Discussiones Mathematicae
Graph Theory 29 (2009) 5–14

STRONGLY PANCYCLIC AND DUAL-PANCYCLIC GRAPHS

Terry A. McKee

Department of Mathematics & Statistics
Wright State University
Dayton, Ohio 45435, USA

Abstract

Say that a cycle C *almost contains* a cycle C^- if every edge except one of C^- is an edge of C. Call a graph G *strongly pancyclic* if every nontriangular cycle C almost contains another cycle C^- and every nonspanning cycle C is almost contained in another cycle C^+. This is equivalent to requiring, in addition, that the sizes of C^- and C^+ differ by one from the size of C. Strongly pancyclic graphs are pancyclic and chordal, and their cycles enjoy certain interpolation and extrapolation properties with respect to almost containment. Much of this carries over from graphic to cographic matroids; the resulting 'dual-pancyclic' graphs are shown to be exactly the 3-regular dual-chordal graphs.

Keywords: pancyclic graph, cycle extendable, chordal graph, pancyclic matroid, dual-chordal graph.

2000 Mathematics Subject Classification: 05C38 (05C62, 05C45).

1. Introduction and Strongly Pancyclic Graphs

Viewing cycles as sets of edges, no cycle can be contained in another cycle. But a cycle C can be said to be *almost contained in* a cycle C^+ (and C^+ to *almost contain* C) if every edge except one of C is an edge of C^+—in other words, if $|C - C^+| = 1$, where, as with all cycles in this paper, C is identified with the edge set $E(C)$. (Almost containment of general sets has been defined analogously.) For instance, if G is the 'house graph' (formed by inserting one chord into a length-5 cycle), then both the resulting length-3 cycle and length-4 cycle are almost contained in the length-5 cycle, but the length-3 cycle is not almost contained in the length-4 cycle. (Throughout this paper, 'graphs' will always be simple graphs, and 'multigraphs' will allow multiple edges and loops.)

A graph G is a *pancyclic graph* if it has cycles of every length between 3 and $|V(G)|$, inclusive. Define G to be a *strongly pancyclic graph* if G is 2-connected and both the following hold:

(0.1) Every nontriangular cycle almost contains a cycle.

(0.2) Every nonhamiltonian cycle is almost contained in a cycle.

The graphs in Figure 1 are not strongly pancyclic (although G_1 and G_3 are pancyclic). Graphs G_1 and G_2 satisfy condition (0.1), but not (0.2) because of the cycle a, b, c, d, a. Graph G_3 satisfies (0.2), but not (0.1) because of the cycle a, b, c, d, a. Inserting an additional edge into either G_1 or G_3 will produce a strongly pancyclic graph.

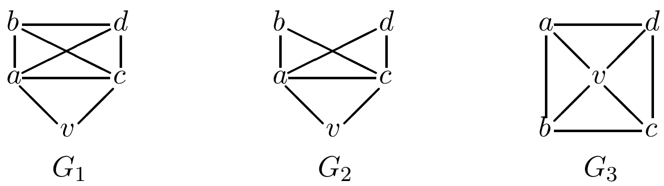

Figure 1. Three graphs that are not strongly pancyclic.

Theorem 1 (and Corollaries 2 and 6) will justify the name 'strongly pancyclic' by characterizing strongly pancyclic graphs by every cycle almost containing and being almost contained in cycles of every possible length. The symbol $\oplus$ denotes the symmetric difference of cycles, the familiar 'ring sum' used with cycle spaces. Recall that a graph is 2-*connected* if and only if every two edges are in a common cycle.

Theorem 1. *A* 2*-connected graph is strongly pancyclic if and only if it satisfies both of the following*:

(1.1) *Every nontriangular cycle almost contains a cycle of length one less.*

(1.2) *Every nonhamiltonian cycle is almost contained in a cycle of length one more.*

Proof. Suppose a graph G is strongly pancyclic with a nontriangular cycle C. We will show condition (1.1) by showing a stronger property by induction on $|C| \geq 4$: *For each $e \in C$, cycle C will almost contain a cycle*

