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k-KERNELS AND SOME OPERATIONS IN DIGRAPHS

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Abstract

Let D be a digraph. $V(D)$ denotes the set of vertices of D; a set $N \subseteq V(D)$ is said to be a k-kernel of D if it satisfies the following two conditions: for every pair of different vertices $u, v \in N$ it holds that every directed path between them has length at least k and for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-directed path of length at most $k - 1$.

In this paper, we consider some operations on digraphs and prove the existence of k-kernels in digraphs formed by these operations from another digraphs.

Keywords: k-kernel, k-subdivision digraph, k-middle digraph and k-total digraph.

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1. Introduction

We refer the reader to [1] for general concepts. In this paper, D denotes a digraph; $V(D)$ is the set of vertices and $A(D)$ denotes the set of arcs.

A directed path is a sequence $P = (x_0, x_1, \ldots, x_n)$ of distinct vertices of D such that $(x_i, x_{i+1}) \in A(D)$ for each $i, 0 \leq i \leq n-1$. The length of P is n and we denote $\ell(P) = n$. For $x, y \in V(D)$, the distance from x to y in D is denoted as $d_D(x,y)$ and defined as: $d_D(x,y) = \min\{\ell(P)|P$ is an $xy-$ directed path $\}$ whenever there exists an xy-directed path in D, otherwise, we define $d_D(x,y) = \infty$. If P is a directed path and $a, b \in V(P)$, then (a, P, b) denotes the ab-directed path contained in P.

A set $N \subseteq V(D)$ is said to be k-independent whenever for any two different vertices $x, y \in N$ we have $d_D(x,y) \geq k$ and $d_D(y,x) \geq k$. N is said to be $(k-1)$-absorbent whenever for each $x \in V(D) - N$ there exists $y \in N$ such that $d_D(x,y) \leq k-1$. The set N is said to be a k-kernel if it is k-independent and $(k-1)$-absorbent.

We note that a 2-kernel is a kernel of a digraph in the sense of J. von Neumann and O. Morgenstern [20]. The problem of the existence of a kernel in a digraph has been studied in [2, 3, 4, 7, 17, 18].

The existence of kernels of digraphs formed by some operations from another digraphs have been studied by several authors, namely: M. Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel [16]; M. Harminc and T. Olejníková [11]; J. Topp [19], H. Galeana-Sánchez and V. Neumann-Lara [7, 8].

The concept of k-kernel was introduced by M. Kwaśnik in [14]. Clearly, this concept generalizes the concept of a kernel of a digraph. It has been studied by several authors: M. Harminc [9], M. Kwaśnik [14, 15], M. Kucharska [12, 13], H. Galeana-Sánchez [5, 6], A. Włoch and I. Włoch [21].

In [10], M. Harminc constructed all kernels of the line digraph of D from the kernels of D and in [19] the author considered some special digraphs: $S(D)$; $Q(D)$, $T(D)$ and $L(D)$ which were called the subdivision digraph, the middle digraph, the total digraph and the line digraph of D, respectively and studied some neccessary or sufficient conditions for the existence or uniqueness of kernels of these digraphs.

In this paper, for a given digraph D and any $k \geq 2$ we define: the k-subdivision $S^k(D)$, a generalization of the subdivision $S(D)$, the digraph $R^k(D)$, the k-middle digraph $Q^k(D)$ and the k-total digraph $T^k(D)$. Also the following results are proved: for any digraph D and for any $k \geq 2$ the digraphs $S^k(D), R^k(D)$ and $Q^k(D)$ have a k-kernel. For any digraph D and for $k \geq 3$ the digraph $T^k(D)$ has a k-kernel.

2. k-Kernels in: $S^k(D)$, $R^k(D)$, $Q^k(D)$ and $T^k(D)$

Let D be a digraph. The *line digraph* $L(D)$ of D is the digraph defined as follows: $V(L(D)) = A(D)$ and $(a = (u,v), b = (z,w)) \in A(L(D))$ if and only if $v = z$ [1].

