*Discussiones Mathematicae*
*Graph Theory 29 (2009) 209–218*

# MULTICOLOR RAMSEY NUMBERS FOR SOME PATHS AND CYCLES

Halina Bielak

*Institute of Mathematics*
*UMCS, Lublin, Poland*

**e-mail:** hbiel@golem.umcs.lublin.pl

**Abstract**

We give the multicolor Ramsey number for some graphs with a path or a cycle in the given sequence, generalizing a results of Faudree and Schelp [4], and Dzido, Kubale and Piwakowski [2, 3].

**Keywords:** cycle, path, Ramsey number.

**2000 Mathematics Subject Classification:** 05C55.

## 1. Introduction

We consider simple graphs with at least two vertices. For given graphs $G_1, G_2, \ldots, G_k$ and $k \geq 2$ multicolor Ramsey number $R(G_1, G_2, \ldots, G_k)$ is the smallest integer $n$ such that in arbitrary $k$-colouring of edges of a complete graph $K_n$ a copy of $G_i$ in the colour $i$ $(1 \leq i \leq k)$ is contained (as a subgraph).

Let $ex(n, F)$ be the Turán number for integer $n$ and a graph $F$, defined as the maximum number of edges over all graphs of order $n$ without any subgraph isomorphic to $F$.

Theorems 1, 2 and 3 presented below are very useful for study multicolour Ramsey numbers for paths and cycles. In this paper we generalize the results presented in Theorems 4 and 5.

**Theorem 1** (Faudree and Schelp [4]). *If $G$ is a graph with $|V(G)| = kp+r$ $(0 \leq k, 0 \leq r < p)$ and $G$ contains no $P_{p+1}$, then $|E(G)| \leq kp(p-1)/2 + r(r-1)/2$ with the equality if and only if $G = kK_p \cup K_r$ or $G = lK_p \cup$*

$(K_{(p-1)/2} + \overline{K}_{(p+1)/2+(k-l-1)p+r})$ *for some* $0 \leq l < k$, *where* $p$ *is odd, and* $k > 0, r = (p \pm 1)/2$.

Let $c(G)$ be the circumference of $G$, i.e., the length of the longest cycle in $G$.

**Theorem 2** (Brandt [1]). *Every non-bipartite graph* $G$ *of order* $n$ *with more than* $\frac{(n-1)^2}{4} + 1$ *edges contains cycles of every length* $t$, *where* $3 \leq t \leq c(G)$.

For positive integers $a$ and $b$, set $r(a, b) = a \bmod b = a - \lfloor \frac{a}{b} \rfloor b$. For integers $n \geq k \geq 3$, set

$$\omega(n,k) = \frac{1}{2}(n-1)k - \frac{1}{2}r(k-r-1), \tag{1}$$

where $r = r(n-1, k-1)$.

**Theorem 3** (Woodall [7]). *Let* $G$ *be a graph of order* $n$ *and size* $m$ *with* $m \geq n$ *and* $c(G) = k$. *Then* $m \leq \omega(n, k)$ *and the result is best possible.*

In 1975 Faudree and Schelp published the following results concerning a multicolor Ramsey number for paths.

**Theorem 4** (Faudree and Schelp [4]). *If* $r_0 \geq 6(r_1 + r_2)^2$, *then* $R(P_{r_0}, P_{r_1}, P_{r_2}) = r_0 + \lfloor \frac{r_1}{2} \rfloor + \lfloor \frac{r_2}{2} \rfloor - 2$ *for* $r_1, r_2 \geq 2$.

*If* $r_0 \geq 6(\sum_{i=1}^{k} r_i)^2$, *then* $R(P_{r_0}, P_{2r_1+\delta}, P_{2r_2}, \ldots, P_{2r_k}) = \sum_{i=0}^{k} r_i - k$ *for* $\delta = 0, 1$, $k \geq 1$ *and* $r_i \geq 1$ $(1 \leq i \leq k)$.

Recently, Dzido, Kubale, and Piwakowski published the following results.

