QUASIPERFECT DOMINATION
IN TRIANGULAR LATTICES

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Abstract

A vertex subset S of a graph G is a perfect (resp. quasiperfect) dominating set in G if each vertex v of $G \setminus S$ is adjacent to only one vertex ($d_v \in \{1, 2\}$ vertices) of S. Perfect and quasiperfect dominating sets in the regular tessellation graph of Schläfli symbol $\{3, 6\}$ and in its toroidal quotients are investigated, yielding the classification of their perfect dominating sets and most of their quasiperfect dominating sets S with induced components of the form K_ν, where $\nu \in \{1, 2, 3\}$ depends only on S.

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1. Introduction

A vertex subset S of a graph G is a *perfect dominating set*, or PDS, in G if each vertex v of the complementary graph $G \setminus S$ of S in G is adjacent to exactly one vertex of S. If G is a regular graph, then $G \setminus S$ is a regular graph, [2, 3]. If in addition S is isolated in G, then S is said to be a 1-*perfect code*, [5], or *efficient dominating set* in G, [1].

In the present work, the graph that spans the regular tessellation of Schläfli symbol $\{3, 6\}$ in an Euclidean plane $\Pi \equiv \mathbb{R}^2$ (with six equilateral triangles at each vertex, as in the center of Figure 4/4 in page 24 of [4]) is referred to as the *triangular lattice* Δ. We will visualize Π as the set of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 = 0$; the vertices of Δ

will be indicated by these Δ-*coordinates* in $\mathbb{R}^3$, as shown in Figure 1. Even though no two Δ-coordinates are orthogonal, they are more manageable for our purposes than orthogonal coordinates are. Notice, for example, that the positive coordinate directions of x_1 and x_2 are separated by an angle of $60°$.
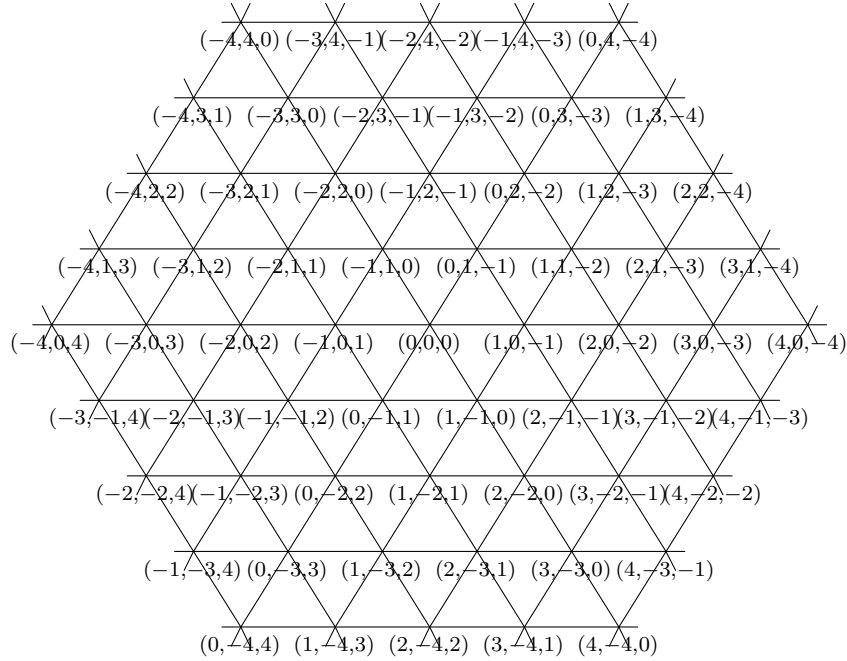

Figure 1. Simplex coordinates in Δ.

Suggested by A. Delgado, the problem of characterizing the PDSs in Δ and its toroidal quotients is completed in Sections 2–3: Up to symmetry, there is just one 1-perfect code in Δ. In order to broaden the subject, the PDS condition is relaxed by defining a *quasiperfect dominating set*, or QPDS, in a graph G as a vertex subset S of G for which each vertex v of $G \setminus S$ is adjacent to $d_v \in \{1, 2\}$ vertices of S.

Section 4 deals with constant $d_v = 2$; the rest of the paper is devoted to characterizing, in Δ and its toroidal quotients, the QPDSs S with induced components of the form K_ν, for fixed $\nu = \nu(S) \in \{1, 2, 3\}$, (only dependent on S). This is left open in Δ for a few unknown possibilities, in Theorem 13 and Conjecture 15. Theorems 1, 6 and 22 characterize all the remaining cases; they have such K_ν's with a minimum graph distance $\delta = 3$, in contrast with one in Section 4, for which $\delta = 2$, (see left of Figure 3).

An ordered triple of pairwise adjacent vertices of Δ, as for example $T_0 = ((0,0,0), (1,0,-1), (0,1,-1))$, will be said to be an *ordered triangle*. There is a bijection Θ from the group $\mathcal{A}(\Delta)$ of automorphisms of Δ onto the collection of ordered triangles. In fact, each $\phi \in \mathcal{A}(\Delta)$ is determined by assigning T_0 to an ordered triangle $\Theta(\phi) = \phi(T_0)$ with the vertices of T_0 sent bijectively (in one of six fashions) onto the vertices of $\Theta(\phi) = \phi(T_0)$. This yields that $\mathcal{A}(\Delta)$ is a semidirect product of the cyclic group D_{12} (of symmetries of Δ around $(0,0,0)$) and the group $\mathbf{Z}^2$ (of parallel translations of Δ expressed, say, in the first two Δ-coordinates: x_1, x_2). This semidirect product, $D_{12} \times_\phi \mathbf{Z}^2$, is given in the Cartesian set product $D_{12} \times \mathbf{Z}^2$ via the homomorphism $\phi : \mathbf{Z}^2 \to \mathcal{A}(D_{12})$ defined by $\phi(m)(r) = mrm^{-1} = mr(-m)$, with multiplication set by $(r,m)(s,n) = (r\phi(m)(s), mn)$, where $r, s \in D_{12}$ and $(m,n) \in \mathbf{Z}^2$.

2. Perfect Domination in Δ

The *minimum distance graph* $M(S)$ of a 1-perfect code S in Δ is the graph whose vertex set is S and whose edges are traced between those pairs of vertices that realize the minimum graph distance, 3, of S in Δ. On the left of Figure 2, such an S and its 5-regular complement $\Delta \setminus S$ in Π are represented, (coinciding with the $(3,3,3,3,6)$-tessellation of Figure 5/1 in page 42 of [4]); its $M(S)$ is shown on the right of Figure 2 (superposable, vertex by corresponding vertex, over the left of Figure 2).
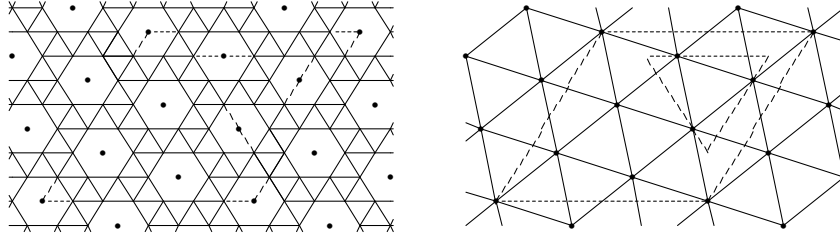

Figure 2. Complement of 1-Perfect code S in Δ and its graph $M(S)$.

