Discussiones Mathematicae
Graph Theory 28 (2008) 535–549

INDEPENDENT CYCLES AND PATHS IN BIPARTITE BALANCED GRAPHS

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Abstract

Bipartite graphs $G = (L, R; E)$ and $H = (L', R'; E')$ are bi-placeabe if there is a bijection $f : L \cup R \to L' \cup R'$ such that $f(L) = L'$ and $f(u)f(v) \notin E'$ for every edge $uv \in E$. We prove that if G and H are two bipartite balanced graphs of order $|G| = |H| = 2p \geq 4$ such that the sizes of G and H satisfy $\parallel G \parallel \leq 2p - 3$ and $\parallel H \parallel \leq 2p - 2$, and the maximum degree of H is at most 2, then G and H are bi-placeable, unless G and H is one of easily recognizable couples of graphs.

This result implies easily that for integers p and $k_1, k_2, \ldots, k_l$ such that $k_i \geq 2$ for $i = 1, \ldots, l$ and $k_1 + \cdots + k_l \leq p - 1$ every bipartite balanced graph G of order $2p$ and size at least $p^2 - 2p + 3$ contains mutually vertex disjoint cycles $C_{2k_1}, \ldots, C_{2k_l}$, unless $G = K_{3,3} - 3K_{1,1}$.

Keywords: bipartite graphs, bi-placing, path, cycle.

2000 Mathematics Subject Classification: 05C38, 05C35.

1. Preliminaries

Let $G = (L, R; E)$ and $G' = (L', R'; E')$ be two bipartite graphs. $|G|$ denotes the order of G and by $\parallel G \parallel$ its size ($|G| = |L \cup R|$, $\parallel G \parallel = |E|$). $\Delta_R(G)$ is the maximum vertex degree $d_G(x)$, when $x \in R$ and $\Delta_L(G)$ the maximum degree $d(y, G)$ when $y \in L$. The maximum vertex degree in G is denoted

Partially supported by AGH local grant No. 11 420 04.

by $\Delta(G)$ $(\Delta(G) = \max\{\Delta_L(G), \Delta_R(G)\})$. The corresponding minimum degrees are denoted by $\delta_R(G), \delta_L(G)$ and $\delta(G)$, respectively. A vertex x with $d(x, G) = 1$ is said to be *pendent*. The set $L(G) = L$ is called the *left hand side set*, and $R(G) = R$ the *right hand side set* of bipartition of the vertex set $V(G) = L \cup R$.

For $x \in V(G)$, $N(x; G)$ denotes the set of the neighbors of the vertex x in G. C_k denotes a cycle of the length k.

G is called (p, q)-*bipartite* if $|L(G)| = p$ and $|R(G)| = q$. If $p = q$ then G is said to be *balanced*. $K_{p,q}$ stands for the complete bipartite graph with $|L(K_{p,q})| = p$ and $|R(K_{p,q})| = q$.

Bi-placement of G and G' is a bijection $f : L \cup R \to L' \cup R'$ such that $f(L) = L'$ and $f(u)f(v) \notin E'$ for every edge $uv \in E$. If there is a bi-placement of G and G' then we say that G and G' are *bi-placeable*.

Note that the bipartite graphs $H = (\{a, b\}, \{c, d, e\}; \{ac, ad, be\})$ and $H' = (\{a', b'\}, \{c', d', e'\}; \{a'c', b'c'\})$ are not bi-placeable, while it is very easy to find a bi-placement of H and $H'' = (\{a'', b''\}, \{c'', d'', e''\}; \{a''c'', a''d''\})$ (see Figure 1).

