# KERNELS AND CYCLES' SUBDIVISIONS IN ARC-COLORED TOURNAMENTS 

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#### Abstract

Let $D$ be a digraph. $D$ is said to be an $m$-colored digraph if the arcs of $D$ are colored with $m$ colors. A path $P$ in $D$ is called monochromatic if all of its arcs are colored alike. Let $D$ be an $m$-colored digraph. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths of $D$ if it satisfies the following conditions: a) for every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them; and b) for every vertex $x \in V(D)-N$ there is a vertex $n \in N$ such that there is an $x n$-monochromatic directed path in $D$.

In this paper we prove that if $T$ is an arc-colored tournament which does not contain certain subdivisions of cycles then it possesses a kernel by monochromatic paths. These results generalize a well known sufficient condition for the existence of a kernel by monochromatic paths obtained by Shen Minggang in 1988 and another one obtained by Hahn et al. in 2004. Some open problems are proposed.


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## 1. General Concepts and Notation

Let $D$ be a digraph. $V(D)$ and $A(D)$ will denote the set of vertices and arcs of $D$. A subdigraph $H$ of $D$ is a spanning subdigraph if $V(H)=V(D)$; if $S \subseteq V(D)$ is non empty then the subdigraph $D[S]$ induced by $S$ is that digraph having vertex set $S$ and whose arc set consists of all those arcs of $D$ joining vertices of $S$. An arc $z_{1} z_{2} \in A(D)$ is called asymmetrical (symmetrical) if $z_{2} z_{1} \notin A(D)\left(z_{2} z_{1} \in A(D)\right)$; the asymmetrical part of $D$ (the symmetrical part of $D$ ) denoted by $\operatorname{Asym}(D)(\operatorname{Sym}(D))$ is the spanning subdigraph of $D$ whose arcs are the asymmetrical (symmetrical) arcs of $D$; $D$ is called an asymmetrical digraph if $\operatorname{Asym}(D)=D$. A digraph is called semicomplete if for every two distinct vertices $u$ and $v$ of $D$, at least one of the $\operatorname{arcs}(u, v)$ or $(v, u)$ is present in $D$. A semicomplete asymmetrical digraph is called a tournament.

An arc $z_{1} z_{2} \in A(D)$ will be called an $S_{1} S_{2}$-arc whenever $z_{1} \in S_{1} \subseteq V(D)$ and $z_{2} \in S_{2} \subseteq V(D)$. By $\left[z_{1}, z_{2}\right]_{T}$ we denote one of the two possible arcs between $z_{1}$ and $z_{2}$. For a directed walk $W$ we will denote its length by $\ell(W)$. And if $z_{1}, z_{2} \in V(W)$ then we denote by $\left(z_{1}, W, z_{2}\right)$ the $z_{1} z_{2}$ - directed walk contained in $W$. We will denote by $C_{n}$ a directed cycle with length $n$ and by $T_{3}$ the transitive tournament on 3 vertices. We call a triangle a transitive tournament of order 3 or a cycle of length 3 . Let $C=(0,1, \ldots, m, 0)$ be a directed cycle of $D$, a pseudodiagonal of $C$ is an arc $f=(i, j) \in A(D)-A(C)$ such that $i \neq j,\{i, j\} \subseteq V(C)$ and $\ell(i, C, j) \leq \ell(C)-1$. A pole of the cycle $C$ is the terminal vertex $y$ of a pseudodiagonal $(x, y)$ of $C$. Throughout the paper all the paths and cycles considered are directed paths and directed cycles.

A set $S \subseteq V(D)$ is independent if $A(D[S])=\emptyset$. For general concepts on digraphs we refer the reader to [1].

## 2. Introduction

### 2.1. Kernels

The concept of a kernel was first presented in [23] (under the name solution) in the context of Game Theory by Von Neumann and Morgenstern as an interesting solution for cooperative $n$-person games with general $n$. In [1] Berge presented it as follows: Suppose that $n$ players wish to select a point $x$ from a set $X$ of situations. As the individual preferences might not be compatible, then it is necessary to introduce the effective preference: the
situation $a$ is effectively preferred to $b$ if there is a set of players who prefer $a$ to $b$ and who are capable of enforcing their preference. Consider the digraph $D$ with $X$ as a set of vertices and such that $(b, a) \in F(D)$ if the situation $a$ is effectively preferred to situation $b$. Von Neumann and Morgenstern suggested that the situations that are preferred must be the elements of a distinguished set $S$ which is an independent (stable) set and such that for every situation $y$ not in $S$ there is a situation $w$ in $S$ which is effectively preferred to $y$. They called such a set $S$ a solution.

The (solution) set defined in this problem is now well known as a kernel of a digraph. More formally, a kernel $N$ of $D$ is an independent set of vertices such that for each $z \in V(D)-N$ there exists a $z N$-arc in $D$. As the reader can see, not every digraph has a kernel and when a digraph contains a kernel, it may not be the only one. This simple observation compels us to ask for sufficient conditions for the existence of a kernel in a digraph. It is well known that if $D$ is finite, the decision problem of the existence of a kernel in $D$ is NP-complete for a general digraph (see [5] and [27]) and for a planar digraph with indegrees less than or equal to 2 , outdegrees less than or equal to 2 and degrees less than or equal to 3 (see [8]). For any tighter constraints the problem is solvable in linear time.