It is convenient to represent the line graph $L(S)$ of $M(S)$ as a plane graph with its vertices as the midpoints of the edges of $M(S)$. By properly coloring the faces of $L(S)$ with seven colors, it can be seen that an upper bound on the chromatic number χ of the graph of Π with a forbidden fixed Euclidean

distance is 7, (which can be realized via the chromatic number, 7, of $M(S)$; actually, Thomassen [6] showed that $\chi = 7$).

The complement of $\Delta \setminus S$ in Π is the disjoint union of 2-dimensional connected components whose closures are regular hexagons and equilateral triangles, both of unit side lengths. The Euclidean distance between such hexagons (resp. triangles) has a lower bound of $\frac{\sqrt{3}}{2}$ (0). The collection of these hexagons and triangles will be denoted by $\mathcal{P}(S)$. An *n-hole* of a graph G is an induced n-cycle of G (no chords, i.e., extra edges). Then the boundary of each hexagon h in $\mathcal{P}(S)$ forms a 6-*hole* of $\Delta \setminus S$. Alternatively, we can say that h is *bordered* by a 6-hole.

A superposed plane representation of both $L(S)$ and $\Delta \setminus S$ can be found on the lower part of Figure 5/2 in page 46 of [4], where the vertices of $L(S)$ are the centroids of those triangles in $\mathcal{P}(S)$ whose vertices are shared by three different hexagons of $\mathcal{P}(S)$. Each edge of $L(G)$ is of length $\sqrt{\frac{7}{3}}$ and perpendicular to the edge of $M(G)$ that crosses it. The edge perpendicular to each edge e of $M(G)$ is realized by the middle third of the segment joining the pair of vertices opposite to e in the triangles incident to e.

Theorem 1. *Up to symmetry, there is exactly one proper* PDS *S in Δ. Moreover, S is a 1-perfect code and its minimum graph distance is $\delta = 3$.*

Proof. We assert that a proper PDS S in Δ is isolated. For otherwise there would be two adjacent vertices v and w in S having a common neighbor in $\Delta \setminus S$, a contradiction. In fact, there exists exactly two proper PDS S in Δ, up to parallel translations and rotations of Π; their complementary graphs $\Delta \setminus S$ are *enantiomorphic* (that is: mirror images of each other, say by reflection on the x_1-axis), so they are isomorphic. One of these PDSs is the one depicted on the left of Figure 2. ∎

Let S be as in Theorem 1. An ordered pair of adjacent vertices of Δ is said to be an *arc*. Let S be a PDS in Δ. Then there exists a bijection θ from $\mathcal{A}(\Delta \setminus S)$ onto the collection of arcs of 6-holes of $\Delta \setminus S$. Assume that $\Delta \setminus S$ has a 6-hole containing the arc $A_0 = ((0, -1, 1), (1, -1, 0))$. Then any $\psi \in \mathcal{A}(\Delta \setminus S)$ is determined by assigning A_0 to $\theta(\psi) = \psi(A_0)$. This allows us to see $\mathcal{A}(\Delta \setminus S)$ as a semidirect product of $\mathbf{Z}_6$ and $\mathbf{Z}^2$, where the generators $(1, 0)$ and $(0, 1)$ of $\mathbf{Z}^2$ are sent onto the parallel translations of $\Delta \setminus S$ along respective vectors $(3, -1 - 2)$ and $(1, 2, -3)$. However, $\mathcal{A}(M(S)) = \mathcal{A}(L(S))$ is isomorphic to $\mathcal{A}(\Delta)$.

Each vertex x of S excludes its six neighbors in Δ from S, which determine a regular 6-hole $S(x)$ in $\Delta \setminus S$; each edge of $S(x)$ is adjacent to a triangle in $\Delta \setminus S$, so there are six triangles in $\Delta \setminus S$ adjacent to $S(x)$ by means of an edge each. If a triangle t of $\Delta \setminus S$ is not adjacent to any $S(x)$, then t is incident to three 6-holes $S(x), S(y), S(z)$, where $x, y, z \in S$ are at unit distance from t. Any such t will be called a $\mathbf{Z}_3$-triangle; (rotations around the barycenter of t offer three automorphisms of $\Delta \setminus S$ composing a subgroup $\mathbf{Z}_3$ of $\mathcal{A}(\Delta \setminus S)$). So, each vertex of a $\mathbf{Z}_3$-triangle is incident to just one 6-hole in $\Delta \setminus S$.

Corollary 2. $V(\Delta)$ *admits a partition into seven translated copies of S at unit distance from each other. Moreover, the total number of* PDS*s isomorphic to S in Δ is* 14. *These* 14 PDS*s compose two enantiomorphic partitions of Δ into* 1-*perfect codes.*

Proof. Given a fixed $S(x)$ in $\Delta \setminus S$, the parallel translations of S taking x to its neighbors in $S(x)$ give the remaining six members of the partition in the first assertion of the corollary. This yields seven of the 14 copies of S mentioned in the second assertion. The other seven copies of S are obtained similarly from a reflected copy of S in Δ through the reflection, say, on the x_1-axis. ∎

Note that the inner dotted lines on the left of Figure 2 form an example of the largest 1-perfect code existing on a triangular subgraph of Δ.

3. PDSs in Toroidal Triangular Lattices

Let us express the vertices of Δ by means of their first two Δ-coordinates: x_1, x_2. The subgraph Δ' of Δ spanned by the edges with constant x_1 or x_2 has a quotient Cartesian product $C_m \times C_n$ of cycles of lengths $m, n \in \mathbf{Z}$ with $3 \leq m$ and $3 \leq n$. From such $C_m \times C_n$, which is a toroidal graph (i.e., embeddable into the flat torus $\mathcal{T}$), we obtain a *triangular lattice* graph $\Delta_{m,n}$ by adjoining to it the *anti-diagonal edges* $\{(i, j + 1), (i + 1, j)\}$ of the *elementary 4-cycles* $((i, j), (i+1, j), (i+1, j+1), (i, j+1))$ of $C_m \times C_n$, where $0 \leq i < m$, $0 \leq j < n$ and additions are taken mod m, n, respectively.

There are 1-perfect codes S in the graphs $\Delta_{7k,7\ell}$, for $0 < k, \ell \in \mathbf{Z}$, with cardinality $\frac{1}{7}|V(\Delta_{7k,7\ell})| = 7k\ell$. An example of such an S in $\Delta_{7,7}$ can be visualized from either side of Figure 2, where the outer dotted lines delineate the boundary of a rhomboidal pattern Υ, which is a cutout of $\mathcal{T}$.