**Theorem 5** (Dzido *et al.* [2, 3]). $R(P_3, C_k, C_k) = 2k - 1$ *for odd* $k \geq 9$, $R(P_4, P_4, C_k) = k + 2$ *for* $k \geq 6$, $R(P_3, P_5, C_k) = k + 1$ *for* $k \geq 8$.

Moreover, some asymptotic results are cited below.

**Theorem 6** (Kohayakawa, Simonovits, Skokan [6]). *There exists an integer* $n_0$ *such that if* $n > n_0$ *is odd, then* $R(C_n, C_n, C_n) = 4n - 3$.

**Theorem 7.** (Figaj, Łuczak [5]). *For even* $n$, $R(C_n, C_n, C_n) = 2n + o(n)$.

## 2. Results

First we prove the following theorem, extending the result of Dzido *et al.* (see Theorem 5).

**Theorem 8.** *Let $t, q$ $(t \geq q \geq 2)$ be positive integers and $m$ be odd integer. Let for even $q$ either $t > \frac{3}{4}q^2 - 2q + 2$ and $m = t + \lfloor \frac{q}{2} \rfloor$ or $t > \frac{1}{8}(3q^2 - 10q + 16)$ and $m \leq t + \lfloor \frac{q}{2} \rfloor - 1$. Let for odd $q$, $t > \frac{1}{4}(3q^2 - 14q + 21)$ and $m \leq t + \lfloor \frac{q}{2} \rfloor - 1$. Then $R(P_q, P_t, C_m) = 2t + 2\lfloor \frac{q}{2} \rfloor - 3$.*

***Proof.*** Let $n = 2t + 2\lfloor \frac{q}{2} \rfloor - 3$ and $a = t + \lfloor \frac{q}{2} \rfloor - 2$. First we prove that $R(P_q, P_t, C_m) \geq 2t + 2\lfloor \frac{q}{2} \rfloor - 3$. Let $K_a$ be (red, blue)-coloured without red $P_q$ and without blue $P_t$. It is possible by $R(P_q, P_t) = a + 1$. So there exists the critical colouring of the graph $H = K_a \cup K_a$. Let the edges of $\overline{H}$ be coloured with green. Since $\overline{H}$ is bipartite graph it does not contain any $C_m$.

Now we prove that $R(P_q, P_t, C_m) \leq 2t + 2\lfloor \frac{q}{2} \rfloor - 3$.

Note that $|E(K_n)| = (2t + 2\lfloor \frac{q}{2} \rfloor - 3)(t + \lfloor \frac{q}{2} \rfloor - 2)$ and $|E(K_{a,a})| = (t + \lfloor \frac{q}{2} \rfloor - 2)^2$.

Let $d = |E(K_n)| - |E(K_{a,a})| = (t + \lfloor \frac{q}{2} \rfloor - 2)(t + \lfloor \frac{q}{2} \rfloor - 1)$.

So

$$(2) \quad d = (t-1)(t+q-4) + \left\lfloor \frac{q}{2} \right\rfloor \left( \left\lfloor \frac{q}{2} \right\rfloor - 1 \right) + 2(t-1) - (t-1) \left( \left\lceil \frac{q}{2} \right\rceil - \left\lfloor \frac{q}{2} \right\rfloor \right).$$

Suppose that we can colour $E(K_n)$ with three colours (red, blue, green) without red $P_q$, blue $P_t$ and green $C_m$. So the red subgraph of $K_n$ has at most $ex(n, P_q)$ edges and the blue subgraph of $K_n$ has at most $ex(n, P_t)$ edges. Now we apply Theorem 1 for $p = t-1$. We have two cases. If $2|q$ and $t = q$ then set $k = 3, r = 0$. In the opposite case, set $k = 2$ and $r = 2\lfloor \frac{q}{2} \rfloor - 1$. Thus, we can write $ex(n, P_t) \leq (t-1)(t-2) + (2\lfloor \frac{q}{2} \rfloor - 1)(\lfloor \frac{q}{2} \rfloor - 1)$.

