# $k$-KERNELS AND SOME OPERATIONS IN DIGRAPHS 

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#### Abstract

Let $D$ be a digraph. $V(D)$ denotes the set of vertices of $D$; a set $N \subseteq V(D)$ is said to be a $k$-kernel of $D$ if it satisfies the following two conditions: for every pair of different vertices $u, v \in N$ it holds that every directed path between them has length at least $k$ and for every vertex $x \in V(D)-N$ there is a vertex $y \in N$ such that there is an $x y$-directed path of length at most $k-1$.

In this paper, we consider some operations on digraphs and prove the existence of $k$-kernels in digraphs formed by these operations from another digraphs.


Keywords: $k$-kernel, $k$-subdivision digraph, $k$-middle digraph and $k$ total digraph.
2000 Mathematics Subject Classification: Primary: 05C20; Secondary: 05C69.

## 1. Introduction

We refer the reader to [1] for general concepts. In this paper, $D$ denotes a digraph; $V(D)$ is the set of vertices and $A(D)$ denotes the set of arcs.

A directed path is a sequence $P=\left(x_{0}, x_{1}, \ldots, x_{n}\right)$ of distinct vertices of $D$ such that $\left(x_{i}, x_{i+1}\right) \in A(D)$ for each $i, 0 \leq i \leq n-1$. The length of $P$ is $n$ and we denote $\ell(P)=n$. For $x, y \in V(D)$, the distance from $x$ to $y$ in $D$ is denoted as $d_{D}(x, y)$ and defined as: $d_{D}(x, y)=\min \{\ell(P) \mid P$ is an $x y$ - directed path $\}$ whenever there exists an $x y$-directed path in $D$, otherwise, we define $d_{D}(x, y)=\infty$. If $P$ is a directed path and $a, b \in V(P)$, then $(a, P, b)$ denotes the $a b$-directed path contained in $P$.

A set $N \subseteq V(D)$ is said to be $k$-independent whenever for any two different vertices $x, y \in N$ we have $d_{D}(x, y) \geq k$ and $d_{D}(y, x) \geq k . N$ is said to be ( $k-1$ )-absorbent whenever for each $x \in V(D)-N$ there exists $y \in N$ such that $d_{D}(x, y) \leq k-1$. The set $N$ is said to be a $k$-kernel if it is $k$-independent and ( $k-1$ )-absorbent.

We note that a 2 -kernel is a kernel of a digraph in the sense of J . von Neumann and O. Morgenstern [20]. The problem of the existence of a kernel in a digraph has been studied in $[2,3,4,7,17,18]$.

The existence of kernels of digraphs formed by some operations from another digraphs have been studied by several authors, namely: M. Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel [16]; M. Harminc and T. Olejníková [11]; J. Topp [19], H. Galeana-Sánchez and V. Neumann-Lara [7, 8].

The concept of $k$-kernel was introduced by M. Kwaśnik in [14]. Clearly, this concept generalizes the concept of a kernel of a digraph. It has been studied by several authors: M. Harminc [9], M. Kwaśnik [14, 15], M. Kucharska [12, 13], H. Galeana-Sánchez [5, 6], A. Włoch and I. Włoch [21].

In [10], M. Harminc constructed all kernels of the line digraph of $D$ from the kernels of $D$ and in [19] the author considered some special digraphs: $S(D) ; Q(D), T(D)$ and $L(D)$ which were called the subdivision digraph, the middle digraph, the total digraph and the line digraph of $D$, respectively and studied some neccessary or sufficient conditions for the existence or uniqueness of kernels of these digraphs.

In this paper, for a given digraph $D$ and any $k \geq 2$ we define: the $k$-subdivision $S^{k}(D)$, a generalization of the subdivision $S(D)$, the digraph $R^{k}(D)$, the $k$-middle digraph $Q^{k}(D)$ and the $k$-total digraph $T^{k}(D)$. Also the following results are proved: for any digraph $D$ and for any $k \geq 2$ the
digraphs $S^{k}(D), R^{k}(D)$ and $Q^{k}(D)$ have a $k$-kernel. For any digraph $D$ and for $k \geq 3$ the digraph $T^{k}(D)$ has a $k$-kernel.