Corollary 3. *The vertex set of a graph $\Delta_{m,n}$ has a 1-perfect code partition $\mathcal{S} = \{S^0 = S, S^1, \ldots, S^6\}$ if and only if m and n are multiples of 7. Each component code S^i is a translate of S and has cardinality $mn/7$.*

Proof. Using that Υ is a minimal cutout for a toroidal embedding of Δ, it can be seen that the natural projection $\Delta \to \Delta_{m,n}$ maps a partition as in Corollary 2 onto a partition as claimed. By allowing larger rhomboidal cutouts formed by $m/7$ horizontally contiguous copies of Υ times $n/7$ diagonally contiguous copies of Υ, the existence of the claimed partitions is ensured. Other values of m and n are clearly incompatible with the existence of isolated PDSs in corresponding graphs $\Delta_{m,n}$. ∎

The group $\mathcal{A}(\Delta_{m,n})$ can be expressed with $\mathbf{Z}^2$ in the remark of the last paragraph in Section 1 replaced by $\mathbf{Z}_m \times \mathbf{Z}_n$. It follows that $\mathcal{A}(\Delta_{m,n} \setminus S)$ is a semidirect product of the rotation group $\mathbf{Z}_6$ and the translation group $\mathbf{Z}_m \times \mathbf{Z}_{n/7}$, or also of $\mathbf{Z}_6$ and $\mathbf{Z}_{m/7} \times \mathbf{Z}_n$.

4.   Semiperfect Domination in Δ and $\Delta_{m,n}$

Let G be a graph and let S be a QPDS in G. If each vertex of $G \setminus S$ is adjacent to *exactly two* vertices of S, then S is a *semiperfect dominating set*, or SPDS, in G.

Theorem 4. *Up to symmetry, there are exactly two SPDSs S in Δ. One of them has minimum graph distance δ between induced components of S equal to 2, while the other has Δ equal to 3.*

Proof. An isolated (resp. non-isolated) SPDS S in Δ with $\delta = 2$ (3) and induced components spanning affine spaces of dimension 0 (1) in Π is shown on the left (center) of Figure 3, with connected (disconnected) 4-regular complement $\Delta \setminus S$ in Π; the two edges at each $v \in V(\Delta \setminus S)$ are separated by an angle of 180° (60°) and their endvertices in S are in different (the same) component(s) of $\Delta[S]$. Up to symmetry, no other SPDSs exist in Δ. ∎

The graph $\Delta \setminus S$ on the left of Figure 3 coincides with the $(3, 6, 3, 6)$-tessellation of Figure 5.1 in page 42 of [4]. Its automorphism group, $D_{12} \times_\phi (2\mathbf{Z})^2$ (isomorphic to $D_{12} \times_\phi \mathbf{Z}^2$), is embedded naturally into $\mathcal{A}(\Delta) = D_{12} \times_\phi \mathbf{Z}^2$.
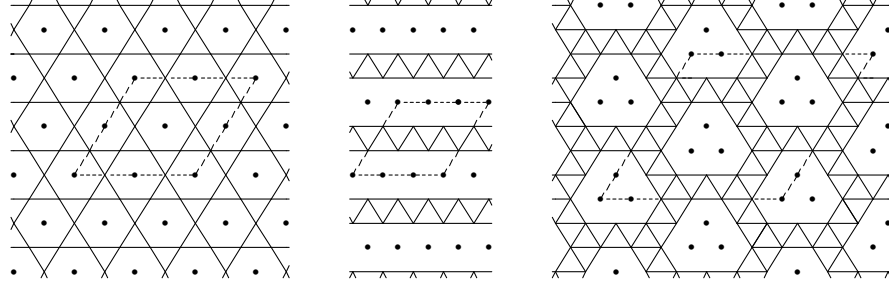
Figure 3. Complements of QPDSs in Δ in Sections 4 and 5.

Now, let S be as in the center of Figure 3. The automorphism group $\mathcal{A}(\Delta\backslash S)$ is the semidirect product of the doubly reflective group $\mathbf{Z}_2^2$ (whose reflection axes are the x_1-axis and the line that passes through $(0,0,0)$ and $(-1,2,-1)$) and the translation group $\mathbf{Z}^2$, with its generators $(1,0)$ and $(0,1)$ of $\mathbf{Z}^2$ sent onto the parallel translations of $\Delta\setminus S$ along respective vectors $(1,0,-1)$ and $(0,3,-3)$.

Corollary 5. (A) *For $1 \leq m, n \in \mathbf{Z}$, $\Delta_{2m,2n}$ contains an isolated SPDS S, with connected induced subgraph $\Delta_{m,n}[S]$.*

   (B) *For $3 \leq m, n \in \mathbf{Z}$, there exists a graph $\Delta_{m,n}$ containing a non-isolated SPDS S if and only if either $3|m$ or $3|n$. Accordingly, the number of induced components of S in $\Delta_{m,n}$, or components of $\Delta_{m,n}[S]$, is either $\frac{m}{3}$ or $\frac{n}{3}$, so each such component is either an n-cycle or an m-cycle.*

Proof. Examples of SPDSs as in items (A) and (B) can be visualized from the left and center of Figure 3, respectively, where the dotted lines delineate the boundaries of rhomboidal cutouts of $\mathcal{T}$. In the first case of (B), for example, there are SPDSs in the graphs $\Delta_{k,3\ell}$ for $3 \leq k \in \mathbf{Z}$ and $1 \leq \ell \in \mathbf{Z}$. These SPDSs have cardinality $\frac{1}{3}|V(\Delta_{k,3\ell})| = 3k\ell$.   ∎

5. K_3-QUASIPERFECT DOMINATION IN Δ AND $\Delta_{m,n}$

Let H be a subgraph of a graph G. A QPDS S of G is an *H-quasiperfect dominating set*, or *H*-QPDS, in G if all the induced components of S in G, that is all the components of $G[S]$, are isomorphic to H.

Theorem 6. *Up to symmetry, there is exactly one K_3-QPDS in Δ. The minimum graph distance between K_3's induced in S is $\delta = 3$.*

Proof. If S is a K_3-QPDS in Δ, then the complement of $\Delta \setminus S$ in Π is the disjoint union of 2-dimensional connected components whose closures are equilateral triangles and equiangular semisymmetric hexagons (the elements of a set $\mathcal{P}(S)$, as in Section 2), illustrated on the left of Figure 3. The Euclidean distance between such hexagons (resp. triangles) has a lower bound of $\frac{\sqrt{3}}{2}$ (0). Notice that the boundary of each such hexagon constitutes a 9-hole in $\Delta \setminus S$. Clearly, $\delta = 3$. ∎

The automorphism group $\mathcal{A}(\Delta \setminus S)$ for the K_3-QPDS S of Δ in the proof of Theorem 6 is a semidirect product of the group S_3 of symmetries of a fixed equiangular semiregular hexagon as above and the group $\mathbf{Z}^2$, where the generators $(1, 0)$ and $(0, 1)$ of $\mathbf{Z}^2$ are sent onto the parallel translations of $\Delta \setminus S$ along respective vectors $(4, -2, -2)$ and $(2, 2, -4)$.