A digraph $D$ such that every induced subdigraph in $D$ has a kernel is called a kernel-perfect digraph (or simply, a KP-digraph). The following sufficient conditions for a digraph to be a KP-digraph are known:

Theorem 1. $D$ is a kernel-perfect digraph if one of the following conditions holds:
(i) D has no cycles of odd length.
(ii) Every directed cycle of odd length in $D$ has at least two symmetric arcs.
(iii) $\operatorname{Asym}(D)$ is acyclic.
(iv) Every directed cycle of odd length in $D$ has at least two consecutive poles.
(v) Every directed cycle in $D$ has at least one symmetrical arc.

These claims were proved respectively by Richardson [26], Duchet [6], Duchet and Meyniel [7], Galeana-Sánchez and Neumann-Lara [12], and by Berge and Duchet [2]. There are many applications of this concept in the context of game theory, logic and decision theory (see [1]), as well as several interesting related results (see also $[2,12]$ and $[13,28,13,3]$ ). A selected bibliography can be found in [9], and we also recommend the survey [4].

### 2.2. Arc colored digraphs and antecedents

Now, let us consider a more realistic variation of the problem first presented by Von Neumann and Morgenstern. If each player can make their own choice then a natural model of this new problem will be an arc colored digraph in which colors and persons will be related by a bijective function. This simple modification to the initial problem brings an interesting generalization of the concept of kernel. In order to present it in a more formal way together with some previous results, first let us introduce some notation.
$D$ is an $m$-colored digraph if the arcs of $D$ are colored with $m$ colors. Let $D$ be an $m$-colored digraph. A directed path (or cycle) is called monochromatic if all of its arcs are colored alike and it is called quasimonochromatic if with at most one exception all of its arcs are colored alike. A subdigraph $H$ of $D$ is called a $k$-colored digraph if all of its arcs are colored with only $k$ colors, in particular for $k=2$ we say that $H$ is bicolor. We will say that a subdigraph $H$ of $D$ is an at most $k$-colored digraph if all of its arcs are colored with at most $k$ colors, in particular for $k=2$ we say that $H$ is at most bicolor; $H$ will be called a polychromatic digraph if all of its arcs are colored with at least 3 colors.

A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths of $D$ if it satisfies the following conditions: a) $N$ is an independent set by monochromatic paths: for every pair of different vertices $u$ and $v$ in $N$ there is no monochromatic path between them in $D$; and b) $N$ is an absorbing set by monochromatic paths: for every vertex $x \in V(D)-N$ there is a vertex $n \in N$ such that there is an $x n$-monochromatic path in $D$. The concept of kernel by monochromatic paths is a generalization of the concept of kernel. Several results arise around this concept (see [15, 16, 17] and [30]) but the following contributions are the foundations of this work.

In [24], Sands et al. have proved that any 2 -colored digraph has a kernel by monochromatic paths, in particular they proved that every 2 -colored tournament $T$ has a vertex $v$ such that for any $x \in V(T)-\{v\}$ there is a monochromatic path from $x$ to $v$ (i.e., $\{v\}$ is a kernel by monochromatic paths of $T$ ). They also raised the following problem: Let $T$ be an $m$-colored tournament such that every cycle of length 3 is a quasimonochromatic cycle. Must $T$ have a kernel by monochromatic paths? This question is already answered for $m \geq 5$ by Shen Minggang in [25] where he proved that if $T$ is an $m$-colored tournament such that every triangle (that is, a transitive tournament of order 3 or a cycle of length 3 ) is a quasimonochromatic sub-
digraph of $T$, then $T$ has a kernel by monochromatic paths (Theorem A). He also proved that this hypothesis is tight for $m \geq 5$.

Later, in [11] Galeana-Sánchez proved that if every cycle of length at most 4 in an $m$-colored tournament is a quasimonochromatic cycle, then every induced subdigraph of $T$ has a kernel by monochromatic paths (Theorem B).

Finally in [22] Hahn et al. proved the following generalization of the previously cited results of Shen Minggang and Galeana-Sánchez: For a tournament $T$, if every triangle is quasimonochromatic or for some $s \geq 4$ each cycle of length $s$ is a quasimonochromatic cycle and no cycle of length less than $s$ is a polychromatic cycle (colored with at least three colors), then $T$ admits an absorbing vertex (Theorem C).

### 2.3. Description of results

In this paper we obtain two sufficient conditions for the existence of a kernel by monochromatic paths not only in an $m$-colored tournament, but in every induced subdigraph of $T$. The first one is a generalization of Theorem C (see [22]) and the second one is a generalization of Theorem A, Shen Minggang's result (see [25]). We also prove that our conditions are not implied by those known previously.

The spirit of our proofs arises from structural properties of arc colored tournaments (see Lemma 1) and these properties are deduced by working with previous results on kernels (see Theorem 1-v) on an new digraph associated with our original tournament, its closure.