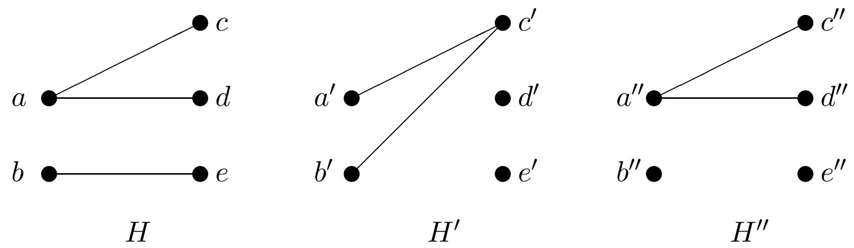


Figure 1. H bi-placeable with H'' and non bi-placeable with H'.

The notion of bi-placeability of bipartite graphs appeared in [7]. To say that *G and G' are bi-placeable* is equivalent to saying that the bipartite graph *$G = (L, R; E)$ is a subgraph of the bipartite graph $\overline{G'} = (L, R; \overline{E'})$* in the sense of [4] $(\overline{E'} = \{xy : x \in L', y \in R', xy \notin E'\})$. The problem of existence of a matching or a hamiltonian cycle in a bipartite graph is, in fact, a problem of bi-placeability of some bipartite graphs. For a survey of results concerning placing of graphs and bi-placing of bipartite graphs we refer the reader to [3, 11] or [12].

The following theorem was proved in [9].

Theorem 1. *Let* $G = (L, R; E)$ *and* $H = (L', R', ; E')$ *be two bipartite balanced graphs of order* $2p$ *such that* $\parallel G \parallel \leq p - 1$ *and* $\parallel H \parallel \leq 2p$. *Then* G *and* H *are bi-placeable unless* $\parallel G \parallel = p - 1$, $\parallel H \parallel = 2p$ *and either*

- $\Delta_L(G) \leq 1$ *and* $H = K_{2,p} \cup K_{p-2,0}$ *or*
- $\Delta_R(G) \leq 1$ *and* $H = K_{p,2} \cup K_{0,p-2}$ *or*
- $G = K_{1,p-1} \cup \overline{K_{p-1,1}}$ *and* $\Delta_L(H) = 2$ *or else*
- $G = K_{p-1,1} \cup \overline{K_{1,p-1}}$ *and* $\Delta_R(H) = 2$.

$G = (L, R; E)$ is said to be $2k$ *freely cyclable* whenever, for any sequence $k_1, \ldots, k_l$ of integers such that $k_i \geq 2$ for $i = 1, \ldots, l$ and $k_1 + \cdots + k_l \leq k$, G contains mutually vertex disjoint cycles $C_{2k_1}, \ldots, C_{2k_l}$. The problem of the existence of a union of independent cycles of prescribed lengths in a graph was considered by many authors (see [1, 5, 6, 8, 10]).

Theorem 1 implies easily the following generalisation of a result of Amar, Fournier and Germa (Theorem 2 in [2]).

Theorem 2. *Let* $G = (L, R; E)$ *be a bipartite balanced graph of order* $2p$ *and size at least* $p^2 - p + 1$. *Then* G *is* $2p$ *freely cyclable unless* $\parallel G \parallel = p^2 - p + 1$ *and* G *contains a penedent vertex.*

In the next section we give a sufficient condition for a $(p, p)-$bipartite graphs to be $2(p-1)$ freely cyclable. Namely, we shall prove that the only balanced bipartite graph of order $2p$ and size at least $p^2 - 2p + 3$ which is not $2(p-1)$ freely cyclable is $K_{3,3}$ minus a perfect matching.

2. Results

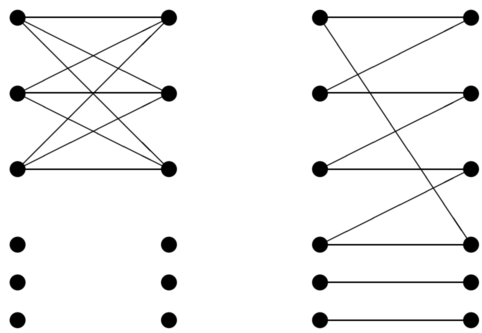
Theorem 3. *Let* $p \geq 2$ *be an integer, and let* $G = (L, R; E)$ *and* $H = (L', R'; E')$ *be two* $(p, p)-$*bipartite graphs such that* $\parallel G \parallel \leq 2p - 3$, $\parallel H \parallel \leq 2p - 2$ *and* $\Delta(H) \leq 2$. *Then* G *and* H *are bi-placeable unless one of the following occurs:*

