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Note

# TRIANGLE-FREE PLANAR GRAPHS WITH MINIMUM DEGREE 3 HAVE RADIUS AT LEAST 3

Seog-Jin Kim

*Mathematics Education Department*
*Konkuk University, Seoul, Korea*

**e-mail:** skim12@konkuk.ac.kr

and

Douglas B. West*

*Department of Mathematics*
*University of Illinois*
*Urbana, IL 61801, USA*

**e-mail:** west@math.uiuc.edu

**Abstract**

We prove that every triangle-free planar graph with minimum degree 3 has radius at least 3; equivalently, no vertex neighborhood is a dominating set.

**Keywords:** planar graph, radius, minimum degree, triangle-free, dominating set.

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In 1975, Plesník [3] determined all triangle-free planar graphs with diameter 2. They are the stars, the complete bipartite graphs $K_{2,n}$, and a third family that can be described in several ways. One can start with the disjoint union $K_2 + K_1$ and add vertices of degree 2 joined to either nonadjacent pair of the original triple, or start with $C_5$ and expand two nonadjacent vertices into larger independent sets, or start with $K_{2,n}$ and apply a "vertex split" to one of the high-degree vertices.

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Each graph in Plesník's characterization has a vertex of degree at most 2. Thus his result implies that every triangle-free planar graph with minimum degree 3 has diameter at least 3 (note that no triangle-free planar graph has minimum degree greater than 3). In this note, we strengthen this statement by proving that every triangle-free graph with minimum degree 3 has radius at least 3. That is, it has no vertex whose neighborhood is a dominating set. There are many triangle-free planar graphs with minimun degree 3 and radius equal to 3.

Our result can also be related to other past work about distances in triangle-free or planar graphs. Erdős, Pach, Pollack, and Tuza [1] studied the maximum radius and diameter among graphs with fixed minimum degree. They also solved these problems in the family of triangle-free graphs. In contrast, we are seeking the minimum radius when the family is further restricted to planar graphs.

For planar graphs, Harant [2] proved an upper bound on the radius when the graph is 3-connected and has no long faces (it is $n/6 + q + \frac{3}{2}$ when the graph has $n$ vertices and no face of length more than $q$). We prove a lower bound on the radius when the graph has no short faces (no triangles), without restriction on connectivity.

We use $\delta(G)$ to denote the minimum degree of $G$, and we write $[v_1, \ldots, v_k]$ to denote a cycle with vertices $v_1, \ldots, v_k$ in order. Our graphs have no loops or multiple edges. A vertex *dominates* (is adjacent to) any subset of its neighbors.

**Theorem 1.** *Every triangle-free planar graph with minimum degree* 3 *has radius at least* 3*.*

***Proof.*** If the radius is 1, then one vertex dominates all others; additional edges would create triangles, so the other vertices cannot reach degree 3. Hence it suffices to forbid radius 2. We assume that our graph $G$ has a vertex $v$ whose neighborhood $U$ dominates the remaining vertices. Let $W = V(G) - U - \{v\}$.

If $v$ lies on no cycle, then each component of $G - v$ is dominated by one vertex of $U$, which cannot happen since $G$ is triangle-free and $\delta(G) = 3$. If $v$ lies on no cycle of length at most 5, then the shortest path in $G - v$ between any two vertices of $U$ has length at least 4, and the center of such a path is undominated by $U$.

Fix a planar embedding of $G$. Define a *trap* to be a cycle of length at most 5 through $v$. Say that a cycle in $G$ is *empty* if no vertex lies inside the region enclosed by it. Let a *flap* in an embedding of $G$ be the subgraph induced by a nonempty trap and the vertices inside it. If a trap $C$ is empty, then we redraw $G$ so that $C$ is the external face, and now $G$ itself is a flap. Hence a flap exists in some embedding of $G$.

We obtain a contradiction by proving that every flap $P$ in an embedding of $G$ contains another flap; this contradicts the finiteness of $G$. The cases appear in Figure 1.

