Discussiones Mathematicae
Graph Theory 28 (2008) 557–561

Note

A RESULT RELATED TO THE LARGEST EIGENVALUE OF A TREE

Gurusamy Rengasamy Vijayakumar

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba, Mumbai 400 005, India

e-mail: vijay@math.tifr.res.in

Abstract

In this note we prove that $\{0, 1, \sqrt{2}, \sqrt{3}, 2\}$ is the set of all real numbers ℓ such that the following holds: every tree having an eigenvalue which is larger than ℓ has a subtree whose largest eigenvalue is ℓ.

Keywords: eigenvalues of a graph, characteristic polynomial.

2000 Mathematics Subject Classification: 05C50, 15A18.

For terminology and notation, we follow [8]. The path with n vertices and the star with n edges are denoted by P_n and $K_{1,n}$, respectively. The largest eigenvalue and the least one of a graph G are denoted by $\Lambda(G)$ and $\lambda(G)$, respectively. Let A be the adjacency matrix of G. Then $|xI - A|$, the characteristic polynomial of G, is denoted by $\phi(G; x)$. In [1], it has been found that $\{-2, -\sqrt{2}, -1, 0\}$ is the set of all real numbers ℓ such that if the least eigenvalue of a graph is less than ℓ, then the least eigenvalue of one of its induced subgraphs is equal to ℓ. A result similar to this one is proved in this note: we determine $\mathcal{L}$ which is defined to be the set of all real numbers ℓ such that the following holds: if T is a tree with $\Lambda(T) > \ell$, then for some subtree U of T, $\Lambda(U) = \ell$. To prove our result, we need the following facts:

(1) If F is a forest and u is a vertex of F, then

$$\phi(F; x) = x\phi(F - u; x) - \sum_{v \in N(u)} \phi(F - u - v; x). \quad \text{(See [8, Page 468].)}$$

(2) $\Lambda(P_5) = \sqrt{3}$. (This fact can be easily derived by using the above formula; for more information in this connection, see [5] and [4, Problems 1.29 and 11.5].)

(3) For each $n \in \mathbb{N}$, $\Lambda(K_{1,n}) = \sqrt{n}$. (By using (1), it can be easily verified that $\phi(K_{1,n}; x) = x^{n-1}(x^2 - n)$; see [8, Pages 453–454] for an alternative method.)

(4) If H is a proper subgraph of a connected graph G, then $\Lambda(H) < \Lambda(G)$. (See [2, Page 178].)

Obviously $0 \in \mathfrak{L}$. Let T be any tree. If $\Lambda(T) > 1$, then K_2 is a subtree of T. Therefore $1 \in \mathfrak{L}$. If $\Lambda(T) > \sqrt{2}$, then $K_{1,2}$ is a subtree of T. Therefore by (3), $\sqrt{2} \in \mathfrak{L}$.

Let T be a tree with $\Lambda(T) > \sqrt{3}$. By (2) and (4), T cannot be a subtree of P_4. Therefore it contains P_5 or $K_{1,3}$; now (2) and (3) imply that T has a subtree whose largest eigenvalue is $\sqrt{3}$. Therefore $\sqrt{3} \in \mathfrak{L}$.

In [7], the family of all graphs G with $\Lambda(G) = 2$ has been determined. By using this family, the following result can be derived.

(5) Every graph G with $\Lambda(G) > 2$ has a (connected) subgraph H with $\Lambda(H) = 2$.

A shorter method of classifying the above mentioned family has been found in [3]; in its process of classification, (5) has been observed; but it has not been stated explicitly. Note that (5) is an easy consequence of the main result of [6]: every signed graph S with $\lambda(S) < -2$ has an induced subgraph R with $\lambda(R) = -2$. Confining (5) to trees we find that $2 \in \mathfrak{L}$.

Summary of what we have observed so far:

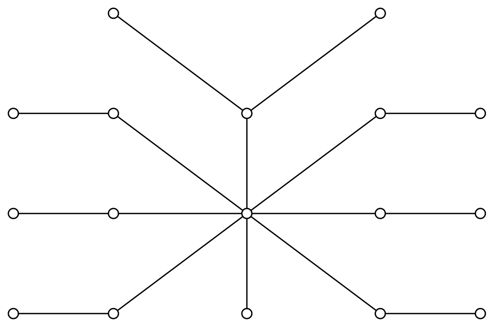
(6) $0, 1, \sqrt{2}, \sqrt{3}, 2 \in \mathfrak{L}$.

Now we proceed to show that $\mathfrak{L}$ does not have elements other than those listed above. As a prelude to this end, we have the following observation.

(7) A real number ℓ does not belong to $\mathfrak{L}$ when $\ell^2 \notin \mathbb{Z}$. (Reason: for any integer $m > \ell^2$, by (3), $\Lambda(K_{1,m}) > \ell$ but for each subtree U of $K_{1,m}$, $\Lambda(U) \neq \ell$.)

