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Note

PATH AND CYCLE FACTORS OF CUBIC BIPARTITE GRAPHS

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Dedicated to Professor Hikoe Enomoto on his 60th Birthday

Abstract

For a set $\mathcal{S}$ of connected graphs, a spanning subgraph F of a graph is called an $\mathcal{S}$-factor if every component of F is isomorphic to a member of S. It was recently shown that every 2-connected cubic graph has a $\{C_n \mid n \geq 4\}$-factor and a $\{P_n \mid n \geq 6\}$-factor, where C_n and P_n denote the cycle and the path of order n, respectively (Kawarabayashi *et al.*, J. Graph Theory, Vol. 39 (2002) 188–193). In this paper, we show that every connected cubic bipartite graph has a $\{C_n \mid n \geq 6\}$-factor, and has a $\{P_n \mid n \geq 8\}$-factor if its order is at least 8.

Keywords: cycle factor, path factor, bipartite graph.

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1. Introduction

We consider finite graphs without loops or multiple edges. A 3-regular graph is called a *cubic graph*. We denote by P_n and C_n the path and the cycle of order n, respectively. For a set $\mathcal{S}$ of connected graphs, a spanning subgraph F of a graph G is called an $\mathcal{S}$-factor of G if every component of F

is isomorphic to one of members in S. Then a $\{C_n \mid n \geq 3\}$-factor is nothing but a 2-factor, which is a spanning 2-regular subgraph.

In this paper we consider cycle-factors and path-factors of cubic graphs, whose components are cycles and paths, respectively. Notice that in a cubic graph, the edge-connectivity is equal to the connectivity. We begin with some known results on these factors.

Theorem 1 (Petersen [5]). *Every 2-connected cubic graph has a $\{C_n \mid n \geq 3\}$-factor.*

Kaneko found a criterion for a graph to have a $\{P_n \mid n \geq 3\}$-factor, and obtained the following theorem as its corollary. Note that a short proof of Kaneko's theorem can be found in [3].

Theorem 2 (Kaneko [2]). *Every connected cubic graph has a $\{P_n \mid n \geq 3\}$-factor.*

Recently Kawarabayashi *et al.* [4] showed the next theorem.

Theorem 3 [4].

(i) *Every 2-connected cubic graph has a $\{C_n \mid n \geq 4\}$-factor.*

(ii) *Every 2-connected cubic graph of order at least six has a $\{P_n \mid n \geq 6\}$-factor.*

In this paper we shall prove the following theorem.

Theorem 4.

(i) *Every connected cubic bipartite graph has a $\{C_n \mid n \geq 6\}$-factor.*

(ii) *Every connected cubic bipartite graph of order at least eight has a $\{P_n \mid n \geq 8\}$-factor.*

We now give some remarks on the above Theorem 4. It follows immediately from Theorem 4 that every connected cubic bipartite graph G of order at most 16 has a Hamiltonian path since G has a $\{C_n \mid n \geq 6\}$-factor, which consists of at most two components, and a graph consisting of two disjoint cycles and one edge joining them has a Hamiltonian path. It is not mentioned in [4] that the conclusion of Theorem 3 is best possible. However, we can easily find 2-connected cubic graphs having no $\{C_n \mid n \geq 5\}$-factors.

An example of such a cubic graph is given in Figure 1 (a), and it has many triangles. So we might expect that a 2-connected triangle-free cubic graph has a $\{C_n \mid n \geq 5\}$-factor. But this is not true as shown in Figure 1 (b), which shows a 2-connected triangle-free cubic graph having no $\{C_n \mid n \geq 5\}$-factor. Moreover, Theorem 4 is sharp in the sense that there exists a connected cubic bipartite graph having no $\{C_n \mid n \geq 8\}$-factor as shown in Figure 1 (c).

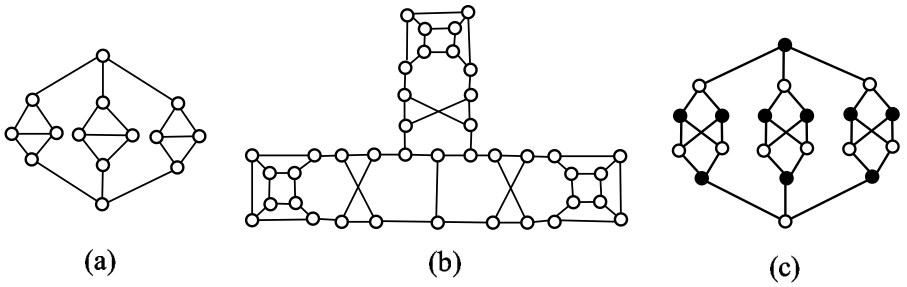

Figure 1. (a) A 2-connected cubic graph having no $\{C_n \mid n \geq 5\}$-factor;
(b) A 2-connected triangle-free cubic graph having no $\{C_n \mid n \geq 5\}$-factor;
(c) A 2-connected cubic bipartite graph having no $\{C_n \mid n \geq 8\}$-factor.

However we have been unable to find a 3-connected cubic graph having no $\{C_n \mid n \geq 5\}$-factor or no $\{P_n \mid n \geq 7\}$-factor. So we propose the following conjecture and problem.

Conjecture 5. Every 3-connected cubic graph of order at least six has a $\{C_n \mid n \geq 5\}$-factor.

Problem 6. Determine the maximum integer $k \geq 6$ for which every 3-connected (or 2-connected) cubic graph of order at least $f(k)$ has a $\{P_n \mid n \geq k\}$-factor, where $f(k)$ is a suitable function of k.

We conclude this section with a conjecture on path-factors of 3-connected cubic graphs.