C_e *with* $|C_e| = |C| - 1$ *and* $e \in C_e$. Suppose $e \in C$. In the $|C| = 4$ basis case, condition (0.1) implies that C almost contains a triangle Δ where $C \oplus \Delta = \Delta'$ is also a triangle; take C_e to be whichever of Δ and Δ' contains e. When $|C| \geq 5$, (0.1) implies that C almost almost contains a cycle C' with $|C'| < |C|$. Let f be the chord of C in $C' - C$, noting that C also almost contains the cycle $C \oplus C' = C''$ and that $f \in C''$. Edge e is in one of C' and C''; without loss of generality, say $e \in C'$. If C'' is a triangle, take $C_e = C'$. If C'' is not a triangle, the inductive hypothesis implies that C'' almost contains a cycle C''_f with $|C''_f| = |C''| - 1$ and $f \in C''_f$; take $C_e = C' \oplus C''_f$.

Now suppose instead that C is any nonhamiltonian cycle of G. By (0.2), C is almost contained in another cycle C^+; say $\{f\} = C - C^+$. Let $C' = C \oplus C^+$, noting that $f \in C'$. Then $|C'| - 3 \geq 0$ applications of the result proved in the preceding paragraph shows that C' will almost contain a triangle C'_f with $f \in C'_f$, and $C \oplus C'_f$ will be the cycle required in (1.2).

The converse follows from each condition (1.i) immediately implying condition (0.i). ■

Corollary 2. *A* 2*-connected graph* G *is strongly pancyclic if and only if, for every cycle* C*, there exist cycles* $C_3, \ldots, C_{|V(G)|}$ *such that each* $|C_i| = i$, $C = C_{|C|}$*, and if* $i < j$*, then* C_i *is almost contained in* C_j*.* ■

The 'domino graph' (formed by inserting one chord into a length-6 cycle so as to form two almost-contained length-4 cycles) shows that condition (0.2) is not by itself equivalent to condition (1.2). Yet (0.1) is equivalent to (1.1), since they are both easily seen to be equivalent to every cycle of length at least four having a chord; this characterizes a graph being *chordal* [8]. (The graphs G_1 and G_2 in Figure 1 are the only chordal graphs on five or fewer vertices that are not strongly pancyclic.) Somewhat similarly, condition (1.2) is a direct translation of a graph being 0*-chord extendable* [3] (called *cycle* 0*-extendable* in [4]). These equivalences together with Theorem 1 imply both Corollary 3 and Corollary 2.

Corollary 3. *A* 2*-connected graph is strongly pancyclic if and only if it is both chordal and* 0*-chord extendable.* ■

Define 2*-trees* recursively, starting from K_2 being a 2-tree, as follows: If G is any 2-tree with $e \in E(G)$ and if Δ is a triangle that is vertex disjoint from G with $e' \in E(\Delta)$, then the graph formed from G and Δ by identifying edges e and e' (along with their endpoints) is another 2-tree. Corollary 4 is related to [2, Theorem 5]: In a hamiltonian chordal graph G, every hamiltonian cycle of G is contained in a 2-tree subgraph of G. It is important that the spanning hamiltonian 2-tree (and the spanning maximal outerplanar subgraph) mentioned in Corollary 4 does not need to be an induced subgraph (just as a spanning tree or spanning cycle does not need to be an induced subgraph).

Corollary 4. *The following are equivalent for every* 2*-connected graph* G:

(4.1) *G is strongly pancyclic.*

(4.2) *Every cycle of G is contained in a subgraph of G that is a hamiltonian* 2*-tree.*

(4.3) *Every cycle of G is contained in a subgraph of G that is maximal outerplanar.*

Proof. First suppose G is 2-connected and every cycle C is contained in a hamiltonian 2-tree subgraph T_C of G. To show condition (0.1), suppose cycle C is nontriangular. Then C will have a chord $e \in E(T_C)$ by [2, Thm 1], and $E(C) \cup \{e\}$ will contain two cycles, each of which is almost contained in C. To show condition (0.2), suppose cycle C is nonhamiltonian. Repeatedly removing from G vertices of degree two in T_C that are not in C will eventually leave a smallest hamiltonian 2-tree T'_C such that $V(C) \subset V(T'_C)$, $E(T'_C) \subseteq E(T_C)$, and $|V(T'_C)| = |C| + 1$. The hamiltonian cycle that spans T'_C will almost contain C.