From Theorem 6 and the left of Figure 3, we see that for $0 < k, \ell \in \mathbf{Z}$ there are K_3-QPDSs in toroidal graphs $\Delta_{6k,6\ell}$ and that they have cardinality $\frac{1}{9}|V(\Delta_{6k,6\ell})| = 3k\ell$. An example of such a K_3-QPDS in a toroidal graph $\Delta_{6,6}$ can be visualized by identifying adequately the dotted lines on the left of Figure 3.

Corollary 7. *There exists a toroidal graph $\Delta_{m,n}$ containing a K_3-QPDS S if and only if m and n are multiples of 6. The number of induced components of such an S in $\Delta_{m,n}$, that is the components of $\Delta_{m,n}[S]$, is $\frac{mn}{12}$.*

Proof. The argument here resembles that of the proof of Corollary 3 but for just one K_3-QPDS, because there is no partition in this case. ∎

The group $\mathcal{A}(\Delta \setminus S)$ of the K_3-QPDS S in $\Delta_{m,n}$ of Theorem 6 is a semidirect product of $\mathbf{Z}_m \times \mathbf{Z}_{n/3}$ and S_3, or also of $\mathbf{Z}_{m/3} \times \mathbf{Z}_n$ and S_3.

6. K_2-Quasiperfect Domination in Δ and $\Delta_{m,n}$

Let S be a K_2-QPDS in Δ. The complement of $\Delta \setminus S$ in Π is the disjoint union of 2-dimensional connected components whose closures are equilateral triangles and elongated hexagons, these containing each a unique induced component K_2 of S in Δ and thus a unique induced edge. The Euclidean distances between such hexagons (resp. triangles) has a lower bound of $\frac{\sqrt{3}}{2}$ (0). The collection of these triangles and hexagons will be denoted by $\mathcal{P}(S)$. Notice that the boundary of each elongated hexagon in $\mathcal{P}(S)$ here

constitutes an 8-hole in $\Delta \setminus S$. It is clear that the minimum graph distance between those components K_2 is $\delta = 3$.
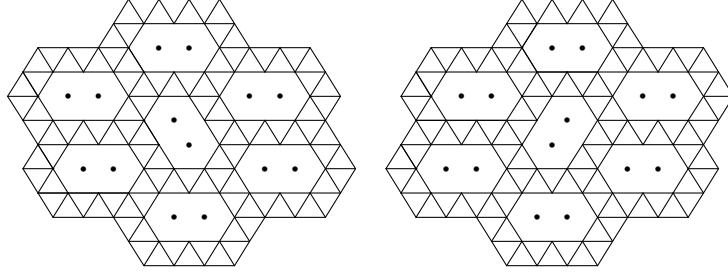

Figure 4. Examples of hexagon types in complements of K_2-QPDSs in Δ.

An hexagon in $\mathcal{P}(S)$ is said to be of *type* $i \in I_3 = \{1, 2, 3\}$ if the vertices of S it contains share the i-th Δ-coordinate, x_i. Figure 4 contains on its left (resp. right) side an hexagon of type 2 (3) bordered by equilateral triangles separating it from six hexagons of type 1. In view of this, we define an auxiliary graph $\Gamma(S)$ whose vertices are the components of $\Delta[S]$, or of some $\Delta_{m,n}[S]$; two such vertices are adjacent $\Gamma(S)$ if and only if the induced components they represent are at a graph distance of 3, the minimum attainable graph distance separating the components of $\Delta[S]$ in Δ. Alternatively, we may consider the vertices of $\Gamma(S)$ as the hexagons of types $\in I_3$, any two of them adjacent if and only if the Euclidean distance between them is $\frac{\sqrt{3}}{2}$.

For the two portions of a $\Delta \setminus S$ in Figure 4, the corresponding portions of their associated graphs $\Gamma(S)$ look like as in the leftmost tables below:

`  1  `
`1   1`
`  2  `
`1   1`
`  1  `
	`1    1    2`
	`  1    1  `
	`1    2    1`
	`  1    1  `
	`2    1    1`
		`2    1    2`
		`  1    1  `
		`1    2    1`
		`  1    1  `
		`2    1    2`

`  1  `
`1   1`
`  3  `
`1   1`
`  1  `
	`3    1    1`
	`  1    1  `
	`1    3    1`
	`  1    1  `
	`1    1    3`
		`3    1    3`
		`  1    1  `
		`1    3    1`
		`  1    1  `
		`3    1    3`

where the type of each vertex represents it, and two vertices so represented are adjacent if they are vertically, diagonally or anti-diagonally contiguous,

meaning they are on a line at 90°, 60° or 120°, respectively, from the x_1-axis. The tables at right and far right of each such table represent two possible extended portions of $\Delta \setminus S$ in each case. We conclude that there are two extreme possibilities for extending these portions to complements of K_2-QPDSs in Δ. One of them has an infinitely extended table with either a unique infinite anti-diagonal that descends or ascends with period (12), as in the bisequence (i.e., doubly infinite sequence) $(\ldots 1212\ldots)$, or a unique infinite diagonal that descends or ascends with period (13), as in the bisequence $(\ldots 1313\ldots)$, with all the hexagon types that are away from this anti-diagonal or diagonal being equal to 1, as in the middle tables of the arrangement above. The other extreme possibility is to have alternate diagonals, (resp. anti-diagonals) of periods (12) (resp. (13)) and (1), as on the rightmost tables above. Between these two pairs of extreme situations lie all the possible complements of K_2-QPDSs in Δ whose induced components pertain to just two hexagon types, one of which is 1.
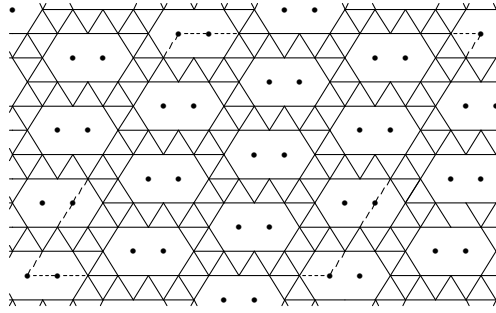

Figure 5. Complement of a type-parallel K_2-QPDS in Δ.

We consider now the K_2-QPDSs S of Δ for which the hexagons in $\mathcal{P}(S)$ have constant type, for example type 1, as in Figure 5. The automorphism group $\mathcal{A}(\Delta \setminus S)$, for one such S, is the semidirect product of the doubly reflective group $\mathbf{Z}_2^2$ of a fixed hexagon of type 1 and the group $\mathbf{Z}^2$, with its generators $(1,0)$ and $(0,1)$ sent onto the parallel translations of $\Delta \setminus S$ along respective vectors $(4,-2,-2)$ and $(3,1,-4)$.