## 2. $\quad k$-KERNELS In: $S^{k}(D), R^{k}(D), Q^{k}(D)$ And $T^{k}(D)$

Let $D$ be a digraph. The line digraph $L(D)$ of $D$ is the digraph defined as follows: $V(L(D))=A(D)$ and $(a=(u, v), b=(z, w)) \in A(L(D))$ if and only if $v=z[1]$.
[19]: For a given digraph $D$, the subdivision digraph $S(D)$ of $D$ is defined by: $V(S(D))=V(D) \cup A(D)$ and

$$
\Gamma^{+}(x)= \begin{cases}\{x\} \times \Gamma_{D}^{+}(x), & \text { whenever } x \in V(D), \\ \{v\}, & \text { whenever } x=(u, v) \in A(D) .\end{cases}
$$

Notice that for a vertex $x$ of the subdivision digraph of $D$ we have the following: If $x$ corresponds to a vertex of $D$, then $x$ is adjacent to the arcs which are incident from $x$ in $D$; and if $x$ corresponds to an arc of $D$, then $x$ is adjacent only to the terminal endpoint of $x$. Also notice that $S(D)$ is obtained from $D$ by changing each arc of $D$ for a directed path of length two.

Let $D$ be a digraph. We define the $k$-subdivision digraph of $D$, denoted $S^{k}(D)$, as follows:

$$
\begin{aligned}
S^{k}(D)= & S(D)-\{(u, a) \mid a \in A(D) \text { and } u \text { is the initial endpoint of } a\} \\
& \cup\left(\bigcup_{a \in A(D)} \beta_{a}\right)
\end{aligned}
$$

for each $a=(u, v) \in A(D), \beta_{a}=\left(a_{0}=u, a_{1}, \ldots, a_{n(a) k+k-1}=a=(u, v)\right)$ is a $u a$-directed path whose length is $\equiv k-1(\bmod k)(n(a) \in \mathbb{N})$ and the following two properties hold:
(i) $V\left(\beta_{a}\right) \cap V(S(D))=\{u, a\}$,
(ii) For any $a, b \in A(D)$ with $a \neq b$ we have $\left(V\left(\beta_{a}\right)-\{u\}\right) \cap V\left(\beta_{b}\right)=\emptyset$.

Notice that $S^{k}(D)$ is obtained from $D$ by substituting each arc of $D$ for a directed path whose length is $\equiv 0(\bmod k)$ (for an example see Figure 1$)$.

We write $V^{0}(D)=\left\{x \in V(D) \mid \delta_{D}^{+}(x)=0\right\}$.


Figure 1
Finally, we define the digraphs $R^{k}(D), Q^{k}(D)$ and $T^{k}(D)$ as follows $R^{k}(D)=S^{k}(D) \cup D, Q^{k}(D)=S^{k}(D) \cup L(D)$ and $T^{k}(D)=S^{k}(D) \cup D \cup L(D)$ (for an example see Figure 2).


Figure 2

Theorem 2.1. For any digraph $D$ and for any integer $k \quad(k \geq 2)$, the $k$-subdivision digraph $S^{k}(D)$ of $D$ has a $k$-kernel.

Proof. Let $D$ and $S^{k}(D)$ be digraphs as in the hypothesis. For each $a \in A(D)$ we denote $\mathfrak{N}_{a}=\left\{a_{i} \in V\left(\beta_{a}\right) \mid i \equiv 0(\bmod k)\right\}$. We will prove that $\mathfrak{N}=V^{0}(D) \cup \bigcup_{a \in A(D)} \mathfrak{N}_{a}$ is a $k$-kernel of $S^{k}(D)$. Observe that $V(D) \subseteq \mathfrak{N}$.

Claim 1. $\mathfrak{N}$ is a $k$-independent set of vertices of $S^{k}(D)$.
Let $x, y \in \mathfrak{N}, x \neq y$. We will prove $d_{S^{k}(D)}(x, y) \geq k$ and $d_{S^{k}(D)}(y, x) \geq k$.
Case 1. $x \in V^{0}(D)$ and $y \in V^{0}(D)$.
Since $\delta_{S^{k}(D)}^{+}(x)=\delta_{D}^{+}(x)=0$ and $\delta_{S^{k}(D)}^{+}(y)=\delta_{D}^{+}(y)=0$, it follows that $d_{S^{k}(D)}(x, y)=d_{S^{k}(D)}(y, x)=\infty$.