The converse follows immediately from Corollary 2, with the observations that each $C_i \oplus C_{i+1}$ is a triangle that has a unique edge in common with C_i and that each C_i is the sum of the first $i-2$ of those triangles.

The equivalence of hamiltonian 2-trees and maximal outerplanar subgraphs is [5, Theorem 7]. ∎

Notice that the 'hamiltonian 2-tree' in Corollary 4 can be equivalently replaced with 'outerplanar 2-tree,' 'G_2-free 2-tree' (with G_2 as in Figure 1), or '2-connected outerplanar subgraph.' Because recognizing whether a graph has a maximal outerplanar subgraph is NP-hard [10], Corollary 4 shows that *recognizing whether a graph is strongly pancyclic is NP-hard.*

2. Interpolation and Strongly Pancyclic Matroids

Corollaries 2 and 4 can be viewed as guaranteeing chains of cycles, each almost containing or contained in the next, that extrapolate both upward and downward from any cycle. Corollaries 6 and 7 will be their interpolation analogs.

Lemma 5. *Suppose G is a chordal graph that contains a cycle C of length $n \leq |V(G)|$ and a triangle Δ that consists of chords $e_1, \ldots, e_k$ of C ($0 \leq k \leq 3$) together with $3-k$ edges of C. Then there exist triangles $\Delta_3^C, \ldots, \Delta_n^C$ such that each Δ_{i+1}^C has a unique edge in common with the length-i cycle $C_i = \Delta_3^C \oplus \cdots \oplus \Delta_i^C$, $3 \leq i \leq n$, where if $i < j$, then C_i is almost contained in C_j, $\Delta = \Delta_3^C$, and $C = C_{|C|}$.*

Proof. Suppose G is a chordal graph that contains a cycle C of length $n \leq |V(G)|$ and a triangle Δ that consists of chords $e_1, \ldots, e_k$ of C ($0 \leq k \leq 3$) together with $3-k$ edges of C. Argue by induction on n, with the $n = 3$ ($k = 0$) basis case immediate. Suppose $n \geq 4$. For each $1 \leq l \leq k$, let $C(l)$ be the cycle that has $C(l) \subset C \cup \{e_l\}$ where e_j is not a chord of $C(l)$ whenever $j \neq l$. Let n_l be the length of $C(l)$, so that $\sum_l n_l = n + 2k - 3$. Since G is chordal, let each Δ_l be any triangle that contains e_l and two additional edges or chords of $C(l)$.

The induction hypothesis on each $C(l)$ implies the existence of triangles $\Delta_3^{C(l)}, \ldots, \Delta_{n_l}^{C(l)}$ such that each $\Delta_{i+1}^{C(l)}$ has a unique edge in common with the length-i cycle $C_i^l = \Delta_3^{C(l)} \oplus \cdots \oplus \Delta_i^{C(l)}$, $3 \leq i \leq n_l$, where if $i < j$, then C_i^l is almost contained in C_j^l, $\Delta_l = \Delta_3^{C(l)}$, and $C(l) = C_{n_l}^l$. Then $\Delta, \Delta_1^{C(1)}, \ldots, \Delta_{n_1}^{C(1)}$, followed by $\Delta_1^{C(2)}, \ldots, \Delta_{n_2}^{C(2)}$ if $k \geq 2$, followed by $\Delta_1^{C(3)}, \ldots, \Delta_{n_3}^{C(3)}$ if $k = 3$, can be taken to be the $n-2$ triangles $\Delta_3^C, \ldots, \Delta_n^C$ required in the lemma. ■

Corollary 6. *If G is a strongly pancyclic graph with triangle Δ and hamiltonian cycle H, then G has cycles $C_3 = \Delta, \ldots, C_{|V(G)|} = H$ such that each $|C_i| = i$ and if $i < j$, then C_i is almost contained in C_j.*

Proof. Suppose G, H, and Δ are as in the statement of the corollary, with $n = |H| = |V(G)|$. Apply Lemma 5 with $C = H$ to get H as the sum of triangles $\Delta_3^C, \ldots, \Delta_n^C$. Whenever $3 \leq i \leq |V(G)|$, set C_i equal to the sum of the first $i-2$ of those triangles. ■

Corollary 7. *Every strongly pancyclic graph G with triangle Δ and hamiltonian cycle H has a hamiltonian 2-tree—or, equivalently, a maximal outerplanar subgraph—that contains both Δ and H.*