[19]: For a given digraph D, the subdivision digraph $S(D)$ of D is defined by: $V(S(D)) = V(D) \cup A(D)$ and

$$\Gamma^+(x) = \begin{cases} \{x\} \times \Gamma^+_D(x), & \text{whenever } x \in V(D), \\ \{v\}, & \text{whenever } x = (u,v) \in A(D). \end{cases}$$

Notice that for a vertex x of the *subdivision digraph* of D we have the following: If x corresponds to a vertex of D, then x is adjacent to the arcs which are incident from x in D; and if x corresponds to an arc of D, then x is adjacent only to the terminal endpoint of x. Also notice that $S(D)$ is obtained from D by changing each arc of D for a directed path of length two.

Let D be a digraph. We define the k-*subdivision* digraph of D, denoted $S^k(D)$, as follows:

$$S^k(D) = S(D) - \{(u,a) | a \in A(D) \text{ and } u \text{ is the initial endpoint of } a\} \cup \left(\bigcup_{a \in A(D)} \beta_a \right)$$

for each $a = (u,v) \in A(D)$, $\beta_a = (a_0 = u, a_1, \ldots, a_{n(a)k+k-1} = a = (u,v))$ is a ua-directed path whose length is $\equiv k - 1 (\operatorname{mod} k)$ $(n(a) \in \mathbb{N})$ and the following two properties hold:

(i) $V(\beta_a) \cap V(S(D)) = \{u, a\}$,

(ii) For any $a, b \in A(D)$ with $a \neq b$ we have $(V(\beta_a) - \{u\}) \cap V(\beta_b) = \emptyset$.

Notice that $S^k(D)$ is obtained from D by substituting each arc of D for a directed path whose length is $\equiv 0 (\operatorname{mod} k)$ (for an example see Figure 1).

We write $V^0(D) = \{x \in V(D) \mid \delta^+_D(x) = 0\}$.

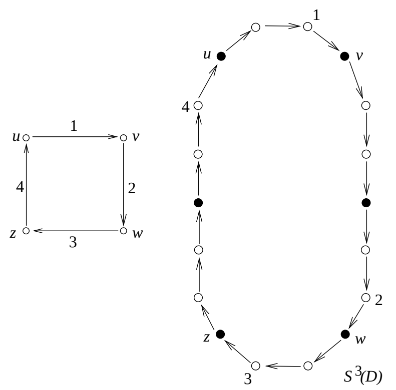

Figure 1

Finally, we define the digraphs $R^k(D)$, $Q^k(D)$ and $T^k(D)$ as follows $R^k(D) = S^k(D) \cup D$, $Q^k(D) = S^k(D) \cup L(D)$ and $T^k(D) = S^k(D) \cup D \cup L(D)$ (for an example see Figure 2).

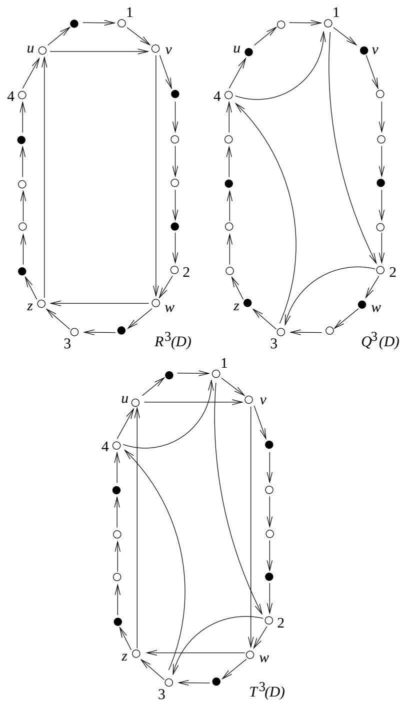

Figure 2

Theorem 2.1. *For any digraph D and for any integer k $(k \geq 2)$, the k-subdivision digraph $S^k(D)$ of D has a k-kernel.*

Proof. Let D and $S^k(D)$ be digraphs as in the hypothesis. For each $a \in A(D)$ we denote $\mathfrak{N}_a = \{a_i \in V(\beta_a) \mid i \equiv 0(\mathrm{mod}\, k)\}$. We will prove that $\mathfrak{N} = V^0(D) \cup \bigcup_{a \in A(D)} \mathfrak{N}_a$ is a k-kernel of $S^k(D)$. Observe that $V(D) \subseteq \mathfrak{N}$.