Case 2. $x \in V^{0}(D)$ and $y \in \bigcup_{a \in A(D)} \mathfrak{N}_{a}$.
Since $\delta_{S^{k}(D)}^{+}(x)=\delta_{D}^{+}(x)=0$, we have $d_{S^{k}(D)}(x, y)=\infty$. Let $c=(u, v) \in$ $A(D)$ such that $y \in \mathfrak{N}_{c}$. From the definition of $S^{k}(D)$ we have $d_{S^{k}(D)}(y, x)=$ $d_{S^{k}(D)}(y, c=(u, v))+d_{S^{k}(D)}(c, x)$. Now since $y=c_{i}$ with $i \equiv 0(\bmod k)$ and $\ell\left(\beta_{c}\right) \equiv k-1(\bmod k)$ it follows that $d_{S^{k}(D)}(y, c=(u, v))=d_{\beta_{c}}(y, c) \geq k-1$. Clearly, $d_{S^{k}(D)}(c, x) \geq 1$ (as $c \in A(D)$ and $x \in V^{0}(D) \subseteq V(D)$ ). Therefore $d_{S^{k}(D)}(y, x) \geq(k-1)+1=k$.

Case 3. $x \in \bigcup_{a \in A(D)} \mathfrak{N}_{a}$ and $y \in V^{0}(D)$.
Proceed exactly as in Case 2 interchanging $x$ with $y$.
Case 4. $x \in \bigcup_{a \in A(D)} \mathfrak{N}_{a}$ and $y \in \bigcup_{a \in A(D)} \mathfrak{N}_{a}$.
Case 4.1. There exists $c=(u, v) \in A(D)$ such that $\{x, y\} \subseteq \mathfrak{N}_{c}$. From the definition of $\mathfrak{N}_{c}$ we have $x=c_{m k}$ and $y=c_{t k}$ for some $0 \leq m \leq$ $n(c), 0 \leq t \leq n(c)$. Assume without loss of generality $t>m$.

From the definition of $S^{k}(D)$ and the fact $x \neq v\left(\right.$ as $\ell\left(\beta_{c}\right) \equiv k-1$ $(\bmod k)$ ) we have: $d_{S^{k}(D)}(x, y)=d_{\beta_{c}}(x, y)=(t-m) k \geq k$. On the other hand, we have $d_{S^{k}(D)}(y, x)=d_{S^{k}(D)}(y, c)+d_{S^{k}(D)}(c, v)+d_{S^{k}(D)}(v, x)$. Since $d_{S^{k}(D)}(y, c)=d_{\beta_{c}}(y, c) \geq k-1$ and $d_{S^{k}(D)}(c, v)=1$, we obtain $d_{S^{k}(D)}(y, x) \geq k$.

Observation 1. Observe that in this case we have the same inequalites when we are working in $Q^{k}(D)$, i.e., $d_{Q^{k}(D)}(y, x) \geq k$, because the definition of $Q^{k}(D)$ implies: $d_{Q^{k}(D)}(y, x)=d_{Q^{k}(D)}(y, c)+d_{Q^{k}(D)}(c, x)$. And clearly, $d_{Q^{k}(D)}(y, c) \geq k-1$ and $d_{Q^{k}(D)}(c, x) \geq 1$.

Case 4.2. $x \in \mathfrak{N}_{a}$ and $y \in \mathfrak{N}_{b}$ for some $a, b \in A(D)$ with $a \neq b$. Assume without loss of generality that $a=(u, v)$ and $b=(w, z)$.
$d_{S^{k}(D)}(x, y)=d_{S^{k}(D)}(x, a)+d_{S^{k}(D)}(a, v)+d_{S^{k}(D)}(v, y)$. From the definition of $\mathfrak{N}_{a}$ we have $d_{S^{k}(D)}(x, a) \geq k-1$ and from the definition of $S^{k}(D)$, $d_{S^{k}(D)}(a, v)=1$. Therefore $d_{S^{k}(D)}(x, y) \geq k$.

