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RESTRAINED DOMINATION IN UNICYCLIC GRAPHS

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a restrained dominating set if every vertex in $V - S$ is adjacent to a vertex in S and to a vertex in $V - S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. A unicyclic graph is a connected graph that contains precisely one cycle. We show that if U is a unicyclic graph of order n, then $\gamma_r(U) \geq \lceil \frac{n}{3} \rceil$, and provide a characterization of graphs achieving this bound.

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1. Introduction

In this paper, we follow the notation of [1]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $S \subseteq V$ is a *dominating set* (**DS**) of G if every vertex in $V - S$ is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a **DS** of G. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [1] includes a chapter on domination. A thorough study of domination appears in [10, 11].

In this paper, we continue the study of a variation of the domination theme, namely that of restrained domination — see [2, 3, 4, 5, 6, 7, 8, 9, 12, 13].

A set $S \subseteq V$ is a *restrained dominating set* (**RDS**) if every vertex in $V - S$ is adjacent to a vertex in S and to a vertex in $V - S$. Every graph has a **RDS**, since $S = V$ is such a set. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a **RDS** of G. A **RDS** of G of cardinality $\gamma_r(G)$ is called a γ_r-set of G.

Throughout, let n and m denote the order and size of G, respectively. A unicyclic graph U of order n is a connected graph that contains exactly one cycle. Thus, U has size n. A vertex of degree one will be called a *leaf*, while a vertex adjacent to a leaf will be called a *remote vertex*. The *open neighborhood* of a vertex u, denoted $N(u)$, is the set $\{v \in V \mid v$ is adjacent to $u\}$, while the *closed neighborhood* of u, denoted $N[u]$, is defined as $N(u) \cup \{u\}$.

A graph G is status labeled if every vertex in V is labeled either A or B. A vertex $v \in V$ has status A (B, respectively) if v is labeled A (B, respectively). The status of a vertex v will be denoted $\mathrm{Sta}(v)$. We define $\mathrm{Sta}(A)$ ($\mathrm{Sta}(B)$, respectively) as the set of vertices in V with status A (B, respectively).

Theorem 1. *Let G be a connected graph of order n and size m. Then $\gamma_r(G) \geq n - \frac{2m}{3}$.*

Proof. Let S be a γ_r-set of G, and consider $H = \langle V - S \rangle$. Let n_1 and m_1 be the order and size of $\langle V - S \rangle$, respectively. Thus, $m_1 = \frac{1}{2}\sum_{v \in V-S} \deg_H(v) \geq \frac{1}{2}(n - \gamma_r(G))$. Let m_2 denote the number of edges between S and $V - S$. Since S is a **DS**, every vertex in $V - S$ is adjacent to at least one vertex in S.

Thus, $m_2 \geq n - \gamma_r(G)$. Hence, $m \geq m_1 + m_2 \geq \frac{1}{2}(n - \gamma_r(G)) + n - \gamma_r(G)$, which implies that $\gamma_r(G) \geq n - \frac{2}{3}m$. ■

The following known result of [4] is an immediate consequence of Theorem 1.

Corollary 2. *Let T be a tree of order n. Then $\gamma_r(T) \geq \lceil \frac{n+2}{3} \rceil$.*

In similar fashion, we derive our first main result.

Corollary 3. *Let U be a unicyclic graph of order n. Then $\gamma_r(U) \geq \lceil \frac{n}{3} \rceil$.*

Domke *et al.* [4] provided a constructive characterization of trees achieving the lower bound given in Corollary 2. Hattingh and Plummer [9] gave a simpler characterization, independent of γ_r-set consideration. In the sequel, we constructively characterize unicyclic graphs achieving the lower bound given in Corollary 3, utilizing constructive operations governed by status labeling.

2. Unicylic Graphs U of Order n with $\gamma_r(U) = \lceil \frac{n}{3} \rceil$

Let $\mathcal{E}$ denote the class of all unicyclic graphs U of order n such that $\gamma_r(U) = \lceil \frac{n}{3} \rceil$. In order to provide the characterization, we state and prove a few observations.

Let $U \in \mathcal{E}$ and let S be a γ_r-set of U.

Observation 1. *If $n \equiv 0 \bmod 3$, then S is independent and every vertex in $V - S$ has degree 2.*

Proof. Assume that $n \equiv 0 \bmod 3$. If $v \in V$ such that $\deg(v) = 1$, then $v \in S$. Thus $\deg(v) \geq 2$, for all $v \in V - S$. Now, let $y \in V - S$. Suppose that $|N(y) \cap (V - S)| \geq 2$. By assumption, $|V - S| = \frac{2n}{3}$. Therefore, $n = m \geq n - \gamma_r(U) + \frac{1}{2}(n - \gamma_r(U) + 1)$, which implies that $\gamma_r(U) \geq \lceil \frac{n+1}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. Suppose that $|N(y) \cap S| \geq 2$. Then $n = m \geq n - \gamma_r(U) + 1 + \frac{1}{2}(n - \gamma_r(U))$, which implies that $\gamma_r(U) \geq \lceil \frac{n+2}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. Thus, every vertex in $V - S$ is adjacent to exactly one vertex of S and adjacent to exactly one vertex of $V - S$.

Since $|V - S| = \frac{2n}{3}$, the vertices in $V - S$ form a matching with exactly $\frac{n}{3}$ edges. Since $m = n$, there are $\frac{2n}{3}$ edges between S and $V - S$. Hence, S is independent. ■

Observation 2. *If* $n \equiv 1 \bmod 3$*, then* S *has exactly one of the following properties*:

1. $m(\langle S\rangle) = 1$*, while every vertex in* $V - S$ *has degree* 2.
2. *There is a vertex* $y \in V - S$ *such that* $\deg(y) = 3$ *and* $|N(y) \cap S| = 2$. *Furthermore,* S *is independent and every vertex in* $V - S - \{y\}$ *has degree* 2.
3. *There are exactly two vertices* $x, y \in V - S$ *such that* $\deg(x) = \deg(y) = 3$*, and* $|N(x) \cap (V - S)| = |N(y) \cap (V - S)| = 2$. *Furthermore,* S *is independent and every vertex in* $V - S - \{x, y\}$ *has degree* 2.
4. *There is exactly one vertex* $y \in V - S$ *such that* $\deg(y) = 4$ *and* $|N(y) \cap (V - S)| = 3$. *Furthermore,* S *is independent and every vertex in* $V - S - \{y\}$ *has degree* 2.

