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Note

A NOTE ON k-UNIFORM SELF-COMPLEMENTARY HYPERGRAPHS OF GIVEN ORDER

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Abstract

We prove that a k-uniform self-complementary hypergraph of order n exists, if and only if $\binom{n}{k}$ is even.

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Let V be a set of n elements. The set of all k-subsets of V is denoted by $\binom{V}{k}$. A *k-uniform hypergraph* H consists of a *vertex-set* $V(H)$ and an *edge-set* $E(H) \subseteq \binom{V(H)}{k}$. Two k-uniform hypergraphs G and H are *isomorphic*, if there is a bijection $\theta : V(G) \to V(H)$ such that $e \in E(G)$ if and only if $\{\theta(x)|x \in e\} \in E(H)$. The *complement* of a k-uniform hypergraph H is the hypergraph $\overline{H}$ such that $V(\overline{H}) = V(H)$ and the edge set of which consists of all k-subsets of $V(H)$ not in $E(H)$ (in other words $E(\overline{H}) = \binom{V(H)}{k} - E$). A k-uniform hypergraph H is called *self-complementary* (*s-c* for short) if it is isomorphic with its complement $\overline{H}$. Isomorphism of a k-uniform self-complementary hypergraph onto its complement is called a *self-complementing permutation* (or *s-c permutation*).

The k-uniform s-c hypergraphs for $k = 3$ and $k = 4$ are studied in [3] and [6], respectively. The 2-uniform self-complementary hypergraphs are exactly self-complementary graphs. This class of graphs has been independently discovered by Ringel [4] and Sachs [5] who proved that an s-c graph of order n exists if and only if $n \equiv 0$ or $n \equiv 1 \pmod 4$ or, equivalently, whenever $\binom{n}{2}$ is even.

We prove a generalisation of this fact for k-uniform hypergraphs.

Theorem 1. *Let n and k be positive integers, $k \leq n$. There is a k-uniform self complementary hypergraph of order n if and only if $\binom{n}{k}$ is even.*

Let us give first some results which will be needed in the proof of Theorem 1.

For positive integers k and n we say that n *contains* k (we write $k \subset n$) if when k has 1 in a certain binary place, then n also has 1 in the corresponding binary place. That is, the binary representation of k can be obtained from that of n by changing some ones to zeros. For example, $6 \subset 14$ since $6 = 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$ and $14 = 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$ and, clearly, $5 \not\subset 14$. In [2] Hatcher and Riley solved a problem proposed by Kimball by proving the lemma which we give below (Moser has pointed out that this result is contained in [1]).

Lemma 1. *$\binom{n}{k}$ is odd if and only if $k \subset n$.*

Any positive integer n may be, in the unique way, written in the form $n = 2^l c$, where c is an odd integer. We denote then $\lambda(n) = l$. For any finite and nonempty set A we shall write $\lambda(A)$ in place of $\lambda(|A|)$, for short.

The following lemma is proved in [7].

Lemma 2. *Let k, m and n be positive integers, and let $\sigma : V \to V$ be a permutation of a set V, $|V| = n$, with orbits $O_1, \ldots, O_m$. σ is a self-complementing permutation of a self-complementary k-uniform hypergraph, if and only if, for every $p \in \{1, \ldots, k\}$ and for every decomposition*

$$k = k_1 + \ldots + k_p$$

of k ($k_j > 0$ for $j = 1, \ldots, p$), and for every subsequence of orbits

$$O_{i_1}, \ldots, O_{i_p}$$

such that $k_j \leq |O_{i_j}|$ for $j = 1, \ldots, p$, there is a subscript $j_0 \in \{1, \ldots, p\}$ such that

$$\lambda(k_{j_0}) < \lambda(O_{i_{j_0}}).$$

Proposition 1. *Let n and k be two non negative integers, $k < n$. The following two conditions are equivalent.*

(1) *$\binom{n}{k}$ is odd.*

(2) *For every non negative integer* l *such that* $k = a2^l + s$, *where* a *is odd and* $0 \le s < 2^l$ *we have* $n \in \{2^l + s, \dots, 2^{l+1} - 1\} \pmod{2^{l+1}}$.