(1) $\Delta_L(G) = p$ *and* $\delta_L(H) > 0$,
(2) $\Delta_R(G) = p$ *and* $\delta_R(H) > 0$,
(3) $p = 3$, G *is a perfect matching* $3K_{1,1}$, *and* $H = K_{2,2} \cup \overline{K_{1,1}}$ (*see Figure* 2),
(4) $p = 6$, $G = K_{3,3} \cup \overline{K_{3,3}}$, $H = C_8 \cup 2K_{1,1}$ (*see Figure* 3).

The couples of graphs G and H described in (1), (2), (3) and (4) will be called *exceptional* or *exceptions* (1), (2), (3) and (4), respectively.

$3K_{1,1}$ $\quad$ $K_{2,2} \cup \overline{K_{1,1}}$

Figure 2. Exceptional couple (3).



$K_{3,3} \cup \overline{K_{3,3}}$ $\quad$ $C_8 \cup 2\overline{K_{1,1}}$

Figure 3. Exceptional couple (4).

Theorem 3 implies easily the following corollary announced already at the end of Section 1.

Corollary 4. *Let G be a balanced bipartite graph of order $|G| = 2p$ and size $\| G \| \geq p^2 - 2p + 3$. Then G is $2(p-1)$ freely cyclable unless $p = 3$ and $G = K_{3,3} - 3K_{1,1} = C_6$.* ∎

For $p \geq 3$ the graph $H_{p,p} = K_{2,p-1} \cup \overline{K_{p-2,1}}$ is $(p,p)-$bipartite of order $2p$ and size $2p-2$ which is not bi-placeable with any union of vertex disjoint cycles $C_{2l_1} \cup \cdots \cup C_{2l_q}$, where $l_1 + \cdots + l_q = p-1$ and $l_1, \ldots, l_q \geq 2$. Hence

Theorem 3 may not be improved by a simple rising the size of the graph G. The graph $\overline{H_{p,p}}$ (the complement of $H_{p,p}$ in $K_{p,p}$) proves that also Corollary 4 is sharp.

3. PROOF OF THEOREM 3

The proof is by induction on p. It is easy to verify that the theorem holds for $p = 2, 3$. Suppose that $p \geq 4$ and the theorem holds for p' provided that $2 \leq p' < p$. Note that without loss of generality, we may assume that $\| G \| = 2p - 3$, $\| H \| = 2p - 2$ and $\Delta(H) = 2$. Then the graph H is a union of a number of (even) cycles and exactly two (possibly trivial) paths. Moreover, since H is balanced, either both paths have odd lengths (even order) or each path has an even length. In the later case, if the end vertices of one path are in L', then the end vertices of the second paths are in R' and vice versa.

We shall consider two cases and several subcases.

Case 1. There is an isolated vertex z in the set $V(H)$.
Without loss of generality we may assume that $z \in R'$. Let x be a vertex of minimal degree in L. It follows immediately that $d(x, G) \leq 1$.

Subcase 1.1. x is an isolated vertex.
Let $y \in R$, $d(y, G) = \Delta_R(G)$, $w \in L'$ and $d(w, H) = 2$. If the graphs $G' = G - \{x, y\}$ and $H' = H - \{w, z\}$ are m.p. then a bi-placement of the graphs G and H is obvious. Hence, we may suppose that the couple G' and H' is one of the exceptions (1)–(4). Note that $H' \neq C_4 \cup \overline{K_{1,1}}$, and that w may be choosen in such a way that $H' \neq C_8 \cup 2K_{1,1}$. Hence we have only two subcases to consider.