Any K_2-QPDS in Δ for which the hexagons in $\mathcal{P}(S)$ have constant type is said to be a *parallel K_2-QPDS*.

Theorem 8. *Up to symmetry, there is only one parallel K_2-QPDS S in Δ.*

Proof. Consider an hexagon of type 1 in $\mathcal{P}(S)$, for an S as in the statement, and delimited by one of its 8-holes. By analyzing the possibilities of neighboring hexagons of type 1 in the formation of such an S, it follows that the case in Figure 5 is the only possible one. ∎

Corollary 9. *For* $0 < k, \ell \in \mathbf{Z}$, *there are parallel* K_2-*QPDSs in the graphs* $\Delta_{10k,10\ell}$ *with cardinality* $\frac{1}{10}|V(\Delta_{10k,10\ell})| = 10k\ell$.

Proof. An example of such a parallel K_2-QPDS in a graph $\Delta_{10,10}$ can be visualized by identifying adequately the dotted lined in Figure 5. ∎
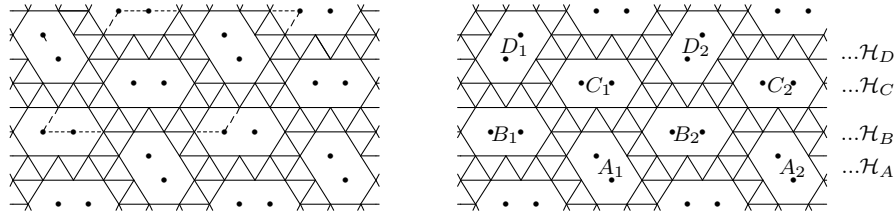

Figure 6. Complements of QPDSs in Δ for Theorems 11 and 16.

Corollary 10. *The vertex set of a graph* $\Delta_{m,n}$ *has a parallel* K_2-*QPDS partition* $\mathcal{S} = \{S^0 = S, S^1, \ldots, S^9\}$ *if and only if* m *and* n *are multiples of* 10. *Each component code* S^i *of such a partition* $\mathcal{S}$ *is a translate of* S *and has cardinality* $mn/10$.

Proof. This follows from the cutout in Figure 5 and the ideas in the proofs of Corollaries 3, 7 and 9. ∎

Now, consider the case of K_2-QPDSs in Δ, or some $\Delta_{m,n}$, with just two types of hexagons, say types $i, j \in I_3$, $i \neq j$, in which an hexagon of type i is *surrounded* by hexagons of type j: its adjacent hexagons in $\Gamma(S)$ are all of type j. In Δ, this implies that each hexagon of type j is surrounded by three hexagons of type i and three alternated hexagons of type j. For example, the upper-right table in the arrangement of six tables above can be extended so that each hexagon of type 2 is surrounded by six hexagons of type 1. We say that a K_2-QPDS S in Δ or some $\Delta_{m,n}$, with exactly two hexagon types $i, j \in I_3$, is (i, j)-*surrounded* if each of its hexagons of type i is surrounded by hexagons of type j. The following result is generalized in Theorem 16.

Theorem 11. *Up to symmetry, there is only one (i,j)-surrounded K_2-QPDS in Δ, for each two types $i, j \in I_3$.*

Proof. The K_2-QPDS in the left (resp. right) side of Figure 4 extends to the sole existing K_2-QPDS as claimed, with $i = 2$ (3) and $j = 1$. ∎

Corollary 12. *There exists a graph $\Delta_{m,n}$ containing a $(2,1)$-surrounded K_2-QPDS S if and only if $6|m$ and $5|n$. The number of hexagons of type 2, resp. 1, for such an S in $\Delta_{m,n}$ is $\frac{mn}{30}$, resp. $\frac{2mn}{30}$.*
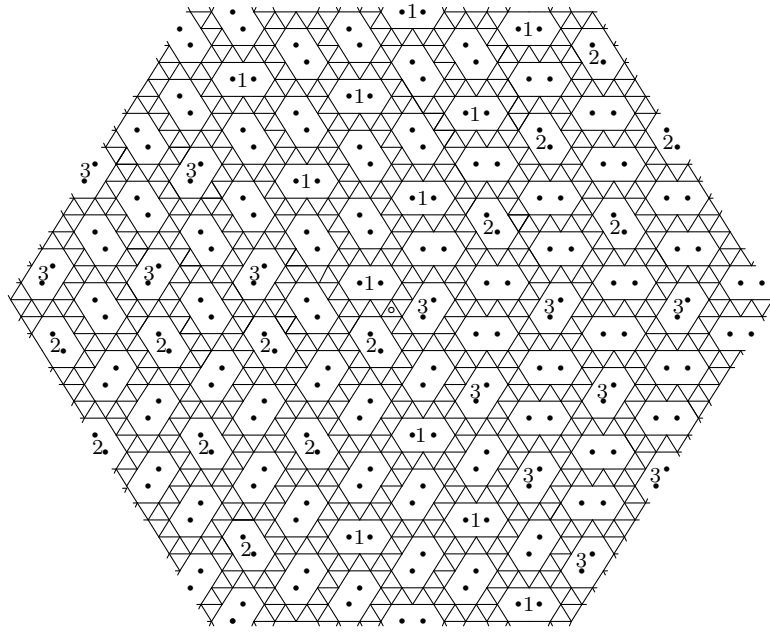
Figure 7. Complement of K_2-QPDS in Δ with $\mathbf{Z}_3$-symmetry.

Proof. This follows the ideas in the proofs of Corollaries 3, 7 and 9, with any of its smallest cases, for example the cutout on the left representation of Figure 6, leading to a K_2-QPDS S in $\Delta_{6,5}$ that produces barely one hexagon of type 2, say J, surrounded by only two hexagons of type 1, each adjacent to J by means of three edges in $\Gamma(S)$. ∎

Theorem 13. *Up to symmetry, there are at least two K_2-QPDSs in Δ with three pairwise adjacent hexagons of different types, covering I_3.*