Proof. Assume that $n \equiv 1 \bmod 3$. Suppose first that, for all $y \in V - S$, $\deg(y) = 2$ and that S is independent. Clearly, $|S| = \frac{n+2}{3}$ and $|V - S| = \frac{2(n-1)}{3}$. There are exactly $\frac{2(n-1)}{3}$ edges between $V - S$ and S, and there are $\frac{n-1}{3}$ edges in $\langle V - S\rangle$. Hence, $n = m = \frac{2(n-1)}{3} + \frac{n-1}{3} = n - 1$, a contradiction. Thus, there is a vertex $y \in V - S$ such that $\deg(y) \geq 3$ or $m(\langle S\rangle) \geq 1$.

Suppose $m(\langle S\rangle) \geq 1$. If $m(\langle S\rangle) \geq 2$, then $n = m \geq n - \gamma_r(U) + 2 + \frac{1}{2}(n - \gamma_r(U))$, implying that $\gamma_r(U) \geq \lceil \frac{n+4}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. Therefore, $m(\langle S\rangle) = 1$.

Suppose there is a vertex $y \in V - S$ such that $\deg(y) \geq 3$. If $|N(y) \cap S| \geq 2$, then $n = m \geq n - \gamma_r(U) + 2 + \frac{1}{2}(n - \gamma_r(U))$, implying that $\gamma_r(U) \geq \lceil \frac{n+4}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. If $|N(y) \cap (V - S)| \geq 2$, then $n = m \geq n - \gamma_r(U) + 1 + \frac{1}{2}(n - \gamma_r(U) + 1)$, implying that $\gamma_r(U) \geq \lceil \frac{n+3}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. Therefore, every vertex in $V - S$ has degree 2. Thus, S has Property 1.

We may assume that S is independent and there is a vertex $y \in V - S$ such that $\deg(y) \geq 3$.

Suppose that $|N(y) \cap S| \geq 2$. If $|N(y) \cap S| \geq 3$, then $n = m \geq n - \gamma_r(U) + 2 + \frac{1}{2}(n - \gamma_r(U))$, implying that $\gamma_r(U) \geq \lceil \frac{n+4}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. Thus, $|N(y) \cap S| = 2$. If $\deg(y) \geq 4$, then $|N(y) \cap (V - S)| \geq 2$, and so $n = m \geq n - \gamma_r(U) + 1 + \frac{1}{2}(n - \gamma_r(U) + 1)$, implying that $\gamma_r(U) \geq \lceil \frac{n+3}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. If $\deg(x) \geq 3$ for $x \in V - S - \{y\}$, then either $\gamma_r(U) \geq \lceil \frac{n+4}{3} \rceil$ or $\gamma_r(U) \geq \lceil \frac{n+3}{3} \rceil$, a contradiction in either case. Thus, S has Property 2.

Suppose that, for all $x \in V-S$ such that $\deg(x) \geq 3$, $|N(x) \cap S| = 1$. If $v \in V-S$ such that $\deg(v) \geq 5$, then $n = m \geq n - \gamma_r(U) + \frac{1}{2}(n - \gamma_r(U) + 3)$, a contradiction. Thus, for all $v \in V-S$, $\deg(v) \leq 4$. Suppose there is a vertex $y \in V-S$ such that $\deg(y) = 4$. Then every vertex in $V-S-\{y\}$ must have degree 2. Thus, S has Property 4.

Therefore, we may assume that, if $y \in V-S$ such that $\deg(y) \geq 3$, then $\deg(y) = 3$, while $|N(y) \cap S| = 1$. Suppose there are three or more vertices $y \in V-S$ such that $\deg(y) = 3$. Then $n = m \geq n - \gamma_r(U) + \frac{1}{2}(n - \gamma_r(U) + 3)$, and so $\gamma_r(U) \geq \lceil \frac{n+3}{3} \rceil$, a contradiction. Suppose there is exactly one $y \in V-S$ such that $\deg(y) = 3$. Recall that there are $\frac{2(n-1)}{3}$ vertices in $V-S$. Moreover, for all $v \in V-S-\{y\}$, $\deg(v) = 2$, and since $|N(y) \cap S| = 1$, there are $\frac{2(n-1)}{3} - 3 > 0$ vertices to be matched in $\langle V-S \rangle$. This is impossible as $\frac{2(n-1)}{3} - 3$ is odd. Thus, there are exactly two vertices $x, y \in V-S$ such that $\deg(x) = \deg(y) = 3$. Thus, S has Property 3. ■

Observation 3. *If* $n \equiv 2 \bmod 3$*, then there is exactly one vertex* $y \in V-S$ *such that* $\deg(y) = 3$ *and* $|N(y) \cap (V-S)| = 2$*. Furthermore,* S *is independent and every vertex in* $V-S-\{y\}$ *has degree* 2.

Proof. Suppose $n \equiv 2 \bmod 3$. If S is dependent, then $n = m \geq n - \gamma_r(U) + 1 + \frac{1}{2}(n - \gamma_r(U))$, and so $\gamma_r(U) \geq \lceil \frac{n+2}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. Suppose that, for all $v \in V-S$, $\deg(v) = 2$. Let $n = 3q+2$, where $q \geq 1$. Then $|S| = q+1$ and $|V-S| = 2q+1$. Notice that $V-S$ must form a matching, and since $|V-S| = 2q+1$ is odd, this is not possible. Thus, there is a $y \in V-S$ such that $\deg(y) \geq 3$. If $|N(v) \cap S| \geq 2$ for some $v \in V-S$, then $\gamma_r(U) \geq \lceil \frac{n+2}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. Thus, $|N(v) \cap S| = 1$ for all $v \in V-S$. Suppose $\deg(y) \geq 4$, or $x \in V-S$ such that $x \neq y$ and $\deg(x) \geq 3$. Then $n = m \geq n - \gamma_r(U) + \frac{1}{2}(n - \gamma_r(U) + 2)$, which implies that $\gamma_r(U) \geq \lceil \frac{n+2}{3} \rceil > \lceil \frac{n}{3} \rceil$, a contradiction. Thus, the observation holds. ■

Let K be the status labeled graph obtained from the complete graph K_2 with vertex set $\{k_1, k_2\}$ by setting $\mathrm{Sta}(k_1) = A$ and $\mathrm{Sta}(k_2) = B$.