Subcase 1.1a. $\Delta_R(G') = p - 1$ and $\delta_R(H') \geq 1$.
There is a vertex $y' \in R(G')$ such that $d(y', G') = p-1$. Hence $d(y, G) = p-1$ and we have $e(G) \geq 2p - 2$, a contradiction.

Subcase 1.1b. $\Delta_L(G') = p - 1$ and $\delta_L(H') \geq 1$.
Let $x_1 \in L(G')$ and $d(x_1, G') = p-1$. If $d(x_1, G) = p$ then the couple G and H form the first exception (1). If $d(x_1, G) = p - 1$ then we can choose the following vertices: y_1 – a pendent vertex in R, w_1 – a pendent vertex in L'. Let $z_1 \in N(w_1, H)$. Then $d(z_1, H) = 2$ and the graphs $G'' = G-\{x, y, x_1, y_1\}$

and $H'' = H - \{w_1, w_2, z, z_1\}$, where w_2 is the second neighbour of z_1, are bi-placeable by Theorem 1.

Let f be a bi-placement of G'' and H''. Then we can extend f to a packing f_* of G and H by letting $f_*(v) = f(v)$, for $v \in V(H'')$, $f_*(w_1) = x_1$, $f_*(w_2) = x$, $f_*(z) = y_1$ and $f_*(z_1) = y$.

Subcase 1.2. $d(x, G) = 1$ and the neighbor y of x is not pendent ($d(y, G) \geq 2$).

So, we can apply the induction hypothesis to the graphs $G'_1 = G - \{x, y\}$ and H'_1, where H'_1 is the graph H' defined in Subcase 1.1. If G'_1 and H'_1 are bi-placeable then it is easy to check that G and H are bi-placeable too.

So we may suppose that the couple G'_1, H'_1 is one of the exceptions. Note that since $\delta_L(G) > 0$, we have $G'_1 \neq K_{3,3} \cup \overline{K_{3,3}}$, and since $\Delta(H) = 2$, we have $H'_1 \neq C_4 \cup \overline{K_{1,1}}$. Hence G'_1 and H'_2 may be the only one of exceptions (1)–(2).

Subcase 1.2a. $\Delta_R(G'_1) = p - 1$ and $\delta_R(H'_1) \geq 1$.
Let $y_1 \in R(G'_1)$, $d(y_1, G) = p - 1$, $x_1 \in N(y_1, G)$ and $d(x_1, G) = 1$. Observe that we may apply the induction hypothesis to the graphs $G_2 = G - \{x_1, y_1\}$ and $H_2 = H'$. From this, we can now map the vertex w to the vertex x_1 and the vertex z to y_1 and we can extend a bi-placement of G_2 and H_2 to a bi-placement of G and H.

Subcase 1.2b. $\Delta_L(G'_1) = p - 1$ and $\delta_L(H'_1) \geq 1$.
Since $\Delta_L(G) \geq p - 1$ and $\delta_L(G) \geq 1$, we have $\| G \| \geq 2p - 2$, a contradiction.

Subcase 1.3. There is no isolated vertex in L and the neighbors of pendent vertices of L are pendent.

Let xy be an isolated edge of G, $x \in L, y \in R$.

Subcase 1.3.1. There is an isolated vertex w in L'.
Note that $H - \{w, z\}$ is a union of vertex disjoint even cycles. Let $x' \in L - \{x\}$ and $y' \in R' - \{y\}$ be choosen in such a way that the sum of degrees $d(x', G) + d(y', G)$ is maximum. One may check easily that $d(x', G) + d(y', G) \geq 4$. Since $p \geq 4$, there exist two nonadjacent vertices $w' \in L' - \{w\}$ and $z' \in R' - \{z\}$. Observe that $d(w', H) = d(z', H) = 2$. The graphs $G'_3 = G - \{x, y, x', y'\}$ and $H'_3 = H - \{w, z, w', z'\}$ verify the induction hypothesis. Moreover, an easy computation shows that $\Delta