Claim 1. $\mathfrak{N}$ is a k-independent set of vertices of $S^k(D)$.
Let $x, y \in \mathfrak{N}, x \neq y$. We will prove $d_{S^k(D)}(x, y) \geq k$ and $d_{S^k(D)}(y, x) \geq k$.

Case 1. $x \in V^0(D)$ and $y \in V^0(D)$.
Since $\delta^+_{S^k(D)}(x) = \delta^+_D(x) = 0$ and $\delta^+_{S^k(D)}(y) = \delta^+_D(y) = 0$, it follows that $d_{S^k(D)}(x, y) = d_{S^k(D)}(y, x) = \infty$.

Case 2. $x \in V^0(D)$ and $y \in \bigcup_{a \in A(D)} \mathfrak{N}_a$.
Since $\delta^+_{S^k(D)}(x) = \delta^+_D(x) = 0$, we have $d_{S^k(D)}(x, y) = \infty$. Let $c = (u, v) \in A(D)$ such that $y \in \mathfrak{N}_c$. From the definition of $S^k(D)$ we have $d_{S^k(D)}(y, x) = d_{S^k(D)}(y, c = (u, v)) + d_{S^k(D)}(c, x)$. Now since $y = c_i$ with $i \equiv 0(\mathrm{mod}\, k)$ and $\ell(\beta_c) \equiv k - 1(\mathrm{mod}\, k)$ it follows that $d_{S^k(D)}(y, c = (u, v)) = d_{\beta_c}(y, c) \geq k - 1$. Clearly, $d_{S^k(D)}(c, x) \geq 1$ (as $c \in A(D)$ and $x \in V^0(D) \subseteq V(D)$). Therefore $d_{S^k(D)}(y, x) \geq (k - 1) + 1 = k$.

Case 3. $x \in \bigcup_{a \in A(D)} \mathfrak{N}_a$ and $y \in V^0(D)$.
Proceed exactly as in Case 2 interchanging x with y.

Case 4. $x \in \bigcup_{a \in A(D)} \mathfrak{N}_a$ and $y \in \bigcup_{a \in A(D)} \mathfrak{N}_a$.

Case 4.1. There exists $c = (u, v) \in A(D)$ such that $\{x, y\} \subseteq \mathfrak{N}_c$.
From the definition of $\mathfrak{N}_c$ we have $x = c_{mk}$ and $y = c_{tk}$ for some $0 \leq m \leq n(c)$, $0 \leq t \leq n(c)$. Assume without loss of generality $t > m$.

From the definition of $S^k(D)$ and the fact $x \neq v$ (as $\ell(\beta_c) \equiv k - 1$ $(\mathrm{mod}\, k)$) we have: $d_{S^k(D)}(x, y) = d_{\beta_c}(x, y) = (t - m)k \geq k$. On the other hand, we have $d_{S^k(D)}(y, x) = d_{S^k(D)}(y, c) + d_{S^k(D)}(c, v) + d_{S^k(D)}(v, x)$. Since $d_{S^k(D)}(y, c) = d_{\beta_c}(y, c) \geq k - 1$ and $d_{S^k(D)}(c, v) = 1$, we obtain $d_{S^k(D)}(y, x) \geq k$.