Let P_{AAB} be the status labeled graph obtained from the path P_3 with consecutive vertices p_1, p_2, p_3 by setting $\mathrm{Sta}(p_1) = \mathrm{Sta}(p_2) = A$ and $\mathrm{Sta}(p_3) = B$. Similarly, let P_{ABA} be the status labeled graph obtained from the path P_3 with consecutive vertices p_1, p_2, p_3 by setting $\mathrm{Sta}(p_1) = \mathrm{Sta}(p_3) = A$ and $\mathrm{Sta}(p_2) = B$.

The following status labeled graphs will serve as the basis for our characterization.

Let B_1 be the status labeled graph obtained from the cycle C_3 with consecutive vertices v_1, v_2, v_3, v_1 by setting $\mathrm{Sta}(v_1) = B$ and $\mathrm{Sta}(v_2) = \mathrm{Sta}(v_3) = A$.

Let B_2 be the status labeled graph obtained from the cycle C_4 with consecutive vertices v_1, v_2, v_3, v_4, v_1 by setting $\mathrm{Sta}(v_1) = \mathrm{Sta}(v_2) = B$ and $\mathrm{Sta}(v_3) = \mathrm{Sta}(v_4) = A$.

Lastly, let B_3 be the status labeled graph obtained from C_5 with consecutive vertices $v_1, v_2, v_3, v_4, v_5, v_1$ by setting $\mathrm{Sta}(v_1) = \mathrm{Sta}(v_3) = B$ and $\mathrm{Sta}(v_2) = \mathrm{Sta}(v_4) = \mathrm{Sta}(v_5) = A$, and joining v_2 to the vertex k_1 of K.

Note that if $U \cong B_i$ for $i \in \{1, 2, 3\}$, then $\mathrm{Sta}(B)$ is a γ_r-set of U of cardinality $\lceil \frac{n}{3} \rceil$.

Let U be a status labeled unicyclic graph. Define the following operations on U:

$\mathcal{O}_1$: Suppose v is a vertex of U such that $\mathrm{Sta}(v) = B$. Join v to the vertex p_1 of P_{AAB}.

$\mathcal{O}_2$: Suppose uv is an edge of U. One of the following is performed:

1. If $\mathrm{Sta}(u) = B$, then delete the edge uv and join the vertex u (v, respectively) to the vertex p_1 (p_3, respectively) of P_{AAB}.
2. If $\mathrm{Sta}(u) = \mathrm{Sta}(v) = A$, then delete the edge uv, join the vertex u (v, respectively) to the vertex p_1 (p_3, respectively) of P_{ABA}.

$\mathcal{O}_3$: Suppose uv is an edge of U, and suppose $\mathrm{Sta}(u) = \mathrm{Sta}(v) = A$. Delete the edge uv, and join u and v to vertex k_1 of K.

Observation 4. *If U' is the status labeled graph obtained by applying one of the above operations on U, then Sta(B) is a* **RDS** *of U'.*

Let $\mathcal{C}$ be the family of status labeled unicyclic graphs U, where U is one of the following six types:

Type 1: U is obtained from B_1 by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Type 2: U is obtained from a **Type 1** graph by joining a vertex v in this **Type 1** graph to a vertex w of K_1, setting $\mathrm{Sta}(w) = B$, and then following this by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Type 3: U is obtained from:

1. a **Type 1** graph by joining some $v \in \mathrm{Sta}(A)$ to the vertex k_1 of K, followed by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.
2. a **Type 1** graph by exactly one application of $\mathcal{O}_3$, followed by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Type 4: U is obtained from:

1. a **Type 3** graph by joining some $v \in \mathrm{Sta}(A)$ to the vertex k_1 of K, followed by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.
2. a **Type 3** graph by exactly one application of $\mathcal{O}_3$, followed by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Type 5: U is obtained from B_2 by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Type 6: U is obtained from B_3 by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Observation 5. *If U is in $\mathcal{C}$, then* $\mathrm{Sta}(B)$ *is a γ_r-set of U of cardinality* $\lceil \frac{n}{3} \rceil$.

Proof. Suppose that U is in $\mathcal{C}$. Then U is of **Type i**, where $1 \leq \mathbf{i} \leq 6$. That $\mathrm{Sta}(B)$ is a **RDS** of U follows from Observation 4, the fact that if an isolated vertex of status B is joined to any vertex of a status labeled unicyclic graph in which $\mathrm{Sta}(B)$ is a **RDS**, then in the resulting unicyclic graph $\mathrm{Sta}(B)$ is still a **RDS**, and the fact that if the vertex k_1 of K is joined to any vertex of status A of a status labeled unicyclic graph in which $\mathrm{Sta}(B)$ is a **RDS**, then in the resulting unicyclic graph $\mathrm{Sta}(B)$ is still a **RDS**.

If U is a **Type 1** graph, then $n(U) \equiv 0 \bmod 3$ and $|\mathrm{Sta}(B)| = \frac{n}{3}$, since B_1 contributes one vertex out of three to $\mathrm{Sta}(B)$, while each of the $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$ contributes one vertex out of three to $\mathrm{Sta}(B)$.

Suppose U is a **Type 2** graph obtained from the **Type 1** graph U' by joining a vertex v in U to a vertex w of K_1, setting $\mathrm{Sta}(w) = B$, and then following this by $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$.

Then $n(U') \equiv 0 \bmod 3$ and U' has exactly $\frac{n(U')}{3}$ vertices of status B, and so $n(U) \equiv 1 \bmod 3$ and $|\mathrm{Sta}(B)| = \frac{n(U)-1}{3} + 1 = \frac{n+2}{3}$, since w contributes one vertex to both $\mathrm{Sta}(B)$ and $n(U)$, while each of the $\ell \geq 0$ applications of $\mathcal{O}_1$ or $\mathcal{O}_2$ contributes one vertex out of three to $\mathrm{Sta}(B)$. As $n \equiv 1 \bmod 3$, we have $\lceil \frac{n}{3} \rceil = \frac{n+2}{3}$, and so $|\mathrm{Sta}(B)| = \lceil \frac{n}{3} \rceil$.

For a **Type 3** graph, $n \equiv 2 \bmod 3$, while $|\mathrm{Sta}(B)| = \frac{n-2}{3} + 1 = \frac{n+1}{3} = \lceil \frac{n}{3} \rceil$.

For a **Type 4** graph, $n \equiv 1 \bmod 3$, while $|\mathrm{Sta}(B)| = \lceil \frac{n-2}{3} \rceil + 1 = \lceil \frac{n+1}{3} \rceil = \lceil \frac{n}{3} \rceil$.