Moreover, by Theorem 1 for $p = q - 1$, we get $ex(n, P_q) \leq \frac{n(q-2)}{2}$. So $ex(n, P_q) \leq (t-1)(q-2) + \frac{1}{2}(2\lfloor \frac{q}{2} \rfloor - 1)(q-2)$.

Let $s = ex(n, P_t) + ex(n, P_q)$. So the red-blue subgraph of $K_n$ has at most $s$ edges and

$$s \leq (t-1)(t+q-4) + (q-1)(q-2) - \begin{cases} 0, & 2|q, \\ \frac{3(q-2)}{2}, & 2 \not| q. \end{cases}$$

By the above fact and (2) we note that $d - s \geq h(q,t)$, where

$$h(q,t) = \left\lfloor \frac{q}{2} \right\rfloor \left( \left\lfloor \frac{q}{2} \right\rfloor - 1 \right) - (q-1)(q-2) + (t-1) + \begin{cases} (t-1), & 2|q, \\ \frac{3(q-2)}{2}, & 2 \not| q. \end{cases}$$

Moreover, $h(q,t) > 0$ if and only if

$$t > \begin{cases} \frac{1}{8}\left(3q^2 - 10q + 16\right), & 2|q, \\ \frac{1}{4}\left(3q^2 - 14q + 21\right), & 2 \not| q. \end{cases}$$

So for $t$ satisfying the above condition the green subgraph $G'$ of $K_n$ has more edges than the graph $K_{a,a}$. Namely, $|E(G')| \geq |E(K_{a,a})| + h(q,t)$. Note that $G'$ is not a bipartite graph. In the opposite case we have at least $t + \lfloor \frac{q}{2} \rfloor - 1 = R(P_t, P_q)$ vertices in a part of the bipartite graph and the proof is done since we get a red $P_q$ or a blue $P_t$.

By definition (1), we get
$\omega(n, m-1) = \omega(2t + 2\lfloor \frac{q}{2} \rfloor - 3, m-1) = (t + \lfloor \frac{q}{2} \rfloor - 2)(m-1) - \frac{1}{2}r(m-2-r)$, where $r = r(n-1, m-2)$. So $\omega(n, m-1) \leq (t + \lfloor \frac{q}{2} \rfloor - 2)(m-1)$.

We would like apply the theorems of Woodall and Brandt. We look for a lower bound of the longest cycle in the green graph $G'$. Thus let $b \geq 0$ be maximum integer $b \geq 0$ such that the following inequalities hold

(i) $b \cdot a < h(q,t)$

and

(ii) $\omega(n, m-1) \leq (t + \lfloor \frac{q}{2} \rfloor - 2)(t + \lfloor \frac{q}{2} \rfloor - 2 + b) < |E(G')|$.

Evidently $b < 2$, else we get a contradiction to the first of the above inequalities. Moreover, if $2|q$ and $t > \frac{1}{4}\left(3q^2 - 8q + 8\right)$, then $b = 1$. For other cases $b = 0$.

Then, by Theorem 3, we get $c(G') \geq (t + \lfloor \frac{q}{2} \rfloor - 1 + b)$. Thus we get a cycle of order at least $(t + \lfloor \frac{q}{2} \rfloor - 1 + b)$ in the green graph $G'$.

Moreover, $\frac{(n-1)^2}{4} + 1 = (t + \lfloor \frac{q}{2} \rfloor - 2)^2 + 1 < |E(G')|$. So, by Theorem 2, the green graph $G'$ is weakly pancyclic. Hence we get a green cycle $C_m$ for $m \leq t + \lfloor \frac{q}{2} \rfloor - 1 + b$, a contradiction. Therefore each (red, blue, green)-colouring of $E(K_n)$ contains a red $P_q$, a blue $P_t$ or a green $C_m$. So we get the upper bound for $R(P_q, P_t, C_m)$. The proof is done. ■

In general case we get the following theorem.