Definition 1. For an $m$-colored digraph $D$, the closure of $D$, denoted by $C(D)$, is the multidigraph such that:

$$
\begin{aligned}
& V(C(D))=V(D) \\
& A(C(D))=A(D) \cup\{u v \mid \text { there is an } u v \text {-monochromatic path in } D\} .
\end{aligned}
$$

Notice that by definition of $C(D)$ it holds that $N \subseteq V(D)$ is a kernel by monochromatic paths of $D$ if and only if $N \subseteq V(C(D))$ is a kernel of $C(D)$. With this we can see that the closure of a digraph $D$ relates in a very natural way kernels by monochromatic paths in this digraph with kernels in its closure. Now, notice that if certain properties which imply that $D$ has a kernel also hold in the closure of $D$, then we can assert that $C(D)$ has a kernel and hence $D$ has a kernel by monochromatic paths (by the definition
of $C(D)$ ). In particular, if the closure of a digraph $D$ satisfies some of the sufficient conditions in Theorem 1, then we get as an immediate application of this theorem that $D$ has a kernel by monochromatic paths. This is an important point to mention because this is not the case with the following results: the sufficient conditions stated in our results hold in tournaments and not in its closure.

## 3. A Preliminary Lemma

The following Lemma gives us structural properties (i.e., existence and color properties) of certain subdigraphs of an arc colored tournament whose closure is not a KP-digraph and such that every $C_{3}$ is a quasimonochromatic cycle.

Lemma 1. Let $T$ be an m-colored tournament. If every $C_{3} \subseteq T$ is a quasimonochromatic cycle and $C(T)$ is not a KP-digraph then there exists a cycle $\gamma=\left(z_{0}, z_{1}, z_{2}=0,1,2, \ldots, p=z_{0}\right) \subseteq C(T)$ such that the following properties hold:
(a) $\ell(\gamma) \geq 4$,
(b) $\gamma \subseteq T$,
(c) $\left(z_{0}, z_{1}\right) \in A(T)$ with color a, $\left(z_{1}, z_{2}\right) \in A(T)$ with color $b$ and there exists a $z_{2} z_{0}$-path $\alpha=\left(z_{2}=0,1,2, \ldots, p=z_{0}\right)(p \geq 2)$ with color $c$, $a \neq b, b \neq c, a \neq c$, let $a=r e d, b=b l u e, c=b l a c k$,
(d) $\left(z_{2}, z_{0}\right) \notin A(T)\left(s o\left(z_{0}, z_{2}\right) \in A(T)\right)$,
(e) There is no $z_{1} z_{0}$-monochromatic path in $T$ and there is no $z_{2} z_{1}$-monochromatic path in $T$,
(f) Every arc between $z_{1}$ and an internal vertex in $\alpha$ is not black.

Proof. Proceeding by contradiction, let us suppose that $C(T)$ is not a KPdigraph so a well known theorem by Berge and Duchet (see Theorem 1-v) asserts that there is a cycle $\Gamma \subseteq \operatorname{Asym}(C(T))$. Let $\Gamma=\left(z_{0}, z_{1}, \ldots, z_{n-1}, z_{n}=\right.$ $\left.z_{0}\right) \subseteq \operatorname{Asym}(C(T))$ be a cycle with minimal length contained in $\operatorname{Asym}(C(T))$. Through the following claims we will discover interesting color properties of this cycle and they will allow us to prove the lemma.

Claim 1. $\ell(\Gamma)=n \geq 3$.
Recall $\Gamma \subseteq \operatorname{Asym}(C(T))$, so $\ell(\Gamma)=n \neq 2$.

## Claim 2. $\Gamma \subseteq T$.

Suppose that there is an arc $\left(z_{i}, z_{i+1}\right) \in \Gamma-T$. Since $T$ is a tournament we have that $\left(z_{i+1}, z_{i}\right) \in T$ and so $\left\{\left(z_{i}, z_{i+1},\left(z_{i+1}, z_{i}\right)\right\} \subseteq \operatorname{Asym}(C(T))\right.$, a contradiction.

Claim 3. $\left(z_{0}, z_{1}\right) \in A(T)$ has color $a,\left(z_{1}, z_{2}\right) \in A(T)$ has color $b, a \neq b$.
Since $\Gamma$ is not a monochromatic cycle (by the contrary: $\left(z_{0}, \Gamma, z_{n-1}\right) \subseteq$ $\operatorname{Asym}(C(T))$ is a monochromatic path, thus $\left(z_{0}, z_{n-1}\right) \in A(C(T))$ and hence $\left(z_{n-1}, z_{0}\right) \in A(\operatorname{Sym}(C(T)) \cap \Gamma)$, a contradiction), then there exist two consecutive arcs in $\Gamma$ colored differently. Say $\left(z_{0}, z_{1}\right) \in A(\Gamma)$ is red and $\left(z_{1}, z_{2}\right) \in A(\Gamma)$ is blue.