Observation 1. *Observe that in this case we have the same inequalites when we are working in $Q^k(D)$, i.e., $d_{Q^k(D)}(y, x) \geq k$, because the definition of $Q^k(D)$ implies: $d_{Q^k(D)}(y, x) = d_{Q^k(D)}(y, c) + d_{Q^k(D)}(c, x)$. And clearly, $d_{Q^k(D)}(y, c) \geq k - 1$ and $d_{Q^k(D)}(c, x) \geq 1$.*

Case 4.2. $x \in \mathfrak{N}_a$ and $y \in \mathfrak{N}_b$ for some $a, b \in A(D)$ with $a \neq b$. Assume without loss of generality that $a = (u, v)$ and $b = (w, z)$.

$d_{S^k(D)}(x, y) = d_{S^k(D)}(x, a) + d_{S^k(D)}(a, v) + d_{S^k(D)}(v, y)$. From the definition of $\mathfrak{N}_a$ we have $d_{S^k(D)}(x, a) \geq k - 1$ and from the definition of $S^k(D)$, $d_{S^k(D)}(a, v) = 1$. Therefore $d_{S^k(D)}(x, y) \geq k$.

Observation 2. *Notice that in this case we have* $d_{Q^k(D)}(x, a) = d_{S^k(D)}(x, a)$ *and* $d_{Q^k(D)}(a, y) \geq 1$ *(as* $a \neq y$*). Thus* $d_{Q^k(D)}(x, y) = d_{Q^k(D)}(x, a) + d_{Q^k(D)}(a, y) \geq k$.

Interchanging x with y we obtain $d_{S^k(D)}(y, x) \geq k$.

Claim 2. $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $S^k(D)$.
Let $x \in V(S^k(D) - \mathfrak{N})$. We will prove that there exists $y \in \mathfrak{N}$ such that $d_{S^k(D)}(x, y) \leq k - 1$. Since $V^0 \subseteq \mathfrak{N}$, it follows from the definition of $S^k(D)$ and the fact $x \in V(S^k(D) - \mathfrak{N})$ that $x \in \bigcup_{a \in A(D)} \beta_a$. Let $c = (u, v) \in A(D)$ be such that $x \in \beta_c$.

Case 1. $x \in \beta_c - \{c_i \mid n(c)k + 1 \leq i \leq n(c)k + (k-1)\}$.
Since $x \notin \mathfrak{N}$ (and then $x \notin \mathfrak{N}_c$), it follows that $x = c_{mk+j}$ for some m and j with $0 \leq m \leq n(c)$ and $1 \leq j \leq k - 1$. From the definition of $S^k(D)$ we have $d_{S^k(D)}(x, c_{(m+1)k}) = d_{\beta_c}(c_{mk+j}, (c_{(m+1)k}) = k - j \leq k - 1$. Clearly, $c_{(m+1)k} \in \mathfrak{N}$.

Case 2. $x \in \{c_i | n(c)k + 1 \leq i \leq n(c)k + (k-1) = (u, v) = c\}$.
Clearly, $d_{S^k(D)}(x, v) = d_{S^k(D)}(x, c) + d_{S^k(D)}(c, v)$; $d_{S^k(D)}(x, c) \leq k - 2$ and $d_{S^k(D)}(c, v) = 1$. Thus $d_{S^k(D)}(x, v) \leq k - 1$ with $v \in \mathfrak{N}$ (recall $V(D) \subseteq \mathfrak{N}$). ∎

Theorem 2.2. *For any digraph D and for any integer k $(k \geq 2)$, the k-middle digraph $Q^k(D)$ of D has a k-kernel.*

Proof. Consider the set $\mathfrak{N} \subseteq V(S^k(D)) = V(Q^k(D))$ defined in the proof of Theorem 2.1. Since $S^k(D)$ is a spanning subdigraph of $Q^k(D)$ and $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $S^k(D)$, it follows that $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $Q^k(D)$.