**Theorem 9.** $R(P_q, P_t, C_m) \geq \lfloor \frac{q}{2} \rfloor - 2 + \max\{t + \lfloor \frac{m}{2} \rfloor, m + \lfloor \frac{t}{2} \rfloor\}$.

***Proof.*** Let $r = \lfloor \frac{q}{2} \rfloor - 3 + \max\{t + \lfloor \frac{m}{2} \rfloor, m + \lfloor \frac{t}{2} \rfloor\}$ and $x = \lfloor \frac{q}{2} \rfloor - 1$. Let $K_{r-x}$ be subgraph of $K_r$ (blue, green)-coloured without blue $P_t$ and without green $C_m$. Such critical colouring exists by $R(P_t, P_m) = r - x + 1$. Let other edges of $K_r$ be coloured with red. The red subgraph does not contain any $P_q$. The proof is done. ∎

Now we extend the result of Faudree and Schelp presented above in Theorem 4.

**Proposition 10.** *Let $t_0 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 2$, $k \geq 2$ be integers and $n = t_0 + \sum_{i=1}^{k} (\lfloor \frac{t_i}{2} \rfloor - 1)$. Let $x = 2$ if $t_0 = t_1 = t_2$ and $2 \not| t_0$, and $x = 0$ in the opposite case. Then $R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k}) \geq n + x$.*

***Proof.*** Let $t_0 = t_1 = t_2$ and $2 \not| t_0$. We define the critical colouring of the graph $K_{n+x-1}$, with $x = 2$. Let $A, B, C, D, E_j, (j = 3, \ldots, k)$ be sets with $|A| = |B| = |C| = |D| = \lfloor \frac{t_0}{2} \rfloor$ and $|E_j| = \lfloor \frac{t_j}{2} \rfloor - 1, (j = 3, \ldots, k)$. Let the edges with ends in the sets $A \cup B$ and $C \cup D$ be coloured with the colour 0, the edges with one end in the set $A$ and the second one in the set $C$ be coloured with the colour 1, the edges with one end in the set $B$ and the second one in the set $D$ be coloured with the colour 1. Other edges with ends in $A \cup B \cup C \cup D$ colour with the colour 2. Let $V_j = A \cup B \cup C \cup D \cup \bigcup_{i=3}^{j-1} E_i, (j = 3, \ldots, k)$. Let colour the edges with both ends in $E_j$ or one end in $E_j$ and the second one in the set $V_j$ with the colour $j, (j = 3, \ldots, k)$. Note that the colouring contains no monochromatic $P_{t_i}$ in the colour $i$.

If the condition $t_0 = t_1 = t_2$ and $2 \not| t_0$ does not hold we define the critical colouring of the graph $K_{n+x-1}$, with $x = 0$. Namely, let $|A| = t_0 + \lfloor \frac{t_1}{2} \rfloor - 2$, $|E_j| = \lfloor \frac{t_j}{2} \rfloor - 1, (j = 2, \ldots, k)$ and $V_j = A \cup \bigcup_{i=2}^{j-1} E_i, (j = 2, \ldots, k)$. Let colour the edges with both ends in $E_j$ or one end in $E_j$ and the second one in the set $V_j$ with the colour $j, (j = 2, \ldots, k)$. The edges with ends in the set $A$ colour critically with colours 0 and 1 (it is possible by $R(P_{t_0}, P_{t_1}) = t_0 + \lfloor \frac{t_1}{2} \rfloor - 1)$). The proof is done. ∎

Now we show some sufficient conditions for $R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k}) = n + x$ with $x = 0$ or $x = 2$ and $n = t_0 + \sum_{i=1}^{k} (\lfloor \frac{t_i}{2} \rfloor - 1)$.