For graphs of **Type 5** and **Type 6**, $n \equiv 1 \bmod 3$, while $|\mathrm{Sta}(B)| = \frac{n-1}{3} + 1 = \frac{n+2}{3} = \lceil \frac{n}{3} \rceil$.

Thus, $\lceil \frac{n}{3} \rceil = |\mathrm{Sta}(B)| \geq \gamma_r(U) \geq \lceil \frac{n}{3} \rceil$, and the observation holds. ■

Let U be a unicyclic graph and denote its unique cycle by C. A *reference path* of U is a path $v = u_0, u_1, \ldots, u_t$, where $v \in C$, u_t is a leaf, and $u_i \notin C$ for $i = 1, \ldots, t$. We are now ready to state our characterization.

Theorem 4. *Let U be a unicyclic graph of order $n \geq 3$. Then $U \in \mathcal{E}$ if and only if U can be status labeled in such a way that it is in $\mathcal{C}$.*

Proof. Suppose $U \in \mathcal{C}$. By Observation 5, $U \in \mathcal{E}$.
Now, assume $U \in \mathcal{E}$ and let S be a γ_r-set of U. We proceed by induction on n. If $n = 3$, then $U = C_3$, and so it can be status labeled as B_1 which is in $\mathcal{C}$. Therefore, assume $n \geq 4$ and, for all $U' \in \mathcal{E}$ such that $3 \leq n(U') < n$, U' can be status labeled so that it is in $\mathcal{C}$. (Henceforth, we will abuse notation slightly by just saying that $U' \in \mathcal{C}$.) Suppose U is a cycle. If $n \equiv 2 \bmod 3$, then Observation 3 is contradicted. Thus, $n \equiv 0$ or $1 \bmod 3$, and so U is of **Type 1** or **Type 5**. Thus, there exists $v \in V(U)$ such that $\deg(v) \geq 3$.

Throughout, S will denote a γ_r-set for U. Before proceeding further, we prove the following two claims.

Claim 1. Suppose $v' = w_0, w_1, \ldots, w_s$ is a reference path of U. If $w_{s-1} \in S$, then $U \in \mathcal{C}$.

Proof. As $w_s \in S$, S is not independent, and so, by Observations 1, 2 and 3, $n = 3q + 1$ for some positive integer q, and Property 1 of Observation 2 is satisfied. Let $U' = U - w_s$, and notice that $S' = S - \{w_s\}$ is a **RDS** of U', while $n(U') = 3q$. Moreover, S' is a **RDS** of U' of size $\lceil \frac{3q+1}{3} \rceil - 1 = q$, whence $q = \frac{3q}{3} \leq \gamma_r(U') \leq |S'| = q$. Thus, $U' \in \mathcal{E}$, and, by the induction assumption, $U' \in \mathcal{C}$. As $n(U') \equiv 0 \bmod 3$, the graph U' is of **Type 1**. U can now be obtained from U' by joining w_s to w_{s-1}, and setting $\mathrm{Sta}(w_s) = B$, and so U is of **Type 2**. ■

Claim 2. Suppose $v' = w_0, w_1, \ldots, w_s$ is a reference path in U. If w_{s-1} is adjacent to a vertex $w'_s \in S - \{w_s\}$, then $U \in \mathcal{C}$.

Proof. As $w'_s, w_s \in S$, $w_{s-1} \notin S$, since otherwise either Observation 1, 2 or 3 will be contradicted. Let $U' = U - w_s$ and notice that $S' = S - \{w_s\}$ is a **RDS** of U'. Then, since $|N(w_{s-1}) \cap S| \geq 2$, Observations 1 and 3 imply that $n = 3q + 1$ for some positive integer q. Therefore, $n(U') = 3q$. Also, S' is a **RDS** of U' of size $\lceil \frac{3q+1}{3} \rceil - 1 = q$, whence $q = \frac{3q}{3} \leq \gamma_r(U') \leq |S'| = q$. Thus, $U' \in \mathcal{E}$, and, by the induction assumption, $U' \in \mathcal{C}$. As $n(U') \equiv 0$

mod 3, the graph U' is of **Type 1**. U can now be obtained from U' by joining w_s to w_{s-1}, and setting $\mathrm{Sta}(w_s) = B$, and so U is of **Type 2**. ■

By Claims 1 and 2, we conclude that if w is a remote vertex of U, then $w \notin S$ and $\deg(w) = 2$.

Let C denote the unique cycle of U. Among all vertices $v \in C$ such that $\deg(v) \geq 3$, choose the reference path $P = v, u_1, \dots, u_t$ for which t is as large as possible. We call a reference path an **R**t path if $\deg(v) = 3$ and $\deg(u_i) = 2$ for $i = 1, \dots, t-1$.

We begin by reducing reference paths to either **R1**, **R2** or **R3**.

Case 1. $t \geq 2$.
Since u_{t-1} is a remote vertex, $\deg(u_{t-1}) = 2$, $u_{t-1} \notin S$ and so $u_{t-2} \notin S$.

Case 1.1. $t = 2$. Note that $v = u_{t-2}$.
Suppose that $\deg(v) \geq 4$. Then v is either a remote vertex or v lies on a reference path v, u'_1, u'_2, where $\{u'_1, u'_2\} \cap \{u_1, u_2\} = \emptyset$, $\deg(u'_1) = 2$ and $u'_1 \notin S$.

As $v \notin S$, Property 4 of Observation 2 must be satisfied. Then $\deg(v) = 4$, $|N(v) \cap (V - S)| \geq 3$, $u_2 \in S$ and $n = 3q + 1$ where q is a positive integer. Let $U' = U - u_1 - u_2$, and notice that $S' = S - \{u_2\}$ is a **RDS** of U'. Then U' has order $n - 2 = 3(q-1) + 2$ and $|S'| = q$. Thus, $U' \in \mathcal{E}$, and Observation 3 holds for U'. Moreover, by the induction assumption, $U' \in \mathcal{C}$. In fact, U' is of **Type 3**. By Observation 3 and 5, $\mathrm{Sta}(B)$ is a $\gamma_r(U')$-set which is independent. If v is a remote vertex, then since the leaf adjacent to v is in $\mathrm{Sta}(B)$, $v \notin \mathrm{Sta}(B)$. If v is not a remote vertex, then $v \in \mathrm{Sta}(B)$ would imply that $u'_1 \in \mathrm{Sta}(B)$, which contradicts the fact that $\mathrm{Sta}(B)$ is independent. Thus, $\mathrm{Sta}(v) = A$. U can now be obtained from U' by joining v to vertex u_1 of $\langle\{u_1, u_2\}\rangle$, and setting $\mathrm{Sta}(u_1) = A$ and $\mathrm{Sta}(u_2) = B$, and so U is of **Type 4**. Thus, if $t = 2$, then $\deg(v) = 3$ and $\deg(u_1) = 2$.