The proof that $\mathfrak{N}$ is k-independent in $Q^k(D)$ is the same as the proof that $\mathfrak{N}$ is k-independent in $S^k(D)$, we only need to recall Observations 1 and 2 given along this proof. ∎

Theorem 2.3. *Let D be any digraph and for any integer k ($k \geq 2$), then the digraph $R^k(D)$ has a k-kernel.*

Proof. Let D, k and $R^k(D)$ be as in the hypothesis. For each $a = (u, v) \in A(D)$ we define $\mathfrak{N}_a$ as follows: $\mathfrak{N}_a$ is the unique k-kernel of $(\beta_a - \{u\}) \cup \{(a = (u, v), v)\}$ whenever $\delta^+_D(v) = 0$. And $\mathfrak{N}_a = \{a_i \in V(\beta_a) | i \equiv 1 (\bmod k)\}$ whenever $\delta^+_D(v) > 0$. We write $B^0 = \{x \in V(D) | \delta^+_D(x) = \delta^-_D(x) = 0\}$. We will prove that $\mathfrak{N} = \bigcup_{a \in A(D)} \mathfrak{N}_a \cup B^0$ is a k-kernel of $R^k(D)$. First, observe that $V^0(D) \subseteq \mathfrak{N}$.

Claim 3. $\mathfrak{N}$ is a k-independent set of $R^k(D)$.
Let $x, y \in \mathfrak{N}$ with $x \neq y$. We will prove that $d_{R^k(D)}(x, y) \geq k$ and $d_{R^k(D)}(y, x) \geq k$. Observe that if $x \in B^0$, then $d_{R^k(D)}(x, y) = d_{R^k(D)}(y, x) = \infty \ \ \forall y \in V(R^k(D))$.

Case 1. There exists $c = (u, v) \in A(D)$ such that $\{x, y\} \subseteq \mathfrak{N}_c$.

Case 1.1. $\delta^+_D(v) = 0$. In this case, we have $\mathfrak{N}_c = \{c_i \in V(\beta_c) | i \equiv 0 \ (\bmod k),\ i > 0\} \cup \{v\}$.

We assume without loss of generality that $x = c_{mk}$ with $1 \leq m \leq n(c)$ and, $y = c_{tk}$ with $m < t$ or $y = v$.

When $y = c_{tk}$, we have $d_{R^k(D)}(x, y) = (t - m)k \geq k$. When $y = v$, we have $d_{R^k(D)}(x, v) = d_{R^k(D)}(x, c) + d_{R^k(D)}(c, v)$. Since $d_{R^k(D)}(x, c) \geq k - 1$ and $d_{R^k(D)}(c, v) = 1$, we conclude $d_{R^k(D)}(x, v = y) \geq k$.

Now, from the definition of $R^k(D)$ we have: $d_{R^k(D)}(y, x) = d_{R^k(D)}(y, v) + d_{R^k(D)}(v, x)$. Since $\delta^+_D(v) = 0$ we have $d_{R^k(D)}(v, x) = \infty$. Thus $d_{R^k(D)}(y, x) \geq k$.

Case 1.2. $\delta^+_D(v) > 0$. In this case we have $\mathfrak{N}_c = \{c_i \in V(\beta_c) | i \equiv 1 \ (\bmod k)\}$. We assume without loss of generality that $x = c_{mk+1}$, $y = c_{tk+1}$ with $0 \leq m < t$. Clearly, $d_{R^k(D)}(x, y) = (t - m)k \geq k$ and $d_{R^k(D)}(y, x) = d_{R^k(D)}(y, c) + d_{R^k(D)}(c, v) + d_{R^k(D)}(v, x)$. From the definition of $R^k(D)$ we have $d_{R^k(D)}(y, c) \geq k - 2$, $d_{R^k(D)}(c, v) = 1$ and $d_{R^k(D)}(v, x) \geq 1$ (because $v \neq x$, be as $m < t$). Thus $d_{R^k(D)}(y, x) \geq k$.