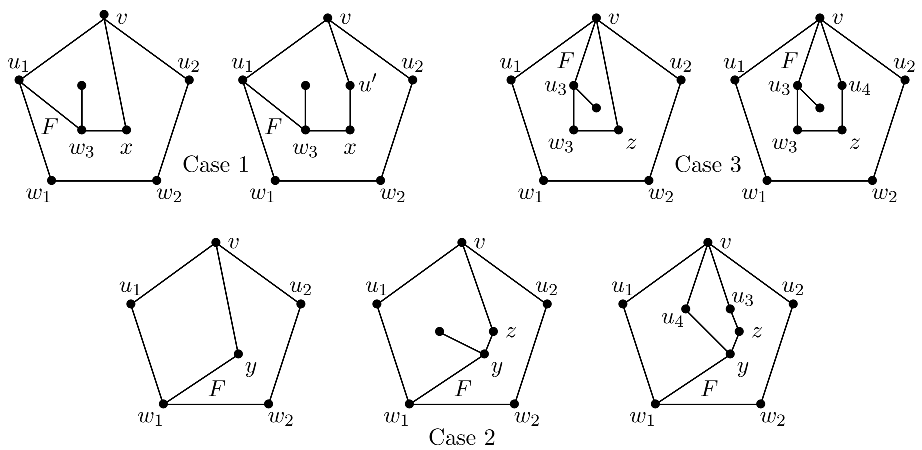


Figure 1. Cases for the proof.

Let $C$ be the external cycle in $P$. Let $u_1$ and $u_2$ be the neighbors of $v$ on $C$ (in $U$). Let $w_1$ and $w_2$ be their neighbors on $C$ other than $v$, respectively, where $w_1 = w_2$ if $C$ has length 4. Note that $w_1, w_2 \in W$, since $U$ is independent. Let $S$ be the set of vertices of $P$ not on $C$; call them the *internal* vertices. If $|S| \leq 2$, then $\delta(G) \geq 3$ forces a triangle, since neighbors of adjacent vertices in $S$ cannot alternate on $C$. Hence we have $|S| \geq 3$.

*Case* 1. *$u_1$ or $u_2$ has an internal neighbor.*
Let $u_1$ have an internal neighbor. Let $w_3$ be the internal neighbor of $u_1$ on the bounded face $F$ of $P$ that contains $w_1$ and $u_1$, and let $x$ be the next vertex reached in following $F$. If $x \in U$, then $[v, u_1, w_3, x]$ is a trap that encloses a smaller flap than $P$, since $w_3$ has a third neighbor inside that trap. If $x \notin U$, then $x$ has a neighbor $u' \in U$, and now $[v, u_1, w_3, x, u']$ encloses a smaller flap.

*Case* 2. *$u_1$ and $u_2$ have no internal neighbors, but $w_1$ or $w_2$ does.* By symmetry, we may assume that $w_1$ has an internal neighbor. Let $y$ be the internal neighbor of $w_1$ following $w_1$ on the bounded face $F$ of $P$ that contains $w_2$ and $w_1$.

If $y \in U$, then there are two cycles formed by $v$, $y$, and part of $C$. Whichever encloses a neighbor of $y$ encloses a smaller flap.

If $y \in W$, then let $z$ be the next vertex after $y$ in following $F$. If $z \in U$, then the cycle $[v, u_1, w_1, y, z]$ encloses a neighbor of $y$ and yields a smaller flap. If $z \in W$, then let $u_3$ be a neighbor of $z$ in $U$. Now $[v, u_1, w_1, y, z, u_3]$ encloses the remaining neighbors of $y$, which must include a vertex $u_4$ in $U$. Since $u_4$ must have another neighbor in the region enclosed by the 6-cycle, $[v, u_1, w_1, y, u_4]$ or $[v, u_4, y, z, u_3]$ is a trap enclosing a smaller flap.

*Case* 3. *None of $\{u_1, w_1, w_2, u_2\}$ has an internal neighbor.* Since the interior is nonempty and $G$ is connected, $v$ has an internal neighbor. Let $u_3$ be the one reached after $u_1$ and $v$ when following the face $F$ of $P$ whose boundary contains all of $C$. Let $w_3$ be the vertex after $u_3$ on $F$ (since $U$ is independent, $w_3 \in W$), and let $z$ be the vertex after $w_3$. If $z \in W$, then we can choose $u_4 \in N(z) \cap U - \{u_3\}$. Otherwise, $z \in U$. In the two cases, $[v, u_3, w_3, z, u_4]$ or $[v, u_3, w_3, z]$ encloses another neighbor of $u_3$ and yields a smaller flap. ■

## References


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