## Note

# A RESULT RELATED TO THE LARGEST EIGENVALUE OF A TREE 

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#### Abstract

In this note we prove that $\{0,1, \sqrt{2}, \sqrt{3}, 2\}$ is the set of all real numbers $\ell$ such that the following holds: every tree having an eigenvalue which is larger than $\ell$ has a subtree whose largest eigenvalue is $\ell$.


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For terminology and notation, we follow [8]. The path with $n$ vertices and the star with $n$ edges are denoted by $P_{n}$ and $K_{1, n}$, respectively. The largest eigenvalue and the least one of a graph $G$ are denoted by $\Lambda(G)$ and $\lambda(G)$, respectively. Let $A$ be the adjacency matrix of $G$. Then $|x I-A|$, the characteristic polynomial of $G$, is denoted by $\phi(G ; x)$. In [1], it has been found that $\{-2,-\sqrt{2},-1,0\}$ is the set of all real numbers $\ell$ such that if the least eigenvalue of a graph is less than $\ell$, then the least eigenvalue of one of its induced subgraphs is equal to $\ell$. A result similar to this one is proved in this note: we determine $\mathcal{L}$ which is defined to be the set of all real numbers $\ell$ such that the following holds: if $T$ is a tree with $\Lambda(T)>\ell$, then for some subtree $U$ of $T, \Lambda(U)=\ell$. To prove our result, we need the following facts:
(1) If $F$ is a forest and $u$ is a vertex of $F$, then

$$
\phi(F ; x)=x \phi(F-u ; x)-\sum_{v \in N(u)} \phi(F-u-v ; x) . \quad \text { (See [8, Page 468].) }
$$

(2) $\Lambda\left(P_{5}\right)=\sqrt{3}$. (This fact can be easily derived by using the above formula; for more information in this connection, see [5] and [4, Problems 1.29 and 11.5].)
(3) For each $n \in \mathbb{N}, \Lambda\left(K_{1, n}\right)=\sqrt{n}$. (By using (1), it can be easily verified that $\phi\left(K_{1, n} ; x\right)=x^{n-1}\left(x^{2}-n\right)$; see [8, Pages 453-454] for an alternative method.)
(4) If $H$ is a proper subgraph of a connected graph $G$, then $\Lambda(H)<\Lambda(G)$. (See [2, Page 178].)

Obviously $0 \in \mathcal{L}$. Let $T$ be any tree. If $\Lambda(T)>1$, then $K_{2}$ is a subtree of $T$. Therefore $1 \in \mathcal{L}$. If $\Lambda(T)>\sqrt{2}$, then $K_{1,2}$ is a subtree of $T$. Therefore by $(3), \sqrt{2} \in \mathcal{L}$.

Let $T$ be a tree with $\Lambda(T)>\sqrt{3}$. By (2) and (4), $T$ cannot be a subtree of $P_{4}$. Therefore it contains $P_{5}$ or $K_{1,3}$; now (2) and (3) imply that $T$ has a subtree whose largest eigenvalue is $\sqrt{3}$. Therefore $\sqrt{3} \in \mathcal{L}$.

In [7], the family of all graphs $G$ with $\Lambda(G)=2$ has been determined. By using this family, the following result can be derived.
(5) Every graph $G$ with $\Lambda(G)>2$ has a (connected) subgraph $H$ with $\Lambda(H)=2$.

A shorter method of classifying the above mentioned family has been found in [3]; in its process of classification, (5) has been observed; but it has not been stated explicitly. Note that (5) is an easy consequence of the main result of [6]: every signed graph $S$ with $\lambda(S)<-2$ has an induced subgraph $R$ with $\lambda(R)=-2$. Confining (5) to trees we find that $2 \in \mathcal{L}$.

Summary of what we have observed so far:
(6) $0,1, \sqrt{2}, \sqrt{3}, 2 \in \mathcal{L}$.

Now we proceed to show that $\mathcal{L}$ does not have elements other than those listed above. As a prelude to this end, we have the following observation.
(7) A real number $\ell$ does not belong to $\mathcal{L}$ when $\ell^{2} \notin \mathbb{Z}$. (Reason: for any integer $m>\ell^{2}$, by $(3), \Lambda\left(K_{1, m}\right)>\ell$ but for each subtree $U$ of $K_{1, m}$, $\Lambda(U) \neq \ell$.