Case 2. $x \in \mathfrak{N}_b$ and $y \in \mathfrak{N}_c$ with $b = (u, v)$, $c = (w, z)$, $b \neq c$.
From the definition of $R^k(D)$ we have $d_{R^k(D)}(x, y) = d_{R^k(D)}(x, b = (u, v)) + d_{R^k(D)}(b, v) + d_{R^k(D)}(v, w) + d_{R^k(D)}(w, y)$. When $\delta^+_D(v) = 0$, we obtain $d_{R^k(D)}(v, w) = \infty$ and then $d_{R^k(D)}(x, y) \geq k$.

When $\delta_D^+(v) > 0$, we obtain $\mathfrak{N}_b = \{b_i \in V(\beta_b) | i \equiv 1(\operatorname{mod} k)\}$ and $d_{R^k(D)}(x, b) \geq k - 2$ also from the definition of $R^k(D)$, $d_{R^k(D)}(b, v) = 1$. If $v \neq w$, then $d_{R^k(D)}(v, w) \geq 1$ and we conclude that $d_{R^k(D)}(x, y) \geq k$. If $v = w$, then $\delta_D^+(w) > 0$, $w \notin \mathfrak{N}_c$ and $w \neq y$; therefore $d_{R^k(D)}(w, y) \geq 1$, and we conclude again that $d_{R^k(D)}(x, y) \geq k$.

Analogously, it can be proved $d_{R^k(D)}(y, x) \geq k$.

Claim 4. $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $R^k(D)$.
We will prove that for any $z \in V(R^k(D) - \mathfrak{N})$ there exists $w \in \mathfrak{N}$ such that $d_{R^k(D)}(z, w) \leq k - 1$.

Let $z \in V(R^k(D) - \mathfrak{N})$. We have observed that $V^0(D) \subseteq \mathfrak{N}$. Thus $z \in \bigcup_{a \in A(D)} V(\beta_a)$. Take $c = (u, v) \in A(D)$ such that $z \in V(\beta_c)$.

Case 1. $\delta_D^+(v) = 0$. In this case, $\mathfrak{N}_c = \{c_i \in V(\beta_c) | i \equiv 0(\operatorname{mod} k), i \geq 1\} \cup \{v\}$. Since $z \notin \mathfrak{N}$, then $z = c_0$ or $z = c_{mk+j}$ with $1 \leq j \leq k - 1$ and $0 \leq m \leq n(c)$.

If $z = c_0 = u$, then from the definition of $R^k(D)$ we have $(z = u, v) \in A(R^k(D))$ and $d_{R^k(D)}(z, v) = 1 \leq k - 1$ with $v \in \mathfrak{N}$. If $z = c_{mk+j}$, then $d_{R^k(D)}(c_{mk+j}, c_{(m+1)k}) = k - j \leq k - 1$ whenever $m \neq n(c)$, and $d_{R^k(D)}(z, v) \leq d_{R^k(D)}(z, c = (u, v)) + d_{R^k(D)}(c = (u, v), v) \leq k - 2 + 1 = k - 1$ whenever $m = n(c)$ (recall that $z = c_{mk+j}, c = c_{n(k)+(k-1)}$ and $d_{R^k(D)}(c = (u, v), v) = 1$).

Case 2. $\delta_D^+(v) > 0$. In this case, $\mathfrak{N}_c = \{c_i \in V(\beta_c) | i \equiv 1(\operatorname{mod} k)\}$.
When $z \in V(\beta_c) - \{c_i | n(c)k + 2 \leq i \leq n(c)k + (k-1)\}$, we have two possibilities: If $z = c_0$, then $d_{R^k(D)}(z, c_1) = 1 \leq k - 1$ with $c_1 \in \mathfrak{N}_c \subseteq \mathfrak{N}$. If $z \neq c_0$, then $z = c_{mk+j}$ with $2 \leq j \leq k, 0 \leq m < n(c)$ and $d_{R^k(D)}(z, c_{(m+1)k+1}) \leq k - 1$ with $c_{(m+1)k+1} \in \mathfrak{N}$.