Case 1.2. $t \geq 3$.
We first show that $\deg(u_{t-2}) = 2$. Suppose, to the contrary, that $\deg(u_{t-2}) \geq 3$. Since $u_{t-2} \notin S$, Observation 1 implies that $n \not\equiv 0 \bmod 3$.

Let $U' = U - u_{t-1} - u_t$. Suppose $n = 3q + 2$ for some positive integer q. Since $u_{t-2} \notin S$, we have, by Observation 3, $\deg(u_{t-2}) = 3$ and $|N(u_{t-2}) \cap (V - S)| \geq 2$, and so $S' = S - \{u_t\}$ is a **RDS** of U'. Thus, $U' \in \mathcal{E}$ and U' must be of **Type 1**. By Observation 1, $\mathrm{Sta}(B)$ is an independent set of U', and so $\mathrm{Sta}(u_{t-2}) = A$. We obtain U by attaching u_{t-1} to u_{t-2}, and setting $\mathrm{Sta}(u_{t-1}) = A$ and $\mathrm{Sta}(u_t) = B$. Hence, U is of **Type 3**.

Suppose $n = 3q+1$ for some positive integer q. Since $u_{t-2} \notin S$ and $\deg(u_{t-2}) \geq 3$, one of the Properties 2, 3 or 4 of Observation 2 must hold. Suppose Property 2 holds. Then $\deg(u_{t-2}) = 3$ and $|N(u_{t-2}) \cap S| = 2$. Then, besides $u_{t-3} \in S$, u_{t-2} is adjacent to exactly one other vertex in S, say w. If $\deg(w) \geq 2$, then, by our choice of the reference path P, w must be adjacent a leaf, which contradicts the fact that S is an independent set. Thus, w is a leaf, and it follows by Claim 2 that $U \in \mathcal{C}$. Hence, suppose either Property 3 or 4 holds. In both cases, u_{t-2} is adjacent to a vertex in $V - S - \{u_{t-1}\}$. It follows that $S' = S - \{u_t\}$ is a **RDS** of U'. Thus, $U' \in \mathcal{E}$ and U' must be of **Type 3**. By Observation 3, $\mathrm{Sta}(B)$ is an independent set of U', and so $\mathrm{Sta}(u_{t-2}) = A$. We obtain U by attaching u_{t-1} to u_{t-2}, and setting $\mathrm{Sta}(u_{t-1}) = A$ and $\mathrm{Sta}(u_t) = B$. Hence, U is of **Type 4**.

We may assume that $\deg(u_{t-2}) = 2$, whence $u_{t-3} \in S$. Note that u_{t-3} is not adjacent to a leaf, since otherwise $U \in \mathcal{C}$ by Claim 1. Suppose u_{t-3} lies on the reference path $v = u_0, \ldots, u_{t-3}, u'_{t-2}, u'_{t-1}$, where $\deg(u'_{t-2}) = 2$. Since $u_{t-3} \in S$, it follows that $\{u_{t-3}, u'_{t-2}, u'_{t-1}\} \subseteq S$, and Observations 1, 2 and 3 cannot be satisfied.

Suppose that $t \geq 4$. We may assume that every reference path that contains u_{t-3} has the form $v, u_1, \ldots, u_{t-3}, u'_{t-2}, u'_{t-1}, u'_t$, where $\deg(u'_{t-2}) = \deg(u'_{t-1}) = 2$, $u_{t-3} \in S$ and $u'_{t-2}, u'_{t-1} \notin S$. Let U' be obtained by removing from U every path of the form u'_{t-2}, u'_{t-1}, u'_t. Then $U' \in \mathcal{E}$. By the induction assumption, U' is of **Type i** for some $\mathbf{i} \in \{1, \ldots, 6\}$. Since u_{t-3} is a leaf of U', $\mathrm{Sta}(u_{t-3}) = B$. It follows that U can be obtained from U' by $\deg(u_{t-3}) - 1$ applications of $\mathcal{O}_1$ by joining u_{t-3} to the vertex u'_{t-2} of each of the deleted paths u'_{t-2}, u'_{t-1}, u'_t, and setting $\mathrm{Sta}(u'_{t-2}) = \mathrm{Sta}(u'_{t-1}) = A$ and $\mathrm{Sta}(u'_t) = B$. Thus, U is of **Type i**.

So suppose $t = 3$. Furthermore, suppose $\deg(v) \geq 4$. We may assume that v lies on more than one reference path of the form v, u'_1, u'_2, u'_3, where $\deg(u'_2) = \deg(u'_1) = 2$, $v \in S$ and $u_1, u_2 \notin S$. Let U' be obtained by removing the vertices u'_1, u'_2 and u'_3. Then $\deg_{U'}(v) \geq 3$, $U' \in \mathcal{E}$, and so U' is of **Type i** for some $\mathbf{i} \in \{1, \ldots, 6\}$. If $v \notin \mathrm{Sta}(B)$, then $\{u_2, u_3\} \subseteq \mathrm{Sta}(B)$, contradicting Observations 1, 2 and 3. Thus, $\mathrm{Sta}(v) = B$. It follows that U can be obtained from U' by applying $\mathcal{O}_1$ once by joining v to the vertex u_1 of the deleted path u_1, u_2, u_3, and setting $\mathrm{Sta}(u_1) = \mathrm{Sta}(u_2) = A$ and $\mathrm{Sta}(u_3) = B$. Thus, U is of **Type i**. Thus, if $t \geq 3$, then $t = 3$ and $\deg(v) = 3$.

Case 2. $t = 1$. By Claim 1, $\deg(v) = 3$, since otherwise $U \in \mathcal{C}$. Thus, if $t = 1$, then $\deg(v) = 3$.

We have now reduced P to either an **R1**, **R2** or **R3** path. We may therefore assume that each reference path of U is either an **R1**, **R2** or **R3** path.

Suppose v_i, u_1 is an **R1** path of U. By Claim 2, $N[v_i] \cap S = \{u_1\}$ since otherwise $U \in \mathcal{C}$. Since $v_i \notin S$ and $\deg(v_i) = 3$, Observation 1 implies that $n \not\equiv 0 \bmod 3$. Let v_{i-1} and v_{i+1} be the neighbors of v_i on the cycle C of U. Since the cycle in U contains at least four vertices, consider the path $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}$ on C. If $v_{i-2} = v_{i+2}$, then U is of **Type 3** or **Type 4**. Thus, $v_{i-2} \neq v_{i+2}$.