Conjecture 7 (Akiyama and Kano [1])**.** Every 3-connected cubic graph of order $3n$ has a $\{P_3\}$-factor.

2. Proof of Theorem 4

For a vertex v of a graph G, we denote by $\deg_G(v)$ the degree of v in G. For two disjoint vertex subsets X and Y of $V(G)$, we denote by $e_G(X,Y)$ the number of edges of G joining X to Y. We denote the order of G by $|G|$, which is equal to $|V(G)|$.

Lemma 8. *Let $r \geq 2$ be an integer. Then every connected r-regular bipartite graph is 2-edge connected. In particular, every connected cubic bipartite graph is 2-connected.*

Proof. Let G be a connected r-regular bipartite graph with bipartition $A \cup B$. Suppose that G has an bridge $e = uw \in E(G)$, $u \in A$, $w \in B$. Then for a component D of $G - e$ containing u but not w, we have

$$r|V(D) \cap A| - 1 = \sum_{x \in V(D) \cap A} \deg_D(x) = \sum_{x \in V(D) \cap B} \deg_D(x) = r|V(D) \cap B|.$$

This is a contradiction. Hence G has no bridge, which implies that G is 2-edge connected. ■

We first prove (i) of Theorem 4.

***Proof of* (i).** Let G be a connected cubic bipartite graph. We prove (i) by induction on the order $|G|$. There exists only one connected cubic bipartite graph of order six, which is $K_{3,3}$, and it has a $\{C_6\}$-factor. So we may assume $|G| \geq 8$.

By Lemma 8, G is 2-connected, and so G has a 2-factor F by Theorem 1, which is a $\{C_n \,|\, n \geq 4\}$-factor. We may assume that F contains a component D isomorphic to C_4 since otherwise F is the desired $\{C_n \,|\, n \geq 6\}$-factor. Let $V(D) = \{a, b, c, d\}$, and as, bt, cu, dw be the edges of $G - E(D)$ incident with $V(D)$ (see Figure 2).

Since $G - E(F)$ is a 1-factor of G, $\{as, bt, cu, dw\}$ is a set of independent edges, and so s, t, u, w are all distinct vertices of G. Let H be the graph obtained from G by removing the four vertices a, b, c, d and their incident edges, and by adding two new vertices x and y together with five new edges sx, ux, ty, wy, xy (see Figure 2).

Then H is a connected cubic bipartite graph, and $|H| = |G| - 2$. Hence H has a $\{C_n \,|\, n \geq 6\}$-factor F_H by induction. We shall obtain the desired $\{C_n \,|\, n \geq 6\}$-factor of G from F_H by considering the following two cases.

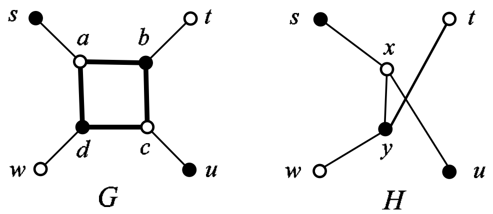

Figure 2. Cubic graphs G and H; Bold lines are edges of D.

Case 1. A component of F_H contains the edge xy.
In this case, without loss generality, we may assume that a component D of F_H contains xy, sx and yw by symmetry. Then $F_H - \{sx, xy, yw\} + \{sa, ab, bc, cd, dw\}$ is the desired $\{C_n \mid n \geq 6\}$-factor of G.

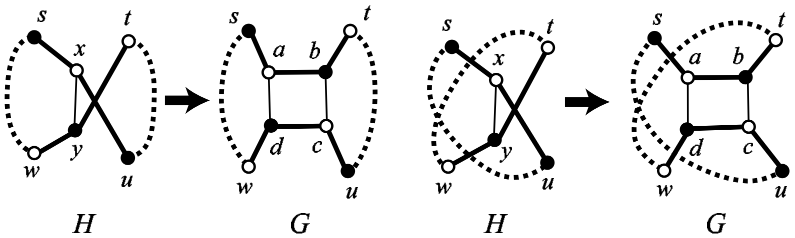

Figure 3. Cubic graphs G and H and their $\{C_n \mid n \geq 6\}$-factors.

Case 2. No component of F_H contains the edge xy.
In this case F_H contains the four edges sx, xu, ty, yw. We first assume that these four edges are contained in the same component D of F_H. By symmetry, we may assume that a cycle D passes through s, x, u and then t, y, w (see Figure 3). Then we can obtain the desired $\{C_n \mid n \geq 6\}$-factor from F_H by removing the edges sx, xu, ty, yw and by adding the edges sa, ab, bt, uc, cd, dw as shown in Figure 3.

Next assume that the four edges sx, xu, ty, yw are contained in two distinct components D_1 and D_2 of F_H. In this case we can obtain the desired $\{C_n \mid n \geq 6\}$-factor of G from F_H by removing sx, xu, ty, yw and by adding sa, ab, bt, wd, dc, cu. Consequently Statement (i) of Theorem 4 is proved. ■

Statement (ii) of Theorem 4 follows immediately from the next Lemma 9 and the statement (i) of Theorem 4. It is shown in [4] that if a 2-connected cubic graph of order at least six has a $\{C_n \,|\, n \geq 4\}$-factor, then it has a $\{P_n \,|\, n \geq 6\}$-factor. This statement can be generalized as the following Lemma 9 without changing the proof.

Lemma 9 [4]**.** *Let $k \geq 3$ be an integer. If a 2-connected cubic graph G of order at least $k+2$ has a $\{C_n \,|\, n \geq k\}$-factor, then G has a $\{P_n \,|\, n \geq k+2\}$-factor.*

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