Claim 4. For any $\left\{z_{i}, z_{j}\right\} \subset V(\Gamma)$ such that $j \notin\{i-1, i+1\}$ it holds that $\left\{\left(z_{i}, z_{j}\right),\left(z_{j}, z_{i}\right)\right\} \subseteq A(C(T))$. Let $\left\{z_{i}, z_{j}\right\} \subset V(\Gamma)$ be such that $j \notin\{i-1$, $i+1\}$. Since $T$ is a tournament, $\left(z_{i}, z_{j}\right) \in A(T)$ or $\left(z_{j}, z_{i}\right) \in A(T)$, without loss of generality let $\left(z_{i}, z_{j}\right) \in A(T)$. Then $\Gamma^{\prime}=\left(z_{i}, z_{j}, z_{j+1}, z_{j+2}, \ldots, z_{i-1}, z_{i}\right)$ $\subseteq T$ is a cycle with $\ell\left(\Gamma^{\prime}\right)<\ell(\Gamma)$. Hence $\Gamma^{\prime} \nsubseteq \operatorname{Asym}(C(T))$ and so $\left(z_{i}, z_{j}\right) \in$ $A(\operatorname{Sym}(C(T))$.

Claim 5. $\left(z_{2}, z_{0}\right) \notin A(T)$.
If $\left(z_{2}, z_{0}\right) \in A(T)$ then there exists $C_{3}=\left(z_{0}, z_{1}, z_{2}, z_{0}\right) \subseteq T$ and it is a quasimonochromatic cycle by hypothesis, so $\left(z_{2}, z_{0}\right) \in A(T)$ is red or blue. If $\left(z_{2}, z_{0}\right) \in A(T)$ is red then $\left(z_{2}, z_{0}, z_{1}\right) \subseteq T$ is a $z_{2} z_{1}$-monochromatic path and $\left(z_{1}, z_{2}\right) \in A(\operatorname{Sym}(C(T)) \cap \Gamma)$, a contradiction. If $\left(z_{2}, z_{0}\right) \in A(T)$ is blue, then $\left(z_{1}, z_{2}, z_{0}\right) \subseteq T$ is a $z_{1} z_{0}$-monochromatic path and $\left(z_{0}, z_{1}\right) \in$ $A(\operatorname{Sym}(C(T)) \cap \Gamma)$, a contradiction again.

Now, by claims (2.3.) and (2.3.) there exist a $z_{2} z_{0}$-monochromatic path in $T$ with length at least 2 . Let $\alpha=\left(z_{2}=0,1,2, \ldots, p=z_{0}\right) \subseteq T$ be a $z_{2} z_{0}$-monochromatic path with minimal length $(p \geq 2)$.

Claim 6. $z_{1} \notin V(\alpha)$.
Otherwise $z_{1} \in V(\alpha)$ and then $\left(z_{2}, \alpha, z_{1}\right)$ is a $z_{2} z_{1}$-monochromatic path in $T$ so $\left(z_{1}, z_{2}\right) \in \operatorname{Sym}(C(T))$, contradiction.

Claim 7. $\alpha$ is neither red nor blue.
If $\alpha$ is red then $\alpha \cup\left(z_{0}, z_{1}\right)$ is a $z_{2} z_{1}$-monochromatic path in $T$ and $\left(z_{2}, z_{1}\right)$ $\in A(\operatorname{Sym}(C(T)) \cap \Gamma)$, a contradiction. If $\alpha$ is blue then $\left(z_{1}, z_{2}\right) \cup \alpha \subseteq T$ is a $z_{1} z_{0}$-monochromatic path in $T$ and $\left(z_{1}, z_{0}\right) \in A(\operatorname{Sym}(C(T)) \cap \Gamma$ ), a contradiction again. Let $\alpha$ be black.

Consider $\gamma=\left(z_{0}, z_{1}, z_{2}\right) \cup \alpha$. Clearly $\gamma$ satisfies the first four properties of our Lemma 1. Let us conclude with the following points.

Claim 8. There is no $z_{1} z_{0}$-monochromatic path in $T$ and there is no $z_{2} z_{1}$ monochromatic path in $T$.
Notice that $\left\{\left(z_{0}, z_{1}\right),\left(z_{1}, z_{2}\right)\right\} \subseteq \operatorname{Asym}(C(T))$.
Claim 9. Every arc between $z_{1}$ and an internal vertex in $\alpha$ is not black. If there exists $i, 1 \leq i \leq p-1$ such that $\left(i, z_{1}\right) \in A(T)$ resp. $\left(z_{1}, i\right) \in$ $A(T))$ is black then $\left(z_{2}=0, \alpha, i\right) \cup\left(i, z_{1}\right) \subseteq T$ (resp. $\left.\left(z_{1}, i\right) \cup\left(i, \alpha, z_{0}\right)\right)$ is a $z_{2} z_{1}$-monochromatic path in $T$ (resp. is a $z_{1} z_{0}$-monochromatic path), a contradiction.

## 4. The Main Results

In order to present our main results we must introduce certain subdigraphs whose arc coloration in an arc colored digraph $D$ will allow us to assert the existence of a kernel by monochromatic paths in $D$ and in every induced subdigraph of this digraph.

