathcal{O} \odot \mathcal{P}_{\mathbf{H}} \not\subseteq \mathcal{S}_5 \odot \mathcal{S}_5$.

Let us recall that a maximal outerplanar graph G with at least 3 vertices is called a 2-*path* of even order $n = 2p$, if G consists of two paths $P_1 = (x_1, x_2, \ldots, x_p)$ and $P_2 = (y_1, y_2, \ldots, y_p)$ and additional edges: $x_i y_i$, $i = 1, \ldots, p$ and $x_j y_{j+1}$ for $j = 1, \ldots, p-1$. A 2-path of odd order is defined as $H = G - x_p$, where G is a 2-path of even order. A maximal outerplanar graph G with at least 3 vertices is called a *fan* of order n, if G is the join of K_1 and a path of order $n-1$.

In [4] it was proved that if G is a 2-path (or a fan) of order 6, then $\mathcal{T}_2 \subseteq \mathcal{O} \odot Forb(A_1, G)$. They also proved that $\mathcal{T}_2 \subseteq \mathcal{O} \odot \mathcal{FT}$ and $\mathcal{T}_2 \subseteq \mathcal{O} \odot \mathcal{PT}$, where classes $\mathcal{FP}$ and $\mathcal{PT}$ are defined as follows:
$\mathcal{PT} = \{G \in \mathcal{I}$ such that each its block is a subgraph of a 2-path$\}$,
$\mathcal{FT} = \{G \in \mathcal{I}$ such that each its block is a subgraph of a fan$\}$.

One can observe that there is a set $\mathbf{H}' = \{H'_0, H'_1, \ldots\}$ of graphs such that each H'_i satisfies the following three conditions:

- $H'_i \in \mathcal{H}_i$,
- H'_i is a child of the graph H'_{i-1}, for $i \geq 1$,
- each its block is a 2-path.

Clearly, we have $\mathcal{P}_{\mathbf{H}'} = \mathcal{PT}$, where $\mathcal{P}_{\mathbf{H}'}$ is defined as a class of graphs such that a graph F belongs to $\mathcal{P}_{\mathbf{H}'}$ if and only if F is a subgraph of a graph from the set $\widetilde{\mathbf{H}'}$.
If we consider the class $\mathcal{FT}$, then it is easy to see that there is a set $\mathbf{H}'' = \{H''_0, H''_1, \ldots\}$ of graphs such that each H''_i satisfies the following three conditions:

- $H''_i \in \mathcal{H}_i$,
- H''_i is a child of the graph H''_{i-1}, for $i \geq 1$,
- each its block is a fan.

Clearly, we have $\mathcal{P}_{\mathbf{H}''} \subset \mathcal{FT}$, where $\mathcal{P}_{\mathbf{H}''}$ is a class of graphs such that a graph $F \in \mathcal{P}_{\mathbf{H}''}$ if and only if F is a subgraph of a graph from the set $\widetilde{\mathbf{H}''}$.

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