**Theorem 11.** *Let $t_0 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 2$, $k \geq 2$ be integers and $n = t_0 + \sum_{i=1}^{k} (\lfloor \frac{t_i}{2} \rfloor - 1)$. Let $x = 2$ if $t_0 = t_1 = t_2$ and $2 \not| t_0$, and $x = 0$*

*in the opposite case, and let* $r_i = (n+x)$ *mod* $(t_i - 1)$ $(i = 0, 1, \ldots, k)$*. The sufficient conditions for* $R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k}) = n + x$ *are as follows:*

(i) $t_0 > t_1,\ 2|t_i$ *for each* $i \geq 1$ *and*

$$t_0 > \max\Big\{\Big(\sum_{i=1}^{k}\Big(\Big\lfloor\tfrac{t_i}{2}\Big\rfloor - 1\Big) + 1\Big)^2 - \sum_{i=1}^{k}\Big(\Big\lfloor\tfrac{t_i}{2}\Big\rfloor - 1\Big), \sum_{i=1}^{k}\Big(\Big\lfloor\tfrac{t_i}{2}\Big\rfloor - 1\Big) + 2\Big\},$$

(ii) $t_0 > t_1,\ 2 \not| t_i$ *for exactly one* $i \geq 1$ *and*

$$t_0 > \max\Big\{2\Big(\sum_{i=1}^{k}\Big(\Big\lfloor\tfrac{t_i}{2}\Big\rfloor - 1\Big) + 1\Big)^2 - \sum_{i=1}^{k}\Big(\Big\lfloor\tfrac{t_i}{2}\Big\rfloor - 1\Big), \sum_{i=1}^{k}\Big(\Big\lfloor\tfrac{t_i}{2}\Big\rfloor - 1\Big) + 2\Big\},$$

(iii) $t_0 \in \{4, 6, 8\},\ t_0 = t_1 > t_2$ *and* $t_i = 2$ *for each* $i = 2, \ldots, k$,

(iv) $t_0 \in \{3, 5\},\ t_0 = t_1 > t_2$ *and* $t_i = 2$ *for each* $i = 2, \ldots, k$,

(v) $t_0 = t_1 = t_2 = 3 > t_3$ *and* $t_i = 2$ *for each* $i = 3, \ldots, k$ *or* $t_0 = t_1 = t_2 = t_3 = 3$ *and* $t_i = 2$ *for each* $i = 4, \ldots, k$,

(vi) $t_i = 2$ *for each* $i = 0, \ldots, k$.

***Proof.*** By Proposition 10 we get the lower bound $n + x \leq R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k})$. Now we prove the upper bound. Evidently, $0 \leq r_i < t_i - 1$. By definition of $n$ and $r_0$ we have

$$\sum_{i=1}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right) + 1 = w \cdot (t_0 - 1) + r_0, \tag{3}$$

where $w \geq 0$ and $0 \leq r_0 \leq t_0 - 2$ are integers.

By Theorem 1 we get $\sum_{i=0}^{k} ex(n + x, P_{t_i}) \leq s$, where $s = \frac{n+x}{2}\sum_{i=0}^{k}(t_i - 2) - \frac{1}{2}\sum_{i=0}^{k} r_i(t_i - 1 - r_i)$. Let $g = \binom{n+x}{2} - s$. Evidently,

$$g = \frac{n+x}{2}\left(n + x - 1 - \sum_{i=0}^{k} t_i + 2k + 2\right) + \frac{1}{2}\sum_{i=0}^{k} r_i(t_i - 1 - r_i). \tag{4}$$

Note that, $g > 0$ is a sufficient condition for $R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k}) \leq n + x$.