### 4.1. Condition I

Definition 2. A (2, $k$-2)-subdivision of $C_{2}$-bicolor is defined to be a cycle of length $k$ containing a monochromatic path of length $k-2$ and a monochromatic path of length 2 .

In particular for $k=3$ we get a bicolor $C_{3}$. The importance of this subdigraph in our condition is that its coloration is less restrictive than the quasimonochromatic coloration of $C_{s}$-cycles in Theorem C. We will use these subdigraphs in order to prove a more general sufficient condition than the condition in Theorem C.

Definition 3. Let $T$ be an $m$-colored tournament. $T$ has property $P I_{k}$ if the following conditions hold for some fixed integer $k \geq 4$ :
(a) Every $C_{k} \subseteq T$ is at most bicolor and is not a (2,k-2)-subdivision of $C_{2}$-bicolor, and
(b) every $C_{t} \subseteq T(t<k)$ is at most bicolor (it is not polychromatic).

Theorem 2. Let $T$ be an $m$-colored tournament. If $T$ satisfies property $P I_{k}$ for some integer $k \geq 4$ then $C(T)$ is a KP-digraph.

Proof. To prove the theorem we proceed by contradiction. Suppose that $C(T)$ is not a KP-digraph, then by Lemma 1 there exists a cycle $\gamma=$ $\left(z_{0}, z_{1}, z_{2}=0,1,2, \ldots, p=z_{0}\right)$ satisfying properties (a) to (f). The following assertions will allow us to obtain a contradiction.

Claim 1. $p>k-2$.
From property (c) of Lemma 1 we have that $\gamma$ is a 3 -colored cycle. Then the assertion holds from the hypothesis of Theorem 2.

Claim 2. For each $i$ with $p-(k-2) \geq i \geq 0$ we have: If $\left(z_{1}, i\right) \in A(T)$ then for every $j$ with $p>i+j(k-2) \geq i$, it holds that $\left(z_{1}, i+j(k-2)\right) \in A(T)$. Let $i$ with $p-(k-2) \geq i \geq 0$ and assume that $\left(z_{1}, i\right) \in A(T)$. If there is some $j, p>i+j(k-2) \geq i$, such that $\left(z_{1}, i+j(k-2)\right) \notin A(T)$ then let $j_{0}=\min \left\{j \mid p>i+j(k-2) \geq i\right.$ and $\left.\left(z_{1}, i+j(k-2)\right) \notin A(T)\right\}$. Since $T$ is a tournament we have $\left(i+j_{0}(k-2), z_{1}\right) \in A(T)$ and it follows from the choice of $j_{0}$ that $\left(z_{1}, i+j_{0}(k-2)-(k-2)\right) \in A(T)$ (notice that $i+j_{0}(k-2)-(k-2) \geq i$ as $\left.\left(z_{1}, i\right) \in A(T)\right)$ then there exists $C_{k}=\left(z_{1}, i+\right.$ $\left.j_{0}(k-2)-(k-2)\right) \cup\left(i+j_{0}(k-2)-(k-2), \alpha, i+j_{0}(k-2)\right) \cup\left(i+j_{0}(k-2), z_{1}\right) \subseteq T$ and it is an at most bicolor cycle by hypothesis, so $\left(z_{1}, i+j_{0}(k-2)-(k-2)\right)$ and $\left(i+j_{0}(k-2), z_{1}\right)$ have the same color and they are not black (by Lemma 1-f), hence $C_{k}$ is a (2, $k$-2)-subdivision of $C_{2}$-bicolor, a contradiction. We conclude that $\left(z_{1}, i+j(k-2)\right) \in A(T)$ for each $j$ with $p>i+j(k-2) \geq i$.

Claim 3. For each $i, p \geq i>k-2$ we have: If $\left(i, z_{1}\right) \in A(T)$ then for every $j, p-(k-2) \geq i-j(k-2)>0$, it holds that $\left(i-j(k-2), z_{1}\right) \in A(T)$.
Let $i$ be such that $p \geq i>k-2$ and $\left(i, z_{1}\right) \in A(T)$. If there exists $j$, $p-(k-2) \geq i-j(k-2)>0$, such that $\left(i-j(k-2), z_{1}\right) \notin A(T)$ then let $j_{0}=\min \left\{j \mid p-(k-2) \geq i-j(k-2)>0\right.$ and $\left.\left(i-j(k-2), z_{1}\right) \notin A(T)\right\}$. As before we have that $\left(z_{1}, i-j_{0}(k-2)\right) \in A(T)(T$ is a tournament) then $\left(z_{1}, i-j_{0}(k-2)+(k-2)=i-\left(j_{0}-1\right)(k-2)\right) \in A(T)$ (as a consequence of Claim 2) since $\left.p-(k-2) \geq i-j_{0}(k-2) \geq 0\right)$, contradicting the choice of $j_{0}$.