Proof. This follows from the proof of Corollary 6, taking the graph formed by $C_3 \cup \cdots \cup C_{|V(G)|}$ to be the hamiltonian 2-tree (maximal outerplanar subgraph). ■

The graph G_1 in Figure 1 is a counterexample to the converses of Corollaries 6 and 7. Also, there would be analogous interpolation results between any triangle Δ and any cycle C with $V(\Delta) \subset V(C)$. But Figure 2 with cycles $C = a, b, f, e, d, a$ and $H = a, b, c, d, e, f, a$ shows that there would not be a analogous interpolation results between any cycle C and any hamiltonian cycle H.

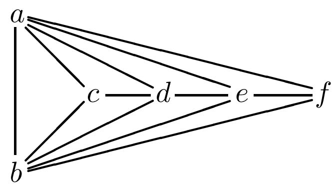

Figure 2. A strongly pancyclic graph in which the interpolation result fails.

There are results for 2-connected simple binary matroids that are analogous to Theorem 1 and the extrapolation in Corollary 2, replacing 'cycle' with 'circuit,' 'length' with 'size,' and 'edge' with 'element.' A matroid M is *simple* if every circuit has size three or more and is 2-connected if every two elements are in a common circuit. A *triangular circuit* of M is a circuit of size three, and a *hamiltonian circuit* of M is a circuit of size $\mathrm{rank}(M)+1$. A *pancyclic matroid*—see [1]—is a simple matroid M that has circuits of every size between 3 and $\mathrm{rank}(M)+1$. Define a *strongly pancyclic matroid* to be a simple, 2-connected matroid M that satisfies conditions (0.1) and (0.2) with 'cycle' replaced with 'circuit.' An element e is a *chord* of a circuit C if there are circuits C_1 and C_2, each almost contained in C, such that $C_1 \cap C_2 = \{e\}$ and $C_1 \oplus C_2 = C$. (See [9] for general matroidal definitions.)

In contrast, there is not an interpolation result for 2-connected simple binary matroids that is analogous to Corollary 6 (the Fano matroid F_7 is a counterexample); indeed, not even for 2-connected simple regular matroids (the non-pancyclic matroid R_{10} is a counterexample). Yet, in addition to all graphic matroids, there are similar interpolation results for cographic matroids, as Section 3 will show using the terminology of cocircuits—cutsets—of graphs.

3. Strongly Dual-Pancyclic Graphs

There are dual versions of many of the preceding results in terms of (edge) cutsets instead of cocircuits, where a *cutset* D of a connected graph G is a minimal subset of $E(G)$ such that $G - D$ is not connected; this is equivalent to using circuits of the (dual) cographic matroid. A graph is 3-*edge-connected* if every cutset has size three or more. A cutset D is a *hamiltonian cutset* if $G - D$ consists of two trees, or, equivalently, if $|D| = |E(G)| - |V(G)| + 2$. In accord with [1], a graph is *dual-pancyclic* if it has cutsets of every size between 3 and $|E(G)| - |V(G)| + 2$. Define G to be a *strongly dual-pancyclic graph* if G is 2-connected and 3-edge connected—a natural condition for duals of simple graphs—and both the following hold:

(0.1*) Every cutset of size greater than 3 almost contains another cutset.

(0.2*) Every cutset of size less than $|E(G)|-|V(G)|+2$ is almost contained in another cutset.

If D is a cutset of a graph G, then (as introduced in [7]) an edge $e \in E(G)-D$ is a *dual-chord* of D if there is a partition $D = D_1 \cup D_2$ such that each $D_i \cup \{e\}$ is a cutset of G. A 2-connected, 3-edge-connected graph G is *dual-chordal* if every every cutset of size four or more has a dual-chord. It is important to realize that we are not assuming planarity; for instance, $K_{3,3}$ is dual-chordal (and strongly dual-pancyclic). Note that strongly dual-pancyclic graphs are automatically dual-chordal.