Proof. Figure 7 can be extended to a K_2-QPDS S for which only the hexagons of $\mathcal{P}(S)$ adjacent to more than three hexagons of a common type are labeled with its type. The set of all the types for this figure has a disposition as in the left frame of the following table, for which the mentioned hexagons (adjacent to more than three hexagons of a common type) are represented by underlined types, as a reference, and the hexagons nearest to the center of $\mathbf{Z}_3$-symmetry are indicated each with a dot over their type.

```
                    1   2   1                 1    3    3    1     3    1     3
          2    2    2   1   2                3    1  1  3  3  3    3  1  1    3  3
     2    1    1    2   1   1         2      1    3  3  1  3  3    1  3  3  3  3  3
     2    2    2    2   1   2   1  2  1      1  3 1  3 1  3 3  3 2 3  2 3  2 3  2
  3  2  3 2  1 2  1 2 1 1   2   1    1       1  1 3  1 3  1 3 3  3 2 3  2 3  2 3  2
3 2  3 2  3 2  2    i   1 1 1   1    1    1  1  1 1  1 3 1 3 3  i  3 2 3  2 3  2 3  2
2 2  2 2  2 2  2    2 3 1 3 1 3 1 3  1    1  2  1 1  1 3 1 1 2  3  2 3  2 3  2 3  2
  3  3 3  3 3  3 3  3   1   3   1            1  2 2  1 1 1 1 1  1 2 2  2 2  2 2  2 2
  2  3 2  3 2  3 1  3   1   1                2  1 1  1 1 2  1 2 1  1 2  1 2 3  2 3
        3   2  3 1  3 1 3   1   3            1  2 1  1 2 1  1 2 1  1 2 2  2 3  2 3
        3   3  3 3  3 3  3                   2  1 1  1 1 2  1 2 1  1 2 2  2 3  2 3
                   1   1   3                 1  1 2  1 1 2  1 2 2  2 2  2 2
```

The right frame of the table offers a (larger) corresponding disposition of a second K_2-QPDS S as in the statement, possessing only six hexagons adjacent to at most three hexagons of a common type. Among these six hexagons, there are three nearest to the center of $\mathbf{Z}_3$-symmetry; each of them constitutes the initial vertex of an infinite type-constant path λ_i in $\Gamma(S)$, $(i \in I_3)$. Each λ_i, whose vertices are indicated in the frame by underlined types, has contiguous edges with two alternate directions. For example, λ_3 has all its vertices labeled $\underline{3}$, starts immediately over the vertex labeled $\dot{3}$ and proceeds first with a vertical edge, then with a diagonal edge (in the upper-left direction), and so on, alternatively. Note that the edge directions are perpendicular to the edges of $\Delta[S]$ in the hexagons that represent vertices of the path, in each λ_i.

The other three of the mentioned six hexagons are each the initial vertex of two infinite type-constant paths sharing their first edge: a rectilinear path η_i (in the direction perpendicular to the edge of $\Delta[S]$ in this initial hexagon) and a zigzagging path ξ_i $(i \in I_3)$. The vertices in both paths and between them have constant type, that we underline in the frame (as we did for the

vertices of λ_i, $i \in I_3$). For example toward the upper-right corner of the frame, a rectilinear path η_3 of vertices labeled $\underline{3}$ (showing five such vertices in the frame) can be seen at an angle of $30°$ over the x_1-axis, sharing its first edge with a zigzagging path ξ_3 of vertices also labeled $\underline{3}$. The remaining vertices of $\Gamma(S)$, which were left not underlined, form constant-type rectilinear paths. Starting from ξ_3 and proceeding clockwise around the center of symmetry, these paths are first vertical 3-paths (four vertices each) having alternate set of vertices with constant types 2, 3. The subgraph induced in $\Gamma(S)$ by the vertices in these 3-paths has parallel delimiting infinite paths whose edges have the non-vertical directions alternating. Counterclockwise from η_3, infinite constant-type rectilinear paths ascending at an angle of $30°$ from vertices next to those of η_3 appear that alternate constant types 1, 3. $\mathbf{Z}_3$-symmetry takes care of the labels of the other vertices in $\Gamma(S)$.   ∎

Corollary 14. *Both K_2-QPDSs in the proof of Theorem* 13 *have automorphism group Z_3, yielding asymmetric (Penrose) tilings of Π. Its second (resp. first) K_2-QPDS S determines an independent vertex set of $\Gamma(S)$ formed by six (resp. an infinite number of) hexagons in $\mathcal{P}(S)$ adjacent each in $\Gamma(S)$ to at most (resp. more than) three hexagons of a common type.*   ∎

The assertion about asymmetric Penrose tilings in Corollary 14 may be compared with a similar assertion in Section 6 of [2] or Section 6 of [3]. In the present section, however, as in Sections 2 and 5 above, the triangles of $\mathcal{P}(S)$ may be thought as forming a 'mortar' that binds – and separates – the hexagons of $\mathcal{P}(S)$ taken as 'bricks', a situation different from that of [2, 3], even though in both situations $\delta = 3$. This mortar and brick interpretation, however, seems to be restricted by the following conjecture.

Conjecture 15. *No K_2-QPDS in Δ contains two different triples of pairwise adjacent hexagons in $\mathcal{P}(S)$ with pairwise different types* 1, 2 *and* 3.

The following definition starts our approach to classifying the K_2-QPDSs in Δ with no triples of pairwise adjacent hexagons in $\mathcal{P}(S)$ having pairwise different types. Such a classification is completed in Theorem 22, below.

Given a K_2-QPDS S in Δ, we say that a *hexagonal row* of type $i \in I_3$ for S is a subset $\mathcal{H}$ of hexagons of type i in $\mathcal{P}(S)$, with $\mathcal{H}$ minimal among the subsets $\mathcal{H}'$ of hexagons of type i in $\mathcal{P}(S)$ having induced automorphism groups $\mathcal{A}(\mathcal{H}') = \{f_k : \Delta \to \Delta; k \in \mathbf{Z}\}$, where f_k is the parallel translation given by $f_k(x_1, x_2, x_3) = (x_1 + 6k, x_2, x_3 - 6k)$, for each $(x_1, x_2, x_3) \in \Delta$.

For example, on the left of Figure 6 the hexagons A_1 and A_2 determine a hexagonal row $\mathcal{H}_A$ of type 2. Above it, hexagonal rows $\mathcal{H}_B, \mathcal{H}_C, \mathcal{H}_D$ of respective types 1, 1, 3 can be determined by the hexagons B_1 and B_2, C_1 and C_2, D_1 and D_2, respectively. We may say here that two 'contiguous' hexagonal rows of type 1, ($\mathcal{H}_B$ and $\mathcal{H}_C$), are 'bordered below' by a hexagonal row of type 2, ($\mathcal{H}_A$), and 'above' by a hexagonal row of type 3, ($\mathcal{H}_D$). This 'sandwiched' disposition may be continuated, leading to a bisequence of such hexagonal rows, which can be described by the upward bisequence $\ldots, 2, 1, 1, 3, 1, 1, 2, 1, 1, 3, 1, 1, 2, 1, 1, 3, \ldots$, (for the types $\ldots, 2, 1, 1, 3, \ldots$, etc. of the hexagonal rows $\ldots, \mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \mathcal{H}_D, \ldots$, etc., respectively), or any other bisequencing of substrings $2, 1, 1$, and $3, 1, 1$, (such as $\ldots, 2, 1, 1, 2, \ldots$, etc., for the left of Figure 6). This takes us to the existence of an infinite number of such *sandwiched* K_2-QPDS in Δ, as in the following theorem. We say that the bisequence just written is a 1^2-*interspersion* of the bisequence $\ldots, 2, 3, 2, 3, 2, 3, \ldots$, because it is obtained from it by interspersing double 1's between each two of its terms.