Now we conclude the proof by analyzing the following two cases.
Case A. $p=m(k-2)$, with $m \in \mathbb{N}$ and $m \geq 2$ (recall that $p>k-2$ ). $\left(z_{1}, z_{2}=0\right) \in A(T)$ so it follows from Claim 2 that $\left(z_{1}, p-(k-2)\right) \in$ $A(T)$, then there exists $C_{k}=\left(z_{1}, p-(k-2)\right) \cup\left(p-(k-2), \alpha, p=z_{0}\right) \cup$ $\left(z_{0}, z_{1}\right) \subseteq T$ and it is an at most bicolor cycle by hypothesis, then we have that $\left(z_{1}, p-(k-2)\right) \in A(T)$ is red (it is not black by Lemma 1-f) and
so $C_{k}=\left(z_{1}, p-(k-2)\right) \cup\left(p-(k-2), \alpha, p=z_{0}\right) \cup\left(z_{0}, z_{1}\right) \subseteq T$ is a $(2$, $k$-2)-subdivision of $C_{2}$-bicolor, a contradiction.

Case B. $p=m(k-2)+r$, with $m, r \in \mathbb{N}, m \geq 1$ and $k-2>r>0$.
Claim 4. $\left(z_{1}, p-r\right) \in A(T)$ and it is red.
$\left(z_{1}, z_{2}=0\right) \in A(T)$ so $\left(z_{1}, m(k-2)=p-r\right) \in A(T)$ (by Claim 2). Then $C_{t}=\left(z_{1}, p-r\right) \cup\left(p-r, \alpha, p=z_{0}\right) \cup\left(z_{0}, z_{1}\right) \subseteq T$ is a cycle with length $t=r+2$ with $t<k$ (as $r<k-2$ ). It is an at most 2 -colored cycle by hypothesis, hence $\left(z_{1}, p-r\right) \in A(T)$ is black or it is red and we conclude it is colored red because of Lemma 1-f.

Claim 5. $\left(r, z_{1}\right) \in A(T)$ and it is blue.
$\left(z_{0}, z_{1}\right) \in A(T)$ so by Claim 3 we have that $\left(p-m(k-2)=r, z_{1}\right) \in A(T)$. Then $C_{t}=\left(z_{1}, z_{2}=0\right) \cup(0, \alpha, p-m(k-2)=r) \cup\left(r, z_{1}\right) \subseteq T$ is a cycle with length $t=r+2<k$ and it is an at most bicolor cycle by hypothesis, so $\left(r, z_{1}\right) \in A(T)$ is black or blue. As a consequence of Lemma 1-f we conclude $\left(r, z_{1}\right) \in A(T)$ is blue.

Claim 6. $\left(z_{0}, r\right) \in A(T)$.
If $\left(r, z_{0}\right) \in A(T)$ then there exists $C_{s}=\left(r, z_{0}, z_{1}, z_{2}=0\right) \cup\left(z_{2}=0, \alpha, r\right) \subseteq T$ a cycle of length $s=r+3 \leq k$ ) and it is an at least 3 -colored cycle ( $k \geq 4$ since $k-3 \geq r \geq 1$ ), a contradiction.

Then there exists the at least 3-colored cycle $C_{q}=\left(z_{0}=p, r, z_{1}, p-r\right) \cup$ $\left(p-r, \alpha, z_{0}\right) \subseteq T(q \leq k$ as $r \leq k-3)$, a contradiction again.

Corollary 1. In combination with the Shen Minggang condition, Theorem 2 is a generalization of the Theorem C by Hahn et al.

### 4.2. Condition II

Definition 4. A (1, 1, k-2)-C-subdivision of a 3 -colored $C_{3}$ with colors 1,2 and 3 , is defined to be a cycle of length $k$ having a monochromatic path of length $k$-2 colored 1 , one arc colored 2 and one arc colored 3.

Definition 5. A ( $1,1, k$-2)-T-subdivision of a 3 -colored $T_{3}$ with colors 1,2 and 3 , can be obtained from such $T_{3}$ by the substitution of one of its arcs with a path colored alike.

Notice that any of the arcs of $T_{3}$ can be replaced by a path, so depending on which arc is replaced, we can get three different 3-colored digraphs.

Definition 6. Let $T$ be an $m$-colored tournament. We say that $T$ satisfies property $P I I_{k}$ for some fixed integer $k \geq 3$ if
(a) There is no $(1,1, t-2)$ - $C$-subdivision in $T$, with $t \leq k$ and $t \geq 3$, and
(b) there is no $(1,1, k-2)$ - $T$-subdivision in $T$.

Theorem 3. Let $T$ be an $m$-colored tournament. If $T$ satisfies property $P I I_{k}$ for some integer $k \geq 3$ then $C(T)$ is a KP-digraph.

Proof. We proceed by contradiction. Suppose that $C(T)$ is not a KPdigraph, then by Lemma 1 there exists a cycle $\gamma=\left(z_{0}, z_{1}, z_{2}=0,1,2, \ldots, p=\right.$ $z_{0}$ ) satisfying properties (a) to (f). The following claims will allow us to obtain a contradiction:

Claim 1. $p>k-2$.
By the contrary, if $p \leq k-2$ then $\gamma \subseteq T$ is a $(1,1, p)$-subdivision of a 3 -colored $C_{3}$ in $T(p+2 \leq k)$, a contradiction. So $p>k-2$.