The dual versions of Theorem 1, Corollary 3, Lemma 5, and Corollary 6 hold by simply translating cycle terminology in their proofs into the corresponding cutset terminology. Replacing Corollaries 4 and 7, Theorem 8 will actually characterize strongly dual-pancyclic graphs within the class of dual-chordal graphs. Notice that 3-regular dual-chordal graphs, since they must also be 3-edge-connected, also have to be 3-connected. Hence, the class of dual-chordal 3-regular graphs mentioned in Theorem 8 is the class of 3-connected, 3-regular dual-chordal graphs that is characterized five ways in [7, §3], one of which is that it is the class of 2-connected, 3-regular graphs that contain no subgraph homeomorphic to—in other words, no subgraph that is a subdivision of—either of the graphs shown in Figure 3. Because the present paper is primarily concerned with the presumably more interesting case of strongly pancyclic graphs, not their duals, we will make free use of results from [7, 6] in the proof of Theorem 8.

Figure 3. The cube and twisted cube (or 4-rung Möbius ladder) graphs that Theorem 8 forbids as homeomorphic subgraphs in strongly dual-pancyclic graphs.

Theorem 8. *A graph is strongly dual-pancyclic if and only if it is dual-chordal and* 3*-regular.*

Proof. First suppose G is dual-chordal and 3-regular, so G is 3-connected and condition (0.1*) holds. Toward proving (0.2*), suppose D is any non-hamiltonian cutset of G. If v is an endpoint of exactly one edge of D and D_v is the set of the three edges of G that are incident with v, then D will be almost contained in the cutset $D \oplus D_v$, as called for in the theorem. So suppose instead that every endpoint of an edge e of D is the endpoint of at least two edges of D. Note that an endpoint of e cannot be on three edges of D, since the minimality of cutsets would then imply $|D| = 3$ and the other endpoint of e would be on only one edge of D. So every endpoint of an edge of D is the endpoint of exactly two edges of D. Therefore there is a cycle C_D with $C_D \subseteq D$ and, since G is 3-connected, $|C_D| \geq 6$. Let v and v' be in, respectively, connected components $H \neq H'$ of $G - D$, and let π_1, π_2, π_3 [and π'_1, π'_2, π'_3] be three v-to-$V(C_D)$ paths in H [or v'-to-$V(C_D)$ paths in H'] whose vertex sets pairwise intersect in precisely the vertex v [or v']. Then the cycle C_D and the six paths π_i and π'_i $(i = 1, 2, 3)$ would form a subgraph of G that is homeomorphic to a cube, contradicting Theorem 9 of [7].

Conversely, suppose G is dual-chordal but not 3-regular (arguing by contraposition that G is not strongly dual-pancyclic, using that strongly dual-pancyclic graphs are dual-chordal). Then Theorem 1 and Corollary 1 of [6] imply that G reduces by 'subgraph contractions' to a graph that contains one of the two subgraphs shown in Figure 4, where the subscripted vertices have exactly the neighbors shown, a and b have additional neighbors (a might be adjacent to b), and G has a cycle that contains none of the subscripted vertices. (In the terminology of [6], in order for G not to be 3-regular in this context, the reduction in [6, Theorem 1] will not involve contracting an '$sc\,C_2$,' and so will involve contracting either a '$c\,C_3$' like the C_3 induced by $\{x_1, x_2, x_3\}$ in the graph on the left or contracting a '$cK_{2,3}$' like the $K_{2,3}$ induced by $\{x_1, x_2, x_3, y_1, y_2\}$ in the graph on the right.)

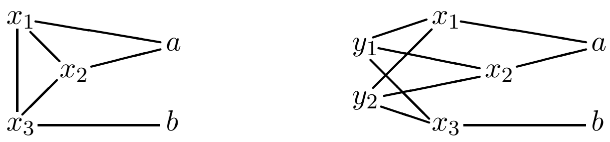

Figure 4. Two subgraphs used in the proof of Theorem 8.

Let $D = \{ax_1, ax_2, x_1x_3, x_2x_3\}$ in the graph on the left and $D = \{ax_1, x_2y_1, x_2y_2, x_3y_1, x_3y_2\}$ in the graph on the right. In either case, D will be a cutset of G that is not a hamiltonian cutset (since $G - D$ will still contain the cycle with no subscripted vertices) and that is not almost contained in another cutset. Thus, G would not be strongly dual-pancyclic. ■

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Received 23 May 2007
Revised 26 November 2008
Accepted 26 November 2008