In what follows, U' is the graph obtained by removing v_i and u_1 and joining v_{i-1} and v_{i+1}. Then $S' = S - \{u_1\}$ is a $\gamma_r(U')$-set of size $\lceil \frac{n-2}{3} \rceil$, and so $U' \in \mathcal{E}$. By the induction hypothesis, $U' \in \mathcal{C}$. If $n \equiv 1 \bmod 3$, then U' is of **Type 3**; if $n \equiv 2 \bmod 3$, then U' is of **Type 1**.

We first show that $\deg(v_{i+1}) = 2$. Suppose, to the contrary, $\deg(v_{i+1}) \geq 3$. Then, since $v_{i+1} \notin S$, we have (cf. Observation 3) $n \equiv 1 \bmod 3$, S is independent, $\deg(v_{i+1}) = 3$, and v_{i+1} lies on either an **R1** or an **R2** path. Suppose v_{i+1} lies on an **R1** path. Then (cf. Observation 3 applied to U'), it follows that $\{v_{i-1}, v_{i+1}\} \subseteq \mathrm{Sta}(A)$. By applying $\mathcal{O}_3$ once, and setting $\mathrm{Sta}(v_i) = A$ and $\mathrm{Sta}(u_1) = B$, we see that U is of **Type 4**.

Suppose v_{i+1} lies on an **R2** path v_{i+1}, u_1', u_2'. Let $U'' = U - u_1' - u_2'$, and $S'' = S - \{u_2'\}$. Then S'' is a $\gamma_r(U'')$-set of size $\lceil \frac{n-2}{3} \rceil$, and so $U'' \in \mathcal{E}$. By the induction hypothesis, $U'' \in \mathcal{C}$. As $n \equiv 1 \bmod 3$, U'' is of **Type 3**. Observation 3 holds for U'', and so $\mathrm{Sta}(B)$ is an independent set, whence $v_i \notin \mathrm{Sta}(B)$, while $N(v_i) \cap \mathrm{Sta}(B) = \{u_1\}$. Thus, $\mathrm{Sta}(v_{i+1}) = A$. We obtain U by attaching u_1' to v_{i+1}, and setting $\mathrm{Sta}(u_1') = A$ and $\mathrm{Sta}(u_2') = B$. Hence, U is of **Type 4**.

Similarly, $\deg(v_{i-1}) = 2$. It now follows that $\{v_{i-2}, v_{i+2}\} \subseteq S$.

Suppose both v_{i-2} and v_{i+2} lie on **R3** paths. To avoid contradicting Observations 2 and 3, vertices v_{i-2} and v_{i+2} cannot lie on an **R1** or **R2** path.

Suppose $n = 3q + 1$ where $q \geq 2$. Observation 3 holds for U'. Thus, $\mathrm{Sta}(B)$ is an independent $\gamma_r(U')$-set, and so $\{v_{i-2}, v_{i+2}\} \subseteq \mathrm{Sta}(B)$, whence $\{v_{i-1}, v_{i+1}\} \subseteq \mathrm{Sta}(A)$. By applying $\mathcal{O}_3$ once, and setting $\mathrm{Sta}(v_i)$ and $\mathrm{Sta}(u_1) = B$, we see that U is of **Type 4**.

Suppose $n = 3q + 2$ where $q \geq 2$. Then U' has order $n - 2 = 3q$, and $|S'| = q$. Thus, $U' \in \mathcal{C}$ and U' must be of **Type 1**. It follows again that $\{v_{i-1}, v_{i+1}\} \subseteq \mathrm{Sta}(A)$. By applying $\mathcal{O}_3$ once, and setting $\mathrm{Sta}(v_i) = A$ and $\mathrm{Sta}(u_1) = B$, we see that U is of **Type 3**.

We may assume that either v_{i-2} or v_{i+2} has degree 2 — suppose $\deg(v_{i+2}) = 2$.

Suppose $\deg(v_{i+3}) \geq 3$. Then Property 3 of Observation 2 holds, S is independent, and so $v_{i+3} \notin S$. If v_{i+3} lies on an **R1** path, then $|N(v_{i+3}) \cap S| \geq 2$, which is a contradiction. Thus, v_{i+3} lies on a **R2** path v_{i+1}, u'_1, u'_2. Let $U'' = U - u'_1 - u'_2$, and $S'' = S - \{u'_2\}$. Then S'' is a $\gamma_r(U'')$-set of size $\lceil \frac{n-2}{3} \rceil$, and so $U'' \in \mathcal{E}$. By the induction hypothesis, $U'' \in \mathcal{C}$. As $n \equiv 1 \bmod 3$, U'' is of **Type 3**. Observation 3 holds for U'', and so $\mathrm{Sta}(B)$ is an independent set, whence $v_i \notin \mathrm{Sta}(B)$, while $N(v_i) \cap \mathrm{Sta}(B) = \{u_1\}$. Thus, $\mathrm{Sta}(v_{i+1}) = A$, $\mathrm{Sta}(v_{i+2}) = B$, while $\mathrm{Sta}(v_{i+3}) = A$. We obtain U by attaching u'_1 to v_{i+3}, and setting $\mathrm{Sta}(u'_1) = A$ and $\mathrm{Sta}(u'_2) = B$. Hence, U is of **Type 4**.

Consider the path $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$, where $v_{i+2} \in S$ and $v_i, v_{i+1}, v_{i+3}, v_{i+4} \notin S$. We form U''' by removing the vertices $v_{i+1}, v_{i+2}, v_{i+3}$ and joining v_i and v_{i+4}. The set $S''' = S - \{v_{i+2}\}$ is a $\gamma_r(U''')$-set of size $\lceil \frac{n(U''')}{3} \rceil$, and so $U''' \in \mathcal{E}$. By the induction hypothesis, $U''' \in \mathcal{C}$ and U''' is of any type except of **Type 1**. By Observations 2 and 3, $\{v_i, v_{i+4}\} \not\subseteq \mathrm{Sta}(B)$. Thus, $\mathrm{Sta}(v_i) = \mathrm{Sta}(v_{i+4}) = A$, $\mathrm{Sta}(v_i) = B$ and $\mathrm{Sta}(v_{i+4}) = A$ or $\mathrm{Sta}(v_i) = A$ and $\mathrm{Sta}(v_{i+4}) = B$. U can now be obtained by reinserting the path $v_{i+1}, v_{i+2}, v_{i+3}$ and labeling the vertices consecutively by either (1) A, B, A (2) A, A, B or (3) B, A, A, and so we have applied $\mathcal{O}_2$ to U'''. Thus, U is of any type except of **Type 1**.