Claim 2. For each $i$ with $p-(k-2)>i \geq 0$ we have: If $\left[z_{1}, i\right]_{T}$ is colored $a(a \neq$ black, by Lemma 1-f) then for each $j$ with $p>i+j(k-2) \geq i$, we have that $\left[z_{1}, i+j(k-2)\right]_{T}$ is also colored $a$.

Let $i, 0 \leq i<p-(k-2)$, be a fixed integer and suppose that $\left[z_{1}, i\right]_{T}$ is colored $a$. Suppose, by contradiction, that there exists $j^{\prime}$ with $p>i+$ $j^{\prime}(k-2) \geq i$, such that $\left[z_{1}, i+j^{\prime}(k-2)\right]_{T}$ is colored $b$, with $a \neq b$. Consider $j_{0}=\min \left\{j^{\prime} \mid p>i+j^{\prime}(k-2) \geq i\right.$ and such that $\left[z_{1}, i+j^{\prime}(k-2)\right]_{T}$ is colored $b$, with $a \neq b\}$. Then $D\left[\left\{z_{1}, j_{0}, j_{0}-1\right\}\right]$ is a 3 -colored triangle, a contradiction.

Claim 3. For each $i$ with $p \geq i>k-2$ the following holds: If $\left[z_{1}, i\right]_{T}$ is colored $a(a \neq$ black $)$ then for every $j, p-(k-2) \geq i-j(k-2)>0$, we have that $\left[z_{1}, i-j(k-2)\right]_{T}$ is colored $a$.
Immediate from the previous claim.
Depending on the length of $\alpha$ we analyze the following two cases.
Case A. $p=m(k-2), m \in \mathbb{N}, m \geq 2(p \geq k-2)$.
$\left(z_{1}, z_{2}=0\right) \in A(T)$ is blue and $\left[z_{1}, p-(k-2)\right]_{T}$ is blue by Claim 2. If $\left(z_{1}, p-\right.$ $(k-2)) \in A(T)$ then $C_{k}=\left(p=z_{0}, z_{1}, p-(k-2)\right) \cup\left(p-(k-2), \alpha, p=z_{0}\right) \subseteq T$ is a $(1,1, k-2)$-subdivision of a 3-colored $C_{3}$ in $T$. If $\left(p-(k-2), z_{1}\right) \in A(T)$ then $T_{k}=\left(p-(k-2), \alpha, p=z_{0}\right) \cup\left(p=z_{0}, z_{1}\right) \cup\left(p-(k-2), z_{1}\right) \subseteq T$ is a $(1,1, k-2)$-subdivision of a 3 -colored $T_{3}$ in $T$, in both cases we get a contradiction.

Case B. $p=m(k-2)+r, m, r \in \mathbb{N}, m \geq 1,1 \leq r<k-2$.
Claim 4. $\left(p-r, z_{1}\right) \in A(D)$ and it is blue: $\left(z_{1}, z_{2}=0\right) \in A(T)$ and it is blue, so $\left[z_{1}, m(k-2)=p-r\right]_{T}$ is also blue (by Claim 2). If $\left(z_{1}, p-r\right) \in A(T)$ then $C_{t}=\left(z_{1}, p-r\right) \cup\left(p-r, \alpha, p=z_{0}\right) \cup\left(z_{0}, z_{1}\right) \subseteq T(t<k)$ is a $(1,1, t-2)$ subdivision of a 3-colored $C_{3}$ in $T$, a contradiction. Then $\left(p-r, z_{1}\right) \in A(D)$ and it is blue.

Let $h=(k-2)-r, A_{1}=\{j(k-2) \mid m \geq j \geq 0\}, A_{2}=\{p-j(k-2) \mid$ $m \geq j \geq 0\}$ and $A=A_{1} \cup A_{2}$. Notice that $A_{1} \cap A_{2}=\emptyset$, as $0<r<k-2$.

Claim 5. For every $i \in A$ we have that $\left(z_{1}, i\right) \in A(T)$.
By contradiction let us define $f=\min \left\{u \in A \mid\left(u, z_{1}\right) \in A(T)\right\}(f>0$ because $\left.\left(z_{1}, 0\right) \in A(T)\right)$. Let $w \in A$ defined as follows: $f-h=w$ whenever $f \in A_{1}$ and $f-r=w$ whenever $f \in A_{2}$. It follows from the definition of $f$ that $\left(z_{1}, w\right) \in A(T)$ (notice that $f \in A_{1}$ implies $w \in A_{2}$; and $f \in A_{2}$ implies $\left.w \in A_{1}\right)$.

If $f \in A_{1}$ then $w \in A_{2},\left(f, z_{1}\right) \in A(T)$ is blue (as $\left(z_{1}, z_{2}\right) \in A(T)$ is blue and because of Claim 2) and $\left(z_{1}, w\right) \in A(T)$ is red $\left(\left(z_{0}, z_{1}\right) \in A(T)\right.$ is red and because of Claim 3). Hence $C_{t}=\left(f, z_{1}\right) \cup\left(z_{1}, w\right) \cup(w, \alpha, f) \subseteq T$ $(h<k-2$ so $t<k)$ is a 3 -colored cycle, a contradiction.