Proof. Put $k = \sum_{i=0} c_i 2^i$ and $n = \sum_{i=0} d_i 2^i$, where $c_i, d_i \in \{0,1\}$ for every i. Let us suppose first that $\binom{n}{k}$ is odd. Then, by Lemma 1, for every i, $c_i = 1$ implies $d_i = 1$. Note that $k = a2^l + s$, where a is odd and $0 \le s < 2^l$, means exactly that $c_l = 1$ and $\sum_{i=0}^{l-1} c_i 2^i = s$. Since $d_i = 1$ whenever $c_i = 1$, we have $\sum_{i=0}^{l} d_i 2^i \ge 2^l + s$ for every l such that $c_l = 1$ (and, clearly, $\sum_{i=0}^{l} d_i 2^i < 2^{l+1}$).

If $\binom{n}{k}$ is even then, again by Lemma 1, there is l_0 such that $c_{l_0} = 1$ and $d_{l_0} = 0$. Hence $k = a2^{l_0} + s$, with a odd and $0 \le s = \sum_{i=0}^{l_0-1} c_i 2^i < 2^{l_0}$, and $n = b2^{l_0+1} + \sum_{i=0}^{l_0-1} d_i 2^i$. Since $\sum_{i=0}^{l_0-1} d_i 2^i < 2^{l_0}$, we have $n \in \{0, \dots, 2^{l_0} - 1\}$ $\pmod{2^{l_0+1}} \subset \{0, \dots, 2^{l_0} + s - 1\} \pmod{2^{l_0+1}}$ and the proposition is proved. ■

Proposition 1 is clearly equivalent to the following.

Proposition 2. *Let* n *and* k *be two non negative integers,* $k < n$. *The following two statements are equivalent.*

(1) $\binom{n}{k}$ *is even.*

(2) *There is a non negative integer* l_0 *such that* $k = a_0 2^{l_0} + s_0$, *where* a_0 *is odd,* $0 \le s_0 < 2^{l_0}$, *and* $n \in \{0, \dots, 2^{l_0} + s_0 - 1\} \pmod{2^{l_0+1}}$. ■

Lemma 3. *Let* l, k, s *and* n *be non negative integers such that* $k < n$, $k = a2^l + s$, *a is odd,* $s < 2^l$. *If* $n \in \{0, \dots, 2^l + s - 1\} \pmod{2^{l+1}}$ *then there is a* k*-uniform self-complementary hypergraph of order* n.

Proof. Let us write n in the form $n = b2^{l+1} + r$, where $0 \le r < 2^l + s$, and let σ be a permutation of an n-set V such that it has b orbits $O_1, \dots, O_b$, each of which having its cardinality equal to 2^{l+1}, and one orbit O_{b+1} with $|O_{b+1}| = r$. Applying Lemma 2 we shall prove that σ is the self-complementing permutation of a self-complementary k-uniform hypergraph.

Suppose, contrary to our claim, that σ is not s-c permutation of any s-c k-uniform hypergraph. Then, by Lemma 2, there is a decomposition of k, $k = k_1 + \ldots + k_p$ and a subsequence $O_{i_1}, \dots, O_{i_p}$ of $O_1, \dots, O_{b+1}$ such that $0 < k_j \le |O_{i_j}|$ and $\lambda(k_j) \ge \lambda(O_{i_j})$ for $j = 1, \dots, p$. Clearly, we have $k_j = |O_{i_j}| = 2^{l+1}$ whenever $i_j \ne b+1$. Hence there exists j_0 such that $i_{j_0} = b+1$ and $k_{j_0} = k - \sum_{j \ne j_0} k_j = (2^l a + s) - (p-1)2^{l+1} = 2^l(a - 2(p-1)) + s$. Observe that $a - 2(p-1) > 0$ is positive and odd, so we have $k_{j_0} \ge 2^l + s > r = |O_{b+1}|$. This contradicts our assumption that $|O_{b+1}| \ge k_{j_0}$. ■

Note that if there is a k-uniform s-c hypergraph of order n then, clearly, $\binom{n}{k}$ is even. Now the proof of Theorem 1 follows by Lemma 3 and Proposition 2. $\blacksquare$

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