The main work of this note is concerned with constructing for each $k \in \mathbb{N}$, a tree T such that (i) $\Lambda(T) > \sqrt{k+4}$ and (ii) for each proper subtree U of T, $\Lambda(U) < \sqrt{k+4}$. If p, q, r are three nonnegative integers, then the tree

$T(p,q,r)$ is formed from $K_{1,p}$, $K_{1,q}$ and r copies of K_2, by joining the vertex of degree p in $K_{1,p}$ with the vertex of degree q in $K_{1,q}$ and joining the latter with one vertex of each K_2. Thus, the degree of the center of $K_{1,q}$ in the new tree is $q+r+1$.



The tree $T(2,1,6)$

In the recursive formula given by (1), taking F to be $T(p,q,r)$ and u to be the vertex of degree $q+r+1$ mentioned above, we get

$$\begin{aligned}\phi(T(p,q,r);x) &= xx^{p-1}(x^2-p)x^q(x^2-1)^r - x^p x^q (x^2-1)^r \\ &\quad - qx^{p-1}(x^2-p)x^{q-1}(x^2-1)^r - rx^{p-1}(x^2-p)x^q x(x^2-1)^{r-1}.\end{aligned}$$

Simplifying we get

$$\begin{aligned}&\phi(T(p,q,r);x) \\ &= x^{p+q-2}(x^2-1)^{r-1}\left[(x^2-1)(x^2-p)(x^2-q) - (r+1)x^4 + (pr+1)x^2\right].\end{aligned}$$

Theorem. *If k is an integer which exceeds* 1, *then* $\sqrt{k+3} \notin \mathcal{L}$.

Proof. The characteristic polynomials of the trees $T(2,1,k)$, $T(2,0,k)$, $T(1,1,k)$ and $T(2,2,k-1)$ given by the above formula can be expressed as follows

$$\begin{aligned}&\phi(T(2,1,k);x) = x(x^2-1)^{k-1}\left\{(x^2-k-3)x^2(x^2-2)-2\right\};\\ &\phi(T(2,0,k);x) = (x^2-1)^{k-1}\left\{(x^2-k-3)\left[x^2(x^2-1)+k\right]+k(k+3)\right\};\\ &\phi(T(1,1,k);x) = (x^2-1)^{k-1}\left\{(x^2-k-3)\left[x^2(x^2-1)+1\right]+k+2\right\};\\ &\phi(T(2,2,k-1);x) = x^2(x^2-1)^{k-2}\left\{(x^2-k-3)(x^2-1)^2+(k-1)\right\}.\end{aligned}$$

Since $\phi(T(2,1,k);\sqrt{k+3}) < 0$ and $\phi(T(2,1,k);\infty) = \infty$, it follows that the largest root of $\phi(T(2,1,k);x)$ exceeds $\sqrt{k+3}$; i.e., $\Lambda(T(2,1,k)) > \sqrt{k+3}$. Let U be a proper subtree of $T(2,1,k)$; note that U is a subgraph of either $T(2,0,k)$ or $T(1,1,k)$ or $T(2,2,k-1)$. Since the largest eigenvalue of each of the latter trees is less than $\sqrt{k+3}$ because this eigenvalue is a root of one of the above polynomials which are positive on the interval $[\sqrt{k+3},\infty)$, by (4) it follows that $\Lambda(U) < \sqrt{k+3}$. Therefore $\sqrt{k+3} \notin \mathcal{L}$. ■

Now combining (6), (7) and the above theorem, we get our result. Since the spectrum of any tree is symmetric about the origin (see [2, Page 178]), the dual of this result, obtained from its statement in the abstract by replacing the words 'larger', 'largest', and the numbers $1, \sqrt{2}, \sqrt{3}, 2$ by 'less', 'least' and $-1, -\sqrt{2}, -\sqrt{3}, -2$ respectively also holds; i.e., for a real number ℓ, each tree T with $\lambda(T) < \ell$ has a subtree U with $\lambda(U) = \ell$ if and only if $\ell \in \{0, -1, -\sqrt{2}, -\sqrt{3}, -2\}$.

Acknowledgement

The author expresses his gratitude to the referee for pointing out some mistakes and for suggesting some modifications.

References

[1] M. Doob, *A surprising property of the least eigenvalue of a graph*, Linear Algebra and Its Applications **46** (1982) 1–7.

[2] C. Godsil and G. Royle, Algebraic Graph Theory (Springer, New York, 2001).

[3] P.W.H. Lemmens and J.J. Seidel, *Equiangular lines*, Journal of Algebra **24** (1973) 494–512.

[4] L. Lovász, Combinatorial Problems and Exercises (North-Holland Publishing Company, Amsterdam, 1979).

[5] A.J. Schwenk, *Computing the characteristic polynomial of a graph*, in: Graphs and Combinatorics, eds. R.A. Bari and F. Harary, Springer-Verlag, Lecture Notes in Math. **406** (1974) 153–172.

[6] N.M. Singhi and G.R. Vijayakumar, *Signed graphs with least eigenvalue* < -2, European J. Combin. **13** (1992) 219–220.

[7] J.H. Smith, *Some properties of the spectrum of a graph*, in: Combinatorial Structures and their Applications, eds. R. Guy, H. Hanani, N. Sauer and J. Schönheim, Gordon and Breach, New York (1970), 403–406.

[8] D.B. West, Introduction to Graph Theory, Second edition (Printice Hall, New Jersey, USA, 2001).

Received 3 October 2007
Revised 10 June 2008
Accepted 10 June 2008