The main work of this note is concerned with constructing for each $k \in \mathbb{N}$, a tree $T$ such that (i) $\Lambda(T)>\sqrt{k+4}$ and (ii) for each proper subtree $U$ of $T, \Lambda(U)<\sqrt{k+4}$. If $p, q, r$ are three nonnegative integers, then the tree
$T(p, q, r)$ is formed from $K_{1, p}, K_{1, q}$ and $r$ copies of $K_{2}$, by joining the vertex of degree $p$ in $K_{1, p}$ with the vertex of degree $q$ in $K_{1, q}$ and joining the latter with one vertex of each $K_{2}$. Thus, the degree of the center of $K_{1, q}$ in the new tree is $q+r+1$.


The tree $T(2,1,6)$
In the recursive formula given by (1), taking $F$ to be $T(p, q, r)$ and $u$ to be the vertex of degree $q+r+1$ mentioned above, we get

$$
\begin{aligned}
\phi(T(p, q, r) ; x) & =x x^{p-1}\left(x^{2}-p\right) x^{q}\left(x^{2}-1\right)^{r}-x^{p} x^{q}\left(x^{2}-1\right)^{r} \\
& -q x^{p-1}\left(x^{2}-p\right) x^{q-1}\left(x^{2}-1\right)^{r}-r x^{p-1}\left(x^{2}-p\right) x^{q} x\left(x^{2}-1\right)^{r-1} .
\end{aligned}
$$

Simplifying we get

$$
\begin{aligned}
& \phi(T(p, q, r) ; x) \\
& =x^{p+q-2}\left(x^{2}-1\right)^{r-1}\left[\left(x^{2}-1\right)\left(x^{2}-p\right)\left(x^{2}-q\right)-(r+1) x^{4}+(p r+1) x^{2}\right] .
\end{aligned}
$$

Theorem. If $k$ is an integer which exceeds 1 , then $\sqrt{k+3} \notin \mathcal{L}$.
Proof. The characteristic polynomials of the trees $T(2,1, k), T(2,0, k)$, $T(1,1, k)$ and $T(2,2, k-1)$ given by the above formula can be expressed as follows

$$
\begin{aligned}
& \phi(T(2,1, k) ; x)=x\left(x^{2}-1\right)^{k-1}\left\{\left(x^{2}-k-3\right) x^{2}\left(x^{2}-2\right)-2\right\} ; \\
& \phi(T(2,0, k) ; x)=\left(x^{2}-1\right)^{k-1}\left\{\left(x^{2}-k-3\right)\left[x^{2}\left(x^{2}-1\right)+k\right]+k(k+3)\right\} ; \\
& \phi(T(1,1, k) ; x)=\left(x^{2}-1\right)^{k-1}\left\{\left(x^{2}-k-3\right)\left[x^{2}\left(x^{2}-1\right)+1\right]+k+2\right\} ; \\
& \phi(T(2,2, k-1) ; x)=x^{2}\left(x^{2}-1\right)^{k-2}\left\{\left(x^{2}-k-3\right)\left(x^{2}-1\right)^{2}+(k-1)\right\} .
\end{aligned}
$$

Since $\phi(T(2,1, k) ; \sqrt{k+3})<0$ and $\phi(T(2,1, k) ; \infty)=\infty$, it follows that the largest root of $\phi(T(2,1, k) ; x)$ exceeds $\sqrt{k+3}$; i.e., $\Lambda(T(2,1, k))>\sqrt{k+3}$. Let $U$ be a proper subtree of $T(2,1, k)$; note that $U$ is a subgraph of either $T(2,0, k)$ or $T(1,1, k)$ or $T(2,2, k-1)$. Since the largest eigenvalue of each of the latter trees is less than $\sqrt{k+3}$ because this eigenvalue is a root of one of the above polynomials which are positive on the interval $[\sqrt{k+3}, \infty)$, by (4) it follows that $\Lambda(U)<\sqrt{k+3}$. Therefore $\sqrt{k+3} \notin \mathcal{L}$.

Now combining (6), (7) and the above theorem, we get our result. Since the spectrum of any tree is symmetric about the origin (see [2, Page 178]), the dual of this result, obtained from its statement in the abstract by replacing the words 'larger', 'largest', and the numbers $1, \sqrt{2}, \sqrt{3}, 2$ by 'less', 'least' and $-1,-\sqrt{2},-\sqrt{3},-2$ respectively also holds; i.e., for a real number $\ell$, each tree $T$ with $\lambda(T)<\ell$ has a subtree $U$ with $\lambda(U)=\ell$ if and only if $\ell \in\{0$, $-1,-\sqrt{2},-\sqrt{3},-2\}$.

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