Let $y$ be the number of odd $t_i$, for $i = 1, \ldots k$. So

$$y = \sum_{i=1}^{k}\left(\left\lceil\frac{t_i}{2}\right\rceil - \left\lfloor\frac{t_i}{2}\right\rfloor\right). \tag{5}$$

Let

$$a = r_0 - \Big( \sum_{i=1}^{k} \Big( \Big\lfloor \frac{t_i}{2} \Big\rfloor - 1 \Big) + y - 1 - x \Big). \tag{6}$$

Then by the definition of $n$ we have

$$\begin{aligned} g = (a - r_0) \frac{t_0 + \sum\limits_{i=1}^{k} \Big( \big\lfloor \frac{t_i}{2} \big\rfloor - 1 \Big) + x}{2} + \frac{1}{2} r_0 (t_0 - 1 - r_0) \\ + \frac{1}{2} \sum_{i=1}^{k} r_i (t_i - 1 - r_i). \end{aligned} \tag{7}$$

Hence, by (7) and (6), we get

$$\begin{aligned} g = \frac{a}{2} t_0 - \frac{1}{2} \Big( \sum_{i=1}^{k} \Big( \Big\lfloor \frac{t_i}{2} \Big\rfloor - 1 \Big) + x \Big)^2 + \frac{1}{2} \Big( \sum_{i=1}^{k} \Big( \Big\lfloor \frac{t_i}{2} \Big\rfloor - 1 \Big) + x \Big) (2x + 1 - y) \\ - \frac{1}{2} r_0 (r_0 + 1) + \frac{1}{2} \sum_{i=1}^{k} r_i (t_i - 1 - r_i). \end{aligned} \tag{8}$$

If $a > 0$ and $g > 0$ then we can find some additional restriction on $t_i$ to obtain the upper bound of Ramsey number for the sequence of paths.

By (6), the assumption $a > 0$ gives

$$r_0 \geq \sum_{i=1}^{k} \Big( \Big\lfloor \frac{t_i}{2} \Big\rfloor - 1 \Big) + y - x. \tag{9}$$

Let us consider three cases.

*Case* 1. Suppose that $t_0 > t_1$. So $x = 0$. Thus, by the value of $n$, we get

$$r_0 = \sum_{i=1}^{k} \Big( \Big\lfloor \frac{t_i}{2} \Big\rfloor - 1 \Big) + 1. \tag{10}$$

By (6), (10) and the assumption $a > 0$, we have $y = 0$ or $y = 1$. Moreover, if $y = 0$ then $a = 2$ and if $y = 1$ then $a = 1$.

By (8),

$$t_0 > \frac{1}{a} \Big( r_0 (r_0 + 1) + \Big( \sum_{i=1}^{k} \Big( \Big\lfloor \frac{t_i}{2} \Big\rfloor - 1 \Big) \Big)^2 - (1 - y) \sum_{i=1}^{k} \Big( \Big\lfloor \frac{t_i}{2} \Big\rfloor - 1 \Big) \Big)$$

is a sufficient condition for $g > 0$.

Thus we get $t_0 > r_0^2 - (r_0 - 1)$ for $y = 0$ and $t_0 > r_0(2r_0 - 1) + 1$ for $y = 1$.

Elementary counting leads to the condition (i) and (ii), respectively.

*Case* 2. Suppose that $t_0 = t_1 > t_2$. Thus $x = 0$ and by (8) we get

$$\begin{aligned}(11)\quad g = \frac{a+r_0}{2}t_0 - \frac{1}{2}\left(\sum_{i=1}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right)\right)^2 + \frac{1}{2}(1-y)\sum_{i=1}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right)\\ - r_0(r_0+1) + \frac{1}{2}\sum_{i=2}^{k} r_i(t_i - 1 - r_i).\end{aligned}$$

If $a + r_0 > 0$ and $g > 0$ then we can find some further restriction on $t_i$ to obtain the above Ramsey number for the sequence of paths.

First, by (6) and the assumption $a + r_0 > 0$, we note that

$$(12)\qquad r_0 > \frac{1}{2}\left(\sum_{i=1}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right) + y - 1\right).$$

Moreover, by (11), if

$$(13)\quad t_0 > \frac{1}{a+r_0}\left(2r_0(r_0+1) + \left(\sum_{i=1}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right)\right)^2 - (1-y)\sum_{i=1}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right)\right)$$

then $g > 0$.