When $z \in \{c_i | n(c)k + 2 \leq i \leq n(c)k + (k-1)\}$, we recall that $\delta_D^+(v) > 0$. Thus there exists $b = (v, w) \in A(D)$. We consider β_b. Consider two possibilities: If $\delta_{R^k(D)}^+(w) > 0$, then $\mathfrak{N}_b = \{b_i \in V(\beta_b) | i \equiv 1(\operatorname{mod} k)\}$; and it follows that $d_{R^k(D)}(z, b_1) = d_{R^k(D)}(z, c) + d_{R^k(D)}(c, v) + d_{R^k(D)}(v, b_1) \leq k - 3 + 1 + 1 = k - 1$ with $b_1 \in \mathfrak{N}$. If $\delta_{R^k(D)}^+(w) = 0$, then $w \in \mathfrak{N}$, and $d_{R^k(D)}(z, w) = d_{R^k(D)}(z, c) + d_{R^k(D)}(c, v) + d_{R^k(D)}(v, w) \leq k - 3 + 1 + 1 = k - 1$. ■

Theorem 2.4. *For any digraph D and for any integer k $(k \geq 3)$, the digraph $T^k(D)$ has a k-kernel.*

Proof. Let k, D and $T^k(D)$ be as in the hypothesis. For each $a = (u, v) \in A(D)$ we define $\mathfrak{N}_a$ as follows: If $\delta_D^+(v) = 0$, then $\mathfrak{N}_a$ is the k-kernel of $(\beta_a - \{u\}) \cup \{v, a = (u, v)\}$, i.e., $\mathfrak{N}_a = \{a_i | 1 \leq i, i \equiv 0 (\operatorname{mod} k)\} \cup \{v\}$. If $\delta_D^+(v) > 0$, then $\mathfrak{N}_a = \{a_i | i \equiv 1(\mod k)\}$. We write $B^0 = \big\{x \in V(D) | \delta_D^+(x) = \delta_D^-(x) = 0\big\}$. We will prove that $\mathfrak{N} = \bigcup_{a \in A(D)} \mathfrak{N}_a \cup B^0$ is a kernel of $T^k(D)$. Observe that $V^0(D) \subseteq \mathfrak{N}$.

Observation 3. *Notice that since $k \geq 3$, we have $a_{n(a)k+1} \neq a = a_{n(a)k+(k-1)}$, therefore $a \notin \mathfrak{N}$, for each $a \in A(D)$.*

Claim 5. $\mathfrak{N}$ is a k-independent set of vertices of $T^k(D)$.
Let $x, y \in \mathfrak{N}$ with $x \neq y$. We will prove that $d_{T^k(D)}(x, y) \geq k$ and $d_{T^k(D)}(y, x) \geq k$. Observe that if $x \in B^0$, then $d_{T^k(D)}(x, y) = d_{T^k(D)}(y, x) = \infty$ for each $y \in V(T^k(D))$.

Case 1. There exists $c = (u, v) \in A(D)$ such that $\{x, y\} \subseteq \mathfrak{N}_c$.

Case 1.1. $\delta_D^+(v) = 0$. In this case, $\mathfrak{N}_c = \{c_i | 1 \leq i, i \equiv 0 (\operatorname{mod} k)\} \cup \{v\}$. Clearly, we may assume $x = c_{mk}$ with $1 \leq m \leq n(c)$ and $y = c_{tk}$ with $1 \leq t \leq n(c)$ and $m < t$ or $y = v$.

If $y = c_{tk}$, then $d_{T^k(D)}(x, y) = (t-m)k \geq k$. If $y = v$, then $d_{T^k(D)}(x, y) = d_{\beta_c}(x, c) + d_{T^k(D)}(c, v) \geq k - 1 + 1 = k$.

Now from the definition of $T^k(D)$, we have $d_{T^k(D)}(y, x) = d_{T^k(D)}(y, c) + d_{T^k(D)}(c, x)$.

If $y \neq v$, then $d_{T^k(D)}(y, c) = d_{\beta_c}(y, c) \geq k - 1$. From Observation 3 $c \neq x$, so $d_{T^k(D)}(c, x) \geq 1$ and we conclude that $d_{T^k(D)}(y, x) \geq k$.