Observation 2. Notice that in this case we have $d_{Q^{k}(D)}(x, a)=d_{S^{k}(D)}(x, a)$ and $d_{Q^{k}(D)}(a, y) \geq 1($ as $a \neq y)$. Thus $d_{Q^{k}(D)}(x, y)=d_{Q^{k}(D)}(x, a)+$ $d_{Q^{k}(D)}(a, y) \geq k$.

Interchanging $x$ with $y$ we obtain $d_{S^{k}(D)}(y, x) \geq k$.
Claim 2. $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $S^{k}(D)$.
Let $x \in V\left(S^{k}(D)-\mathfrak{N}\right)$. We will prove that there exists $y \in \mathfrak{N}$ such that $d_{S^{k}(D)}(x, y) \leq k-1$. Since $V^{0} \subseteq \mathfrak{N}$, it follows from the definition of $S^{k}(D)$ and the fact $x \in V\left(S^{k}(D)-\mathfrak{N}\right)$ that $x \in \bigcup_{a \in A(D)} \beta_{a}$. Let $c=(u, v) \in A(D)$ be such that $x \in \beta_{c}$.

Case 1. $x \in \beta_{c}-\left\{c_{i} \mid n(c) k+1 \leq i \leq n(c) k+(k-1)\right\}$.
Since $x \notin \mathfrak{N}$ (and then $x \notin \mathfrak{N}_{c}$ ), it follows that $x=c_{m k+j}$ for some $m$ and $j$ with $0 \leq m \leq n(c)$ and $1 \leq j \leq k-1$. From the definition of $S^{k}(D)$ we have $d_{S^{k}(D)}\left(x, c_{(m+1) k}\right)=d_{\beta_{c}}\left(c_{m k+j},\left(c_{(m+1) k}\right)=k-j \leq k-1\right.$. Clearly, $c_{(m+1) k} \in \mathfrak{N}$.

Case 2. $x \in\left\{c_{i} \mid n(c) k+1 \leq i \leq n(c) k+(k-1)=(u, v)=c\right\}$.
Clearly, $d_{S^{k}(D)}(x, v)=d_{S^{k}(D)}(x, c)+d_{S^{k}(D)}(c, v) ; d_{S^{k}(D)}(x, c) \leq k-2$ and $d_{S^{k}(D)}(c, v)=1$. Thus $d_{S^{k}(D)}(x, v) \leq k-1$ with $v \in \mathfrak{N}($ recall $V(D) \subseteq \mathfrak{N})$.

Theorem 2.2. For any digraph $D$ and for any integer $k(k \geq 2)$, the $k$ middle digraph $Q^{k}(D)$ of $D$ has a $k$-kernel.

Proof. Consider the set $\mathfrak{N} \subseteq V\left(S^{k}(D)\right)=V\left(Q^{k}(D)\right)$ defined in the proof of Theorem 2.1. Since $S^{k}(D)$ is a spanning subdigraph of $Q^{k}(D)$ and $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $S^{k}(D)$, it follows that $\mathfrak{N}$ is a $(k-1)$ absorbent set of vertices of $Q^{k}(D)$.

The proof that $\mathfrak{N}$ is $k$-independent in $Q^{k}(D)$ is the same as the proof that $\mathfrak{N}$ is $k$-independent in $S^{k}(D)$, we only need to recall Observations 1 and 2 given along this proof.

Theorem 2.3. Let $D$ be any digraph and for any integer $k(k \geq 2)$, then the digraph $R^{k}(D)$ has a $k$-kernel.

Proof. Let $D, k$ and $R^{k}(D)$ be as in the hypothesis. For each $a=(u, v) \in$ $A(D)$ we define $\mathfrak{N}_{a}$ as follows: $\mathfrak{N}_{a}$ is the unique $k$-kernel of $\left(\beta_{a}-\{u\}\right) \cup\{(a=$ $(u, v), v)\}$ whenever $\delta_{D}^{+}(v)=0$. And $\mathfrak{N}_{a}=\left\{a_{i} \in V\left(\beta_{a}\right) \mid i \equiv 1(\bmod k)\right\}$ whenever $\delta_{D}^{+}(v)>0$. We write $B^{0}=\left\{x \in V(D) \mid \delta_{D}^{+}(x)=\delta_{D}^{-}(x)=0\right\}$. We will prove that $\mathfrak{N}=\bigcup_{a \in A(D)} \mathfrak{N}_{a} \cup B^{0}$ is a $k$-kernel of $R^{k}(D)$. First, observe that $V^{0}(D) \subseteq \mathfrak{N}$.