Theorem 16. *For any bisequence ξ whose terms are types 2 and 3, there exists a sandwiched K_2-QPDS S in Δ with associated bisequence obtained by a 1^2-interspersion of ξ.*

Proof. This follows from the discussion previous to the statement. Observe that Theorem 11 is a particular case of this result. ∎

Corollary 17. *If a bisequence ξ as in Theorem 16 has period 23, then there exists a graph $\Delta_{m,n}$ containing a K_2-QPDS S' in Δ which is a quotient of S via the corresponding projection map $\rho_{m,n} : \Delta \to \Delta_{m,n}$ if and only if $6|m$ and $50|n$.*
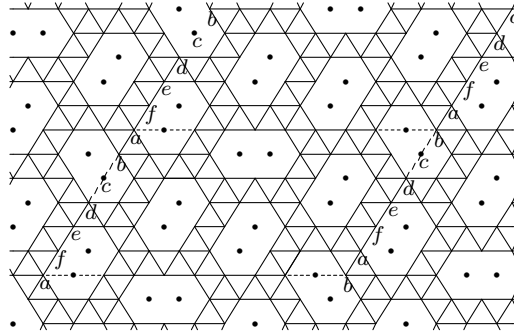

Figure 8. Illustration for Corollary 17.

Proof. Observe Figure 8, which is tilted an angle of $60°$ with respect to the K_2-QPDS represented on the left of Figure 6, so it should be tilted back for a representation of a K_2-QPDS as in the statement, corresponding to the permutation (3,1,2) of the hexagon types. In Figure 8, a rhomboidal cutout with a base 50-edges long horizontally and 6-edges upward in anti-diagonal can be obtained by continuating the sub-rhomboid with base (or top) ab by means of horizontal translations to the right. The successive continuating sub-rhomboids here can be realized by horizontal translation of respective sub-rhomboids (not traced, for clarity) with bases (or tops) bc, cd, de, ef and fa. ∎
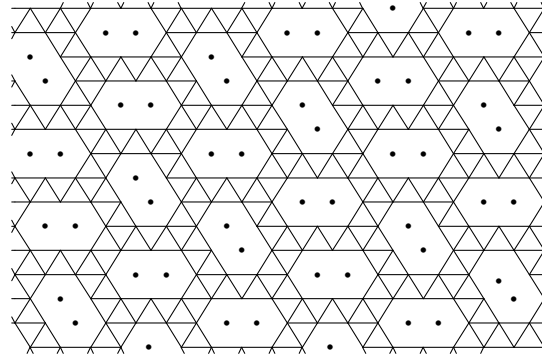

Figure 9. Pairs of adjacent type 2 hexagons surrounded by type 1 hexagons.

There are K_2-QPDSs S in Δ with hexagons in $\mathcal{P}(S)$ only of fixed types $i, j \in I_3$, where $i \neq j$, but those of type i given only in pairs of hexagons adjacent in $\Gamma(S)$ and each such pair surrounded in $\Gamma(S)$ by hexagons of type j. Such a situation is illustrated in Figure 9 with $i = 2$ and $j = 1$. A table of types as in the corresponding $\Gamma(S)$ looks locally as follows:

```
2     1     1     1     2     2
   2     2     1     1     1     2
1     1     2     2     1     1
   1     1     1     2     2     1
2     2     1     1     1     2
   1     2     2     1     1     1
1     1     2     2     2     1
   2     1     1     1     2     2
1     2     2     1     1     1
   1     1     2     2     1     1
2     1     1     1     2     2
   2     2     1     1     1     2
```

In such a table, a type i is *low* if it has immediately above again a type i. Then, the three descending type sequences that start at a low type i,

following the diagonal, vertical and anti-diagonal directions, are periodic with periods forming an ordered triple that we called the *triple period* $\tau(i,j)$.

Starting with the example above, then rotating $\Delta \subset \Pi$ an angle of $\pm 120\circ$, and finally reflecting Δ on the line in Π containing the edge of $\Delta[S]$ in an hexagon of type i, the following triple periods $\tau(i,j)$ are found:

$$\tau(2,1) = ((1^3 2^2),(1^2 212),(121^2 2)), \quad \tau(1,3) = ((313^2 1),(3^2 131),(3^3 1^2)),$$

$$\tau(3,2) = ((2^2 323),(2^3 3^2),(2^2 323)), \quad \tau(2,3) = ((3^2 232),(3^3 2^2),(3^2 232)),$$

$$\tau(1,2) = ((2^3 1^2),(2^2 121),(212^2 1)), \quad \tau(3,1) = ((131^2 3),(1^2 313),(1^3 3^2)).$$

It can be seen that similar triple periods can be obtained by rotating Δ an angle of $\pm 120°$ and by reflecting Δ on a coordinate axis. Also, it can be seen that there is a graph $\Delta_{m,n}$ containing a K_2-QPDS S' which is a quotient of S as in Figure 9 via the correspondence $\rho_{m,n}$ in Corollary 17 if and only if $50|m$ and $5|n$.

A path in $\Gamma(S)$ is *diagonal* if its edges are in a diagonal disposition in the sense of our figures and tables. Similar definitions for *anti-diagonal* and *vertical* paths hold. The following two theorems generalize the discussion above, where instead of working with any pair of types $i,j \in I_3$, we restrict to $i = 2$ and $j = 1$, since the remaining five cases given by rotating Δ angles of $\pm 120°$ and by reflecting it on a coordinate axis differ from this case just by elementary symmetries.

Theorem 18. *There exists exactly one K_2-QPDS S in Δ with hexagons in $\mathcal{P}(S)$ only of types $1, 2$, its type 2 hexagons given only in maximal diagonal t-paths of $\Gamma(S)$ and the resulting $(t+1)$-tuples of type 2 hexagons surrounded by type 1 hexagons, where $t > 0$. Moreover, the hexagons of type 1 in $\mathcal{P}(S)$ are adjacent to either three or four hexagons of type 2 in $\mathcal{P}(S)$.*

Corollary 19. *There is a graph $\Delta_{m,n}$ containing a K_2-QPDS S' in Δ which is a quotient of the K_2-QPDS S of Theorem 18 via $\rho_{m,n}$ if and only if $(10 + 20t)|m$ and $5|n$.*

Proof. The proofs of Theorem 18 and Corollary 19 follow the discussion previous to their statement and the ideas in the proofs of Theorem 16 and Corollary 17. Observe that Corollary 17 yields K_2-QPDSs in $\Delta_{m,n}$ which are quotients of the K_2-QPDS in Theorem 16 via $\rho_{m,n}$, for $30|m$ and $5|n$, where $30 = 10 + 20t$ with $t = 1$. However, because of the further symmetry in Theorem 16 and Corollary 17 with respect to that of Theorem 18 and the

present corollary, the equivalent condition in Corollary 17 realizes a smaller minimal value of m, namely $m = 6$. ∎