Therefore, we may assume that U has no **R1** paths.

Suppose U has at least one **R2** path v_i, u_1, u_2. By Claim 1, $u_1 \notin S$, and so $v_i \notin S$. Without loss of generality, assume $v_{i-1} \in S$. By Observations 1, 2 and 3, U can have at most two **R2** paths. Then Observation 2 or 3 holds. If U has a cycle of three or five vertices, then we are done. If U has a cycle of four vertices, we have a contradiction. Thus, U has a cycle on at least six vertices.

Suppose U has exactly two **R2** paths, and let v_j, u'_1, u'_2 be the other **R2** path. Then, as before, $v_j, u'_1 \notin S$. Thus, Property 3 of Observation 2 holds, and so $n = 3q + 1$ where $q \geq 3$. Moreover, $v_{i+1} \notin S$, and so $v_{i+2} \in S$, $v_{i+3}, v_{i+4} \notin S$, while $v_{i+5} \in S$. Note that $v_{i-1} = v_{i+5}$ is possible.

Suppose $j = i + 1$. Let r' $(0 \leq r' \leq 1)$ denote the number of **R3** paths attached to v_{i+2}. We form U' by removing the vertices $v_{i+2}, v_{i+3}, v_{i+4}$, and the $3r'$ vertices of the possible **R3** path, and then joining v_{i+1} and v_{i+5}. Then the order of U' is $n - 3 - 3r' = 3(q - r' - 1) + 1$, and $\gamma_r(U') = q - r'$. Thus, $U' \in \mathcal{E}$ and Observation 2 holds. Hence, $v_{i+1}, v_i, u'_1 \in \mathrm{Sta}(A)$, and therefore $\mathrm{Sta}(v_{i+5}) = B$. Then U' must be of **Type i**, where $\mathbf{i} \in \{2, 4, 5, 6\}$. Remove the edge $v_{i+1}v_{i+5}$, reinsert the path $v_{i+2}, v_{i+3}, v_{i+4}$ and label the vertices consecutively B, A, A. By applying $\mathcal{O}_1$ to v_{i+2} (if necessary), we obtain U. Hence, U is of **Type i**, where $\mathbf{i} \in \{2, 4, 5, 6\}$.

Thus, v_{i+1} is not on an **R2** path.

Suppose v_{i+2} is not on an **R3** path, and suppose $j = i + 3$. We form U' by removing the vertices $v_{i+1}, v_{i+2}, v_{i+3}, u'_1, u'_2$, and then joining v_i and v_{i+4}. Then the order of U' is $n - 5 = 3(q - 2) + 2$, and $\gamma_r(U') = q - 1$. Thus, $U' \in \mathcal{E}$, U' is of **Type 3**, and Observation 3 holds. Thus, $\mathrm{Sta}(v_i) = A$.

Suppose that $\mathrm{Sta}(v_{i+4}) = B$. Remove the edge v_iv_{i+4}, reinsert the path $v_{i+1}, v_{i+2}, v_{i+3}$ and label the vertices consecutively B, A, A. We obtain U by attaching u'_1 to v_{i+3}, and setting $\mathrm{Sta}(u'_1) = A$ and $\mathrm{Sta}(u'_2) = B$. Hence, U is of **Type 4**.

Thus, $\mathrm{Sta}(v_{i+4}) = A$. Remove the edge v_iv_{i+4}, reinsert the path v_{i+1}, v_{i+2}, v_{i+3} and label the vertices consecutively A, B, A. We obtain U by attaching u'_1 to v_{i+3}, and setting $\mathrm{Sta}(u'_1) = A$ and $\mathrm{Sta}(u'_2) = B$. Hence, U is of **Type 4**.

Thus, $j \neq i+3$. We form U' by removing the vertices $v_{i+1}, v_{i+2}, v_{i+3}$, and then joining v_i and v_{i+4}. The order of U' is $n-3 = 3(q-1)+1$, and $\gamma_r(U') = q$. Thus, $U' \in \mathcal{E}$, U' is of **Type i**, where $\mathbf{i} \in \{2, 4, 5, 6\}$. By Property 3 of Observation 2, $\mathrm{Sta}(v_i) = A$. Suppose $\mathrm{Sta}(v_{i+4}) = B$. Remove the edge v_iv_{i+4}, reinsert the path $v_{i+1}, v_{i+2}, v_{i+3}$ and label the vertices consecutively B, A, A. Thus, U is of **Type i**, where $\mathbf{i} \in \{2, 4, 5, 6\}$.

Thus, $\mathrm{Sta}(v_{i+4}) = A$. Remove the edge v_iv_{i+4}, reinsert the path v_{i+1}, v_{i+2}, v_{i+3} and label the vertices consecutively A, B, A. Thus, U is of **Type i**, where $\mathbf{i} \in \{2, 4, 5, 6\}$.

Now, suppose that v_{i+2} is on an **R3** path.

Suppose $j \in \{i + 3, i + 4\}$. Let $U' = U - u'_1 - u'_2$. Then the order of U' is $n - 2 = 3q - 1 = 3(q - 1) + 2$, and $\gamma_r(U') = q$. Thus, $U' \in \mathcal{E}$, U' is of **Type 3**, and Observation 3 holds. Hence, $\mathrm{Sta}(v_{i+2}) = B$, and so $\mathrm{Sta}(v_{i+1}) = \mathrm{Sta}(v_{i+3}) = A$, whence $\mathrm{Sta}(v_{i+4}) = A$. We obtain U by attaching u'_1 to v_j, and setting $\mathrm{Sta}(u'_1) = A$ and $\mathrm{Sta}(u'_2) = B$. Hence, U is of **Type 4**.