If $y = v$, then $d_{T^k(D)}(y, x) = \infty$, as $\delta_D^+(v) = 0$.

Case 1.2. $\delta_D^+(v) > 0$. In this case, $\mathfrak{N}_c = \{c_i \in \beta_c \,|\, i \equiv 1 (\operatorname{mod} k)\}$ and clearly, we may assume $x = c_{mk+1}$, $y = c_{tk+1}$ with $0 \leq m < t \leq n(c)$. Therefore $d_{T^k(D)}(x, y) = (t-m)k \geq k$. Now from the definition of $T^k(D)$ we have $d_{T^k(D)}(y, x) = d_{T^k(D)}(y, c) + d_{T^k(D)}(c, x)$. Clearly, $d_{T^k(D)}(y, c) \geq k-2$.

Since $c \in A(D), c = (u, v)$ and $x \neq v$, we have $(c, x) \notin A(T^k(D))$ (recall the definition of $T^k(D)$).

Hence $d_{T^k(D)}(c, x) \geq 2$. We conclude that $d_{T^k(D)}(y, x) \geq k$.

Case 2. $x \in \mathfrak{N}_b$ and $y \in \mathfrak{N}_c$ for $b = (u, v)$, $c = (w, z)$ with $\{b, c\} \subseteq A(D)$, $b \neq c$. From the definition of $T^k(D)$ we have $d_{T^k(D)}(x, y) = d_{T^k(D)}(x, b) + d_{T^k(D)}(b, y)$.

Case 2.1. $\delta_D^+(v) = 0$. In this case, $x = b_{mk}$ with $1 \leq m \leq n(b)$ or $x = v$. If $x = b_{mk}$, then $d_{T^k(D)}(x, b) \geq k - 1$; and from Observation 3 $b \neq y$ which implies $d_{T^k(D)}(b, y) \geq 1$. We conclude that $d_{T^k(D)}(x, y) \geq k$. If $x = v$, then $d_{T^k(D)}(x, y) = \infty$ (as $\delta_D^+(v) = \delta_{T^k(D)}^+(v) = 0$).

Case 2.2. $\delta_D^+(v) > 0$. In this case, $\mathfrak{N}_b = \{b_i \in V(\beta_b) | i \equiv 1 (\operatorname{mod} k)\}$. From the definition of $T^k(D)$ we have $d_{T^k(D)}(x, y) = d_{T^k(D)}(x, b) + d_{T^k(D)}(b, y)$. Clearly, $d_{T^k(D)}(x, b) \geq k - 2$. Since $b \notin \mathfrak{N}$ (from Observation 3) and $y \in \mathfrak{N}$, then $y \neq b$. Moreover, $k \geq 3$ implies $n(b)k + 1 \neq n(b)k + (k - 1)$ and $y \neq v$. Finally, $d(b, y) = 1$ implies $y \in A(D)$ and by Observation 3 also $y \notin \mathfrak{N}$, a contradiction. Therefore $d_{T^k(D)}(b, y) \geq 2$. We conclude that $d_{T^k(D)}(x, y) \geq k$. Analogously, it can be proved that $d_{T^k(D)}(y, x) \geq k$.

Claim 6. $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $T^k(D)$.
Clearly, $R^k(D)$ is an spanning subdigraph of $T^k(D)$ and we have proved (Theorem 2.3) that $\mathfrak{N}$ is a k-kernel of $R^k(D)$, in particular $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $R^k(D)$. Thus $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $T^k(D)$. ∎

Observe that the set of black vertices in Figs. 1 and 2 is a 3-kernel.

Remark 2.1. It is easy to prove that for $D = \vec{C}_4$ (the directed cycle of length 4) and $k = 2$, the k-total digraph of D, $T^k(D)$ has no k-kernel. Thus the assertion given in Theorem 2.4 cannot be improved.

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