By analogy, if $f \in A_{2}$ then $w \in A_{1},\left(z_{1}, w\right) \in A(T)$ if blue $\left(\left(z_{1}, z_{2}\right) \in\right.$ $A(T)$ is blue and by Claim 2) and $\left(f, z_{1}\right) \in A(T)$ is red $\left(\left(z_{0}, z_{1}\right) \in A(T)\right.$ is red and by Claim 3). Hence $C_{t}=\left(f, z_{1}\right) \cup\left(z_{1}, w\right) \cup(w, \alpha, f) \subseteq T(t<k$ as $r<k-2)$ is a 3 -colored cycle, a contradiction again.

In particular we have that $\left(z_{1}, p-r=m(k-2)\right) \in A(T)$, contradicting Claim 4.

Corollary 2. Theorem 3 is a generalization of the Theorem $A$ by Shen Minggang.

## 5. Remarks and Open Problems

Remark 1. If we only ask in Theorem 2 for every $C_{k} \subseteq T$ to be an at most bicolor cycle and not to be a $(2, k-2)$-subdivision of $C_{2}$-bicolor, then the result does not hold.

Proof. Consider tournaments in Figure 1a (for $k=4$ ) and Figure 1b (for $k=6)$. Even though every $C_{k}$ in such tournaments is an at most bicolor cycle and is not a $(2, k-2)$-subdivision of $C_{2}$-bicolor, there exists a 3 -colored $C_{3}=$
$\left(z_{0}, z_{1}, z_{2}, z_{0}\right)$ and both tournaments do not have a kernel by monochromatic paths.

Remark 2. If we omit the last hypothesis (c) in Theorem 3 then the result will be false. To prove it consider the digraph in Figure 1a (same argument as in the previous remark).

Remark 3. The Sands et al. condition, the Shen Minggang condition and the Galeana-Sánchez condition do not imply the conditions of Theorem 2 and Theorem 3.

Proof. To prove it consider digraphs in Figure 2 to 5.
Figure 2. The tournament in Figure 2 proves that the Sands et al. Condition does not imply Condition I: Though $T$ is a 2 -colored tournament, there exists $C_{6}=\left(z_{0}, z_{1}, z_{2}, z_{3}, z_{4}, z_{5}, z_{0}\right)$ a $(2,4)$-subdivision of a bicolor $C_{2}$.

Figure 3. The tournament in Figure 3 proves that the Shen Minggang Condition does not imply Condition II: Though every triangle is a 2 -colored subdigraph (notice that if there exists a 3 -colored triangle then it must contain the $\left.\operatorname{arc}\left(z_{1}, z_{2}\right)\right)$, there exists $C_{6}=\left(z_{0}, z_{1}, z_{2}, z_{3}, z_{4}, z_{5}, z_{0}\right)$ a $(1,1,4)$ subdivision of a 3 -colored $C_{3}$.

Figure 4. The tournament in Figure 4 proves that the Galeana-Sánchez Condition does not imply Condition I: Though every cycle of length 3 and 4 in $T$ is a quasimonochromatic cycle (notice that every non quasimonochromatic cycle of length 4 must contain two arcs colored 1), there exists $C_{5}=\left(z_{0}, z_{1}, z_{2}, z_{3}, z_{4}, z_{0}\right)$ a $(2,3)$-subdivision of a bicolor $C_{2}$.

Figure 5. The tournament in Figure 5 proves that the Galeana-Sánchez Condition does not imply Condition II: Though every cycle of length 3 and 4 in $T$ is a quasimonochromatic cycle (notice that there are only 2 arcs not colored 3 and both are adjacent to $z_{1}$, so they can not be simultaneously in a cycle), there exists $P_{5}=\left(z_{2}, z_{3}, z_{4}, z_{0}, z_{1}\right) \cup\left(z_{2}, z_{1}\right)$, a (1, 1, 3)-subdivision of a 3 -colored $T_{3}$.

Open problem 1. Let $T$ be an $m$-colored tournament. If there is some fixed integer $k \geq 4$ such that every $C_{s} \subseteq T(s \leq k)$ is at most bicolor then $C(T)$ is a KP-digraph.

Open problem 2. Let $T$ be an $m$-colored tournament. If there is some fixed integer $k \geq 3$ such that there is no $(1,1, t-2)$-subdivision of a 3-colored $C_{3}$ in $T(3 \leq t \leq k)$ then $C(T)$ is a KP-digraph.


Open problem 3. Let $T$ be an $m$-colored tournament. If there is some fixed integer $k \geq 3$ such that there is no ( $1,1, k-2$ )-subdivision of a 3 colored $T_{3}$ in $T$ and there is no $(1,1, t-2)$-subdivision of a 3 -colored $C_{3}$ in $T(3 \leq t<k)$ then $C(T)$ is a KP-digraph.

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