Thus, $j \notin \{i + 3, i + 4\}$. Let r' $(0 \leq r' \leq 1)$ denote the number of **R3** paths attached to v_{i+5}. We form U' by removing the vertices $v_{i+3}, v_{i+4}, v_{i+5}$, and the $3r'$ vertices of the **R3** paths on v_{i+5}, and then joining v_{i+2} and v_{i+6}. Note that $v_{i+6} \neq v_i$, since $j \notin \{i, \ldots, i + 5\}$. Now, the order of U' is $n - 3 - 3r' = 3(q - 1 - r') + 1$ and $\gamma_r(U') = q - r'$. Thus, $U' \in \mathcal{E}$,

U' is of **Type i**, where $\mathbf{i} \in \{2,4,5,6\}$, and Property 3 of Observation 2 holds. Hence, $\mathrm{Sta}(v_{i+2}) = B$, and so $\mathrm{Sta}(v_{i+1}) = \mathrm{Sta}(v_{i+6}) = A$. Remove the edge $v_{i+2}v_{i+6}$, reinsert the path $v_{i+3}, v_{i+4}, v_{i+5}$, and label the vertices consecutively A, A, B. By applying $\mathcal{O}_1$ to v_{i+5} (if necessary), we obtain U. Hence, U is of **Type i**, where $\mathbf{i} \in \{2,4,5,6\}$.

Thus, v_i, u_1, u_2 is the only **R2** path of U.

Suppose $n = 3q + 2$ for some $q \geq 2$, and so Observation 3 holds. Since $v_{i-1} \in S$, $v_{i+1} \notin S$, and so $v_{i+2} \in S$, $v_{i+3} \notin S, v_{i+4} \notin S$, while $v_{i+5} \in S$. Note that $v_{i-1} = v_{i+5}$ is possible.

Suppose v_{i+2} is not on an **R3** path. We form U' by removing the vertices $v_{i+1}, v_{i+2}, v_{i+3}$, and then joining v_i and v_{i+4}. The order of U' is $n - 3 = 3(q-1) + 2$, and $\gamma_r(U') = q$. Thus, $U' \in \mathcal{E}$, U' is of **Type 3**. By Observation 3, $\mathrm{Sta}(v_i) = A$. Suppose $\mathrm{Sta}(v_{i+4}) = B$. Remove the edge v_iv_{i+4}, reinsert the path $v_{i+1}, v_{i+2}, v_{i+3}$ and label the vertices consecutively B, A, A. Thus, U is of **Type 3**. Hence, $\mathrm{Sta}(v_{i+4}) = A$. Remove the edge v_iv_{i+4}, reinsert the path $v_{i+1}, v_{i+2}, v_{i+3}$ and label the vertices consecutively A, B, A. Thus, U is of **Type 3**.

So suppose v_{i+2} is on an **R3** path. Let r' $(0 \leq r' \leq 1)$ denote the number of **R3** paths attached to v_{i+5}. We form U' by removing the vertices $v_{i+3}, v_{i+4}, v_{i+5}$, and the $3r'$ vertices of the **R3** paths on v_{i+5}, and then joining v_{i+2} and v_{i+6}. Note that $v_{i+6} = v_i$ is possible. Now, the order of U' is $n - 3 - 3r' = 3(q - 1 - r') + 2$ and $\gamma_r(U') = q - r'$. Thus, $U' \in \mathcal{E}$, U' is of **Type 3**, and Observation 3 holds. Hence, $\mathrm{Sta}(v_{i+2}) = B$, and so $\mathrm{Sta}(v_{i+1}) = \mathrm{Sta}(v_{i+6}) = A$. Remove the edge $v_{i+2}v_{i+6}$, reinsert the path $v_{i+3}, v_{i+4}, v_{i+5}$, and label the vertices consecutively A, A, B. By applying $\mathcal{O}_1$ to v_{i+5} (if necessary), we obtain U. Hence, U is of **Type 3**.

Suppose $n = 3q + 1$ for some $q \geq 2$, and so Property 2 of Observation 2 holds. Consider the path $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$, where $\{v_{i+1}, v_{i+4}\} \subseteq S$ and $\{v_i, v_{i+2}, v_{i+3}\} \cap S = \emptyset$. Let r' $(0 \leq r' \leq 1)$ denote the number of **R3** paths on v_{i+1}. We form U' by removing the vertices $v_{i+1}, v_{i+2}, v_{i+3}$, and the $3r'$ vertices of the **R3** paths, and then joining v_i and v_{i+4}. The order of U' is $n - 3 - 3r' = 3(q - r' - 1) + 1$ and $\gamma_r(U') = q - r'$. Thus, $U' \in \mathcal{E}$, Property 2 of Observation 2 holds, while U' is of **Type i**, where $\mathbf{i} \in \{2,4,5,6\}$. Thus, $\mathrm{Sta}(v_i) = A$, and $\mathrm{Sta}(v_{i-1}) = B = \mathrm{Sta}(v_{i+4})$. Remove the edge v_iv_{i+4}, reinsert the path $v_{i+1}, v_{i+2}, v_{i+3}$, and label the vertices consecutively B, A, A. By applying $\mathcal{O}_1$ to v_{i+1} (if necessary), we obtain U. Thus, U is of **Type i**, where $\mathbf{i} \in \{2,4,5,6\}$.

Thus, we may assume that U has only **R3** paths, and so $V-S$ has only degree two vertices. Therefore, Observation 1 or Observation 2 holds, respectively. So $n = 3q+1$ ($3q$, respectively), where $q \geq 2$. If U has a cycle on three, four or six vertices, then we are done. If U has a cycle on five vertices, then we reach a contradiction. Let v_i be a vertex that lies on an **R3** path. Consider the path $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$, where $v_{i+1} \notin S, v_{i+2} \notin S$ and $v_{i+3} \in S$. Let r' $(0 \leq r' \leq 1)$ be the number of **R3** paths attached to v_{i+3}. We form U' by removing $v_{i+1}, v_{i+2}, v_{i+3}$, and the $3r'$ vertices on the **R3** paths on v_{i+3}, and then joining v_i and v_{i+4}. Then U' has order $n-3-3r' = 3(q-r'-1)+1$ ($n-3-3r' = 3(q-r'-1)$, respectively), and $\gamma_r(U') = q-r'$ ($\gamma_r(U') = q-r'-1$, respectively). Thus, $U' \in \mathcal{E}$, and U' is of **Type i**, where $\mathbf{i} \in \{2,4,5,6\}$ (**Type 1**, respectively). Thus, Observation 2 (Observation 1, respectively) holds. Hence, $\mathrm{Sta}(v_i) = B$. Remove the edge v_iv_{i+4}, reinsert the path $v_{i+1}, v_{i+2}, v_{i+3}$, and label the vertices consecutively A, A, B. Thus, U is of **Type i**, where $\mathbf{i} \in \{2,4,5,6\}$, or U is of **Type 1** and the proof is complete. ■

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