If, for the case $t = 1$ in Theorem 18 and Corollary 19, we express its triple triple as $\tau^t = \tau^1 = ((1^3 2^2), (1(12)^2), (121^2 2))$, then we can generalize for $t = 2$ with the triple period $\tau^2 = ((1^4 2^3), (1(12)^3), ((12)21^2))$, and so on for any value of t:

$$\tau^t = ((1^{t+1} 2^t), (1(12)^t), (2^{-1}(21)^t)).$$

Now, consider that instead of a t-path of type 2 hexagons, we deal with *doubly infinite* paths of type 2 hexagons in $\mathcal{P}(S)$ surrounded by hexagons of type 1. One such a situation is shown on the left of Figure 10, in which diagonal paths $P_3(2)$ of $\mathcal{P}(S)$ are interspersed with diagonal paths $P_3(1)$, (where the subindex 3 indicates $P_3(i)$ is a *diagonal path* in $\Gamma(S)$, i.e., with segments between barycenters of hexagons adjacent in $P_3(i)$ perpendicular to *anti-diagonals* in Δ, thus the subindex 3, associated with anti-diagonals; and where the type 1 or 2 of the composing hexagons is set between parentheses). Such a K_2-QPDS will be denoted $S_3(1, 2)$, expressing the periodicity of contiguity and alternation of paths $P_3(2)$ and $P_3(1)$. The triple period τ of such an $S_3(1, 2)$ could be generalized to the form $S_3(\eta)$, where η is any bisequence formed by types 1 and 2, as on the left of Figure 10, in which part of the complement of the K_2-QPDS $S_3(1^2 2^2)$ is shown. Observe that periodicity here is not forced, as nonperiodic bisequences are realizable. These bisequences can be associated with doubly infinite $\{0, 1\}$-sequences, as was done in Theorem 1 and Corollary 2 of [2]. (For the case of periodic η, we have Corollary 21 for K_2-QPDSs on toroidal graphs, below).
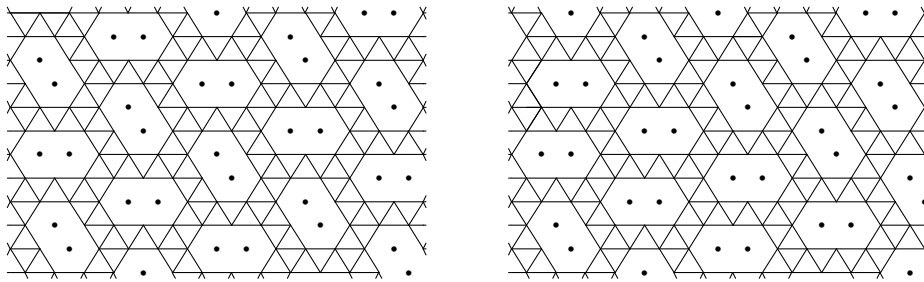

Figure 10. QPDSs with types 1 and 2 hexagons.

Theorem 20. *The family of K_2-QPDSs $S_3(\eta)$ in Δ, where η varies in the set of doubly infinite $\{1,2\}$-sequences, is in one-to-one correspondence with the set of points in the real interval $[0,1]$.*

Proof. We modify the representations of QPDSs in Δ by projecting Π onto $\mathbb{R}^2$, so we can take the Δ-coordinates x_1, x_2 in Figure 1 as orthogonal. In this perspective, each hexagon of $\mathcal{P}(S)$ will be displayed with the vertices of its delimiting 6-hole labeled with its type, so each pair of contiguous hexagons with common type 1 or 2 looks like:

<div align="center">

```
111        22
1111       222
 111        222
             22
```

</div>

In this disposition, vertical sequencing represents vertex adjacency, in the representation of Figure 1, along straight paths in Δ at an angle of $\pi/6$ over the horizontal paths. For example, the K_2-QPDS $S_3(\eta)$, where η has period (12) (of length $I = 2$), becomes representable as in the following display:

<div align="center">

```
...22222111122222111111...
...122222111111222221111...
...111122222111122222111111...
...211112222211111222221111...
...221111112222211112222211111...
...222221111222221111112222211111...
```

</div>

If the horizontal and vertical (or, in Δ, diagonal) periods created in such a display are indicated by $\phi(I)$ and $\psi(I)$, respectively, where I is the length of the period of η, then the display shows that $(\phi(I), \psi(I)) = (\phi(2), \psi(2)) = (20, 5)$. A similar representation corresponds for any $S_3(\eta)$ as in the statement. ∎

Corollary 21. *There exists a graph $\Delta_{m,n}$ containing a K_2-QPDS $S_3'(\eta)$ that behaves as a quotient of a K_2-QPDS $S_3(\eta)$ in Δ via $\rho_{m,n}$, where η is a bisequence as in Theorem 20 with period $1^{i_1} 2^{j_1} 1^{i_2} 2^{j_2} \ldots 1^{i_t} 2^{j_t}$, if and only if $m | \phi(I)$ and $n | \psi(I)$, with $I = i_1 + \ldots + i_t + j_1 + \ldots + j_t$ and $(\phi, \psi)(2) = (20, 5)$, $(\phi, \psi)(I) = (10I, 5I)$ if I is even > 2 and $(\phi, \psi)(I) = (10I, 10I)$ if I is odd > 2.*

Proof. In the final display of the proof of Theorem 20, each horizontal line shows a period composed by the types 1 and 2, and the last shown line repeats the first one, making it clear that $(\phi, \psi)(2) = (20, 5)$. The remaining cases in the statement are managed similarly. ∎

We gather all the results on K_2-QPDSs in Δ obtained above in the following theorem. A K_2-QPDS in Δ is *parallel* if it is as in Theorem 8. Say that a K_2-QPDS in Δ is *ξ-sandwiched* if it is as in Theorem 16, for some bisequence ξ on two types in I_3; *t-linear* if it is as in Theorem 18, up to symmetry, for some $t > 0$; an *$S_i(\eta)$* if it is as in Theorem 20, where $i \in I_3$ and η is a bisequence on the types of $I_3 \setminus \{i\}$. We remark that an (i, j)-surrounded K_2-QPDS is 1-linear.

Theorem 22. *Up to symmetry, a K_2-QPDS in Δ not as in Theorem* 13 *or Corollary* 15 *is either* **(a)** *parallel, or* **(b)** *t-linear ($t > 0$), or* **(c)** *ξ-sandwiched, or* **(d)** *an $S_i(\xi)$, where ξ is a bisequence on the two types $\neq i$.* ∎

It remains to identify completely for which values of integers m, n there exist graphs $\Delta_{m,n}$ containing K_2-QPDSs which are quotients of the K_2-QPDSs classified in Theorem 22, though some of those integer pairs were determined as corollaries above.

Other cases of interest to consider in Δ and its toroidal quotients are H-QPDSs where H is a finite path of length larger than 1.

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