By definition of $r_0$, (3) and (12), we get

$$(14)\qquad \begin{aligned} t_0 - 2 \geq r_0 &= \sum_{i=1}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right) - w\cdot(t_0 - 1)\\ &> \frac{1}{2}\left(\sum_{i=1}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right) + y - 1\right).\end{aligned}$$

Let us assume that $w > 0$. Then, by $t_0 = t_1$, we get

$$(15)\qquad \begin{aligned} \frac{1}{2}\left(\sum_{i=2}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right) + 2 - y\right) &> \left\lceil\frac{t_0}{2}\right\rceil + \frac{1}{2}\left\lfloor\frac{t_0}{2}\right\rfloor\\ &> \frac{1}{2}\left(\sum_{i=2}^{k}\left(\left\lfloor\frac{t_i}{2}\right\rfloor - 1\right) + y + 2\right).\end{aligned}$$

The left-side inequality in (15) follows by the right-side inequality from (14). The right-side inequality in (15) follows by the most left and the most right relation in (14). Hence we get a contradiction.

Let us assume that $w = 0$. Then, by (3) and $t_0 = t_1$, we get $r_0 = \lfloor \frac{t_0}{2} \rfloor + \sum_{i=2}^{k} \left( \lfloor \frac{t_i}{2} \rfloor - 1 \right)$. By (14) we get $y = 0$ or $y = 1$. So, by (13) and (6), we get $t_0 > \frac{1}{r_0+2-y}(2r_0(r_0+1) + (r_0-1)(r_0-2+y))$.

Considering the case we get $t_0 > 3r_0 - 7 + 16/(r_0+2)$ for $y = 0$ and $t_0 > 3r_0 - 3 + 4/(r_0+1)$ for $y = 1$. Elementary counting leads to the condition (iii) and (iv), respectively.

*Case* 3. Suppose that $t_0 = t_1 = t_2$. If the condition (v) holds then $n = 3, x = 2$. If the condition (vi) holds then $n = 2, x = 0$. Thus, by (4), we get $g > 0$ for these cases and the result holds. The proof is done. ■

We conclude with the following result for three paths.

**Corollary 12.** *Let $m, t, q$ ($m \geq t \geq q \geq 2$) be positive integers. Let either $m > \frac{1}{2}((t+q)^2 - 7(t+q) + 14)$ and $2 \not| (t+q)$ or $m > \frac{1}{4}((t+q)^2 - 6(t+q) + 12)$ and $2|t$ and $2|q$. Then $R(P_q, P_t, P_m) = m + \lfloor \frac{t}{2} \rfloor + \lfloor \frac{q}{2} \rfloor - 2$.*

***Proof.*** If $2 \not| (t+q)$ then we apply Theorem 11 (ii). If $2|t$ and $2|q$ then we apply Theorem 11 (i) for $m > 2$ and Theorem 11 (vi) for $m = q = t = 2$. ■

## REFERENCES


[1] S. Brandt, *A sufficient condition for all short cycles*, Discrete Appl. Math. **79** (1997) 63–66.

[2] T. Dzido, *Multicolor Ramsey numbers for paths and cycles*, Discuss. Math. Graph. Theory **25** (2005) 57–65.

[3] T. Dzido, M. Kubale and K. Piwakowski, *On some Ramsey and Turán-type numbers for paths and cycles*, Electr. J. Combin. **13** (2006) R55.

[4] R.J. Faudree and R.H. Schelp, *Path Ramsey numbers in multicolorngs*, J. Combin. Theory (B) **19** (1975) 150–160.

[5] A. Figaj and T. Łuczak, *The Ramsey number for a triple of long even cycles*, J. Combin. Theory (B) **97** (2007) 584–596.

[6] Y. Kohayakawa, M. Simonovits and J. Skokan, *The 3-colored Ramsey numbers of odd cycles*, Electr. Notes Discrete Math. **19** (2005) 397–402.

[7] D.R. Woodall, *Maximal circuits of graphs* I, Acta Math. Acad. Sci. Hungar. **28** (1976) 77–80.

Received 13 December 2007
Revised 4 July 2008
Accepted 23 October 2008