Claim 3. $\mathfrak{N}$ is a $k$-independent set of $R^{k}(D)$.
Let $x, y \in \mathfrak{N}$ with $x \neq y$. We will prove that $d_{R^{k}(D)}(x, y) \geq k$ and $d_{R^{k}(D)}(y, x) \geq k$. Observe that if $x \in B^{0}$, then $d_{R^{k}(D)}(x, y)=d_{R^{k}(D)}(y, x)=$ $\infty \quad \forall y \in V\left(R^{k}(D)\right)$.

Case 1. There exists $c=(u, v) \in A(D)$ such that $\{x, y\} \subseteq \mathfrak{N}_{c}$.
Case 1.1. $\delta_{D}^{+}(v)=0$. In this case, we have $\mathfrak{N}_{c}=\left\{c_{i} \in V\left(\beta_{c}\right) \mid i \equiv 0\right.$ $(\bmod k), i>0\} \cup\{v\}$.

We assume without loss of generality that $x=c_{m k}$ with $1 \leq m \leq n(c)$ and, $y=c_{t k}$ with $m<t$ or $y=v$.

When $y=c_{t k}$, we have $d_{R^{k}(D)}(x, y)=(t-m) k \geq k$. When $y=v$, we have $d_{R^{k}(D)}(x, v)=d_{R^{k}(D)}(x, c)+d_{R^{k}(D)}(c, v)$. Since $d_{R^{k}(D)}(x, c) \geq k-1$ and $d_{R^{k}(D)}(c, v)=1$, we conclude $d_{R^{k}(D)}(x, v=y) \geq k$.

Now, from the definition of $R^{k}(D)$ we have: $d_{R^{k}(D)}(y, x)=d_{R^{k}(D)}(y, v)+$ $d_{R^{k}(D)}(v, x)$. Since $\delta_{D}^{+}(v)=0$ we have $d_{R^{k}(D)}(v, x)=\infty$. Thus $d_{R^{k}(D)}(y, x)$ $\geq k$.

Case 1.2. $\delta_{D}^{+}(v)>0$. In this case we have $\mathfrak{N}_{c}=\left\{c_{i} \in V\left(\beta_{c}\right) \mid i \equiv 1\right.$ $(\bmod k)\}$. We assume without loss of generality that $x=c_{m k+1}, y=c_{t k+1}$ with $0 \leq m<t$. Clearly, $d_{R^{k}(D)}(x, y)=(t-m) k \geq k$ and $d_{R^{k}(D)}(y, x)=$ $d_{R^{k}(D)}(y, c)+d_{R^{k}(D)}(c, v)+d_{R^{k}(D)}(v, x)$. From the definition of $R^{k}(D)$ we have $d_{R^{k}(D)}(y, c) \geq k-2, d_{R^{k}(D)}(c, v)=1$ and $d_{R^{k}(D)}(v, x) \geq 1$ (because $v \neq x$, be as $m<t)$. Thus $d_{R^{k}(D)}(y, x) \geq k$.

Case 2. $x \in \mathfrak{N}_{b}$ and $y \in \mathfrak{N}_{c}$ with $b=(u, v), c=(w, z), b \neq c$.
From the definition of $R^{k}(D)$ we have $d_{R^{k}(D)}(x, y)=d_{R^{k}(D)}(x, b=(u, v))+$ $d_{R^{k}(D)}(b, v)+d_{R^{k}(D)}(v, w)+d_{R^{k}(D)}(w, y)$. When $\delta_{D}^{+}(v)=0$, we obtain $d_{R^{k}(D)}(v, w)=\infty$ and then $d_{R^{k}(D)}(x, y) \geq k$.

When $\delta_{D}^{+}(v)>0$, we obtain $\mathfrak{N}_{b}=\left\{b_{i} \in V\left(\beta_{b}\right) \mid i \equiv 1(\bmod k)\right\}$ and $d_{R^{k}(D)}(x, b)$ $\geq k-2$ also from the definition of $R^{k}(D), d_{R^{k}(D)}(b, v)=1$. If $v \neq w$, then $d_{R^{k}(D)}(v, w) \geq 1$ and we conclude that $d_{R^{k}(D)}(x, y) \geq k$. If $v=w$, then $\delta_{D}^{+}(w)>0, w \notin \mathfrak{N}_{c}$ and $w \neq y$; therefore $d_{R^{k}(D)}(w, y) \geq 1$, and we conclude again that $d_{R^{k}(D)}(x, y) \geq k$.

Analogously, it can be proved $d_{R^{k}(D)}(y, x) \geq k$.
Claim 4. $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $R^{k}(D)$.
We will prove that for any $z \in V\left(R^{k}(D)-\mathfrak{N}\right)$ there exists $w \in \mathfrak{N}$ such that $d_{R^{k}(D)}(z, w) \leq k-1$.

Let $z \in V\left(R^{k}(D)-\mathfrak{N}\right)$. We have observed that $V^{0}(D) \subseteq \mathfrak{N}$. Thus $z \in \bigcup_{a \in A(D)} V\left(\beta_{a}\right)$. Take $c=(u, v) \in A(D)$ such that $z \in V\left(\beta_{c}\right)$.

Case 1. $\delta_{D}^{+}(v)=0$. In this case, $\mathfrak{N}_{c}=\left\{c_{i} \in V\left(\beta_{c}\right) \mid i \equiv 0(\bmod k), i \geq 1\right\}$ $\cup\{v\}$. Since $z \notin \mathfrak{N}$, then $z=c_{0}$ or $z=c_{m k+j}$ with $1 \leq j \leq k-1$ and $0 \leq m \leq n(c)$.

If $z=c_{0}=u$, then from the definition of $R^{k}(D)$ we have $(z=u, v) \in$ $A\left(R^{k}(D)\right)$ and $d_{R^{k}(D)}(z, v)=1 \leq k-1$ with $v \in \mathfrak{N}$. If $z=c_{m k+j}$, then $d_{R^{k}(D)}\left(c_{m k+j}, c_{(m+1) k}\right)=k-j \leq k-1$ whenever $m \neq n(c)$, and $d_{R^{k}(D)}(z, v) \leq d_{R^{k}(D)}(z, c=(u, v))+d_{R^{k}(D)}(c=(u, v), v) \leq k-2+1=k-1$ whenever $m=n(c)$ (recall that $z=c_{m k+j}, c=c_{n(k)+(k-1)}$ and $d_{R^{k}(D)}(c=$ $(u, v), v)=1)$.

Case 2. $\delta_{D}^{+}(v)>0$. In this case, $\mathfrak{N}_{c}=\left\{c_{i} \in V\left(\beta_{c}\right) \mid i \equiv 1(\bmod k)\right\}$. When $z \in V\left(\beta_{c}\right)-\left\{c_{i} \mid n(c) k+2 \leq i \leq n(c) k+(k-1)\right\}$, we have two possibilities: If $z=c_{0}$, then $d_{R^{k}(D)}\left(z, c_{1}\right)=1 \leq k-1$ with $c_{1} \in \mathfrak{N}_{c} \subseteq \mathfrak{N}$. If $z \neq c_{0}$, then $z=c_{m k+j}$ with $2 \leq j \leq k, 0 \leq m<n(c)$ and $d_{R^{k}(D)}\left(z, c_{(m+1) k+1}\right) \leq$ $k-1$ with $c_{(m+1) k+1} \in \mathfrak{N}$.

When $z \in\left\{c_{i} \mid n(c) k+2 \leq i \leq n(c) k+(k-1)\right\}$, we recall that $\delta_{D}^{+}(v)>0$. Thus there exists $b=(v, w) \in A(D)$. We consider $\beta_{b}$. Consider two possibilities: If $\delta_{R^{k}(D)}^{+}(w)>0$, then $\mathfrak{N}_{b}=\left\{b_{i} \in V\left(\beta_{b}\right) \mid i \equiv 1(\bmod k)\right\}$; and it follows that $d_{R^{k}(D)}\left(z, b_{1}\right)=d_{R^{k}(D)}(z, c)+d_{R^{k}(D)}(c, v)+d_{R^{k}(D)}\left(v, b_{1}\right) \leq$ $k-3+1+1=k-1$ with $b_{1} \in \mathfrak{N}$. If $\delta_{R^{k}(D)}^{+}(w)=0$, then $w \in \mathfrak{N}$, and $d_{R^{k}(D)}(z, w)=d_{R^{k}(D)}(z, c)+d_{R^{k}(D)}(c, v)+d_{R^{k}(D)}(v, w) \leq k-3+1+1=k-1$.

Theorem 2.4. For any digraph $D$ and for any integer $k(k \geq 3)$, the digraph $T^{k}(D)$ has a $k$-kernel.

Proof. Let $k, D$ and $T^{k}(D)$ be as in the hypothesis. For each $a=(u, v) \in$ $A(D)$ we define $\mathfrak{N}_{a}$ as follows: If $\delta_{D}^{+}(v)=0$, then $\mathfrak{N}_{a}$ is the $k$-kernel of $\left(\beta_{a}-\{u\}\right) \cup\{v, a=(u, v)\}$, i.e., $\mathfrak{N}_{a}=\left\{a_{i} \mid 1 \leq i, i \equiv 0(\bmod k)\right\} \cup$ $\{v\}$. If $\delta_{D}^{+}(v)>0$, then $\mathfrak{N}_{a}=\left\{a_{i} \mid i \equiv 1(\bmod k)\right\}$. We write $B^{0}=$ $\left\{x \in V(D) \mid \delta_{D}^{+}(x)=\delta_{D}^{-}(x)=0\right\}$. We will prove that $\mathfrak{N}=\bigcup_{a \in A(D)} \mathfrak{N}_{a} \cup B^{0}$ is a kernel of $T^{k}(D)$. Observe that $V^{0}(D) \subseteq \mathfrak{N}$.

Observation 3. Notice that since $k \geq 3$, we have $a_{n(a) k+1} \neq a=a_{n(a) k+(k-1)}$, therefore $a \notin \mathfrak{N}$, for each $a \in A(D)$.

Claim 5. $\mathfrak{N}$ is a $k$-independent set of vertices of $T^{k}(D)$.
Let $x, y \in \mathfrak{N}$ with $x \neq y$. We will prove that $d_{T^{k}(D)}(x, y) \geq k$ and $d_{T^{k}(D)}(y, x) \geq k$. Observe that if $x \in B^{0}$, then $d_{T^{k}(D)}(x, y)=d_{T^{k}(D)}(y, x)$ $=\infty$ for each $y \in V\left(T^{k}(D)\right)$.

Case 1. There exists $c=(u, v) \in A(D)$ such that $\{x, y\} \subseteq \mathfrak{N}_{c}$.
Case 1.1. $\delta_{D}^{+}(v)=0$. In this case, $\mathfrak{N}_{c}=\left\{c_{i} \mid 1 \leq i, i \equiv 0(\bmod k)\right\} \cup\{v\}$. Clearly, we may assume $x=c_{m k}$ with $1 \leq m \leq n(c)$ and $y=c_{t k}$ with $1 \leq t \leq n(c)$ and $m<t$ or $y=v$.

If $y=c_{t k}$, then $d_{T^{k}(D)}(x, y)=(t-m) k \geq k$. If $y=v$, then $d_{T^{k}(D)}(x, y)=$ $d_{\beta_{c}}(x, c)+d_{T^{k}(D)}(c, v) \geq k-1+1=k$.

Now from the definition of $T^{k}(D)$, we have $d_{T^{k}(D)}(y, x)=d_{T^{k}(D)}(y, c)+$ $d_{T^{k}(D)}(c, x)$.

If $y \neq v$, then $d_{T^{k}(D)}(y, c)=d_{\beta_{c}}(y, c) \geq k-1$. From Observation 3 $c \neq x$, so $d_{T^{k}(D)}(c, x) \geq 1$ and we conclude that $d_{T^{k}(D)}(y, x) \geq k$.

If $y=v$, then $d_{T^{k}(D)}(y, x)=\infty$, as $\delta_{D}^{+}(v)=0$.
Case 1.2. $\delta_{D}^{+}(v)>0$. In this case, $\mathfrak{N}_{c}=\left\{c_{i} \in \beta_{c} \mid i \equiv 1(\bmod k)\right\}$ and clearly, we may assume $x=c_{m k+1}, y=c_{t k+1}$ with $0 \leq m<t \leq n(c)$. Therefore $d_{T^{k}(D)}(x, y)=(t-m) k \geq k$. Now from the definition of $T^{k}(D)$ we have $d_{T^{k}(D)}(y, x)=d_{T^{k}(D)}(y, c)+d_{T^{k}(D)}(c, x)$. Clearly, $d_{T^{k}(D)}(y, c) \geq k-2$.

Since $c \in A(D), c=(u, v)$ and $x \neq v$, we have $(c, x) \notin A\left(T^{k}(D)\right)$ (recall the definition of $\left.T^{k}(D)\right)$.

Hence $d_{T^{k}(D)}(c, x) \geq 2$. We conclude that $d_{T^{k}(D)}(y, x) \geq k$.
Case 2. $x \in \mathfrak{N}_{b}$ and $y \in \mathfrak{N}_{c}$ for $b=(u, v), c=(w, z)$ with $\{b, c\} \subseteq A(D)$, $b \neq c$. From the definition of $T^{k}(D)$ we have $d_{T^{k}(D)}(x, y)=d_{T^{k}(D)}(x, b)+$ $d_{T^{k}(D)}(b, y)$.

Case 2.1. $\delta_{D}^{+}(v)=0$. In this case, $x=b_{m k}$ with $1 \leq m \leq n(b)$ or $x=v$. If $x=b_{m k}$, then $d_{T^{k}(D)}(x, b) \geq k-1$; and from Observation $3 b \neq y$ which implies $d_{T^{k}(D)}(b, y) \geq 1$. We conclude that $d_{T^{k}(D)}(x, y) \geq k$. If $x=v$, then $d_{T^{k}(D)}(x, y)=\infty\left(\right.$ as $\left.\delta_{D}^{+}(v)=\delta_{T^{k}(D)}^{+}(v)=0\right)$.

Case 2.2. $\delta_{D}^{+}(v)>0$. In this case, $\mathfrak{N}_{b}=\left\{b_{i} \in V\left(\beta_{b}\right) \mid i \equiv 1(\bmod k)\right\}$. From the definition of $T^{k}(D)$ we have $d_{T^{k}(D)}(x, y)=d_{T^{k}(D)}(x, b)+$ $d_{T^{k}(D)}(b, y)$. Clearly, $d_{T^{k}(D)}(x, b) \geq k-2$. Since $b \notin \mathfrak{N}$ (from Observation 3 ) and $y \in \mathfrak{N}$, then $y \neq b$. Moreover, $k \geq 3$ implies $n(b) k+1 \neq n(b) k+(k-1)$ and $y \neq v$. Finally, $d(b, y)=1$ implies $y \in A(D)$ and by Observation 3 also $y \notin \mathfrak{N}$, a contradiction. Therefore $d_{T^{k}(D)}(b, y) \geq 2$. We conclude that $d_{T^{k}(D)}(x, y) \geq k$. Analogously, it can be proved that $d_{T^{k}(D)}(y, x) \geq k$.

Claim 6. $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $T^{k}(D)$.
Clearly, $R^{k}(D)$ is an spanning subdigraph of $T^{k}(D)$ and we have proved (Theorem 2.3) that $\mathfrak{N}$ is a $k$-kernel of $R^{k}(D)$, in particular $\mathfrak{N}$ is a $(k-1)$ absorbent set of vertices of $R^{k}(D)$. Thus $\mathfrak{N}$ is a $(k-1)$-absorbent set of vertices of $T^{k}(D)$.
Observe that the set of black vertices in Figs. 1 and 2 is a 3 -kernel.
Remark 2.1. It is easy to prove that for $D=\vec{C}_{4}$ (the directed cycle of length 4) and $k=2$, the $k$-total digraph of $D, T^{k}(D)$ has no $k$-kernel. Thus the assertion given in Theorem 2.4 cannot be improved.

## Acknowledgement

We acknowledge and thank the referees for a thorough review and their numerous useful suggestions which improved substantially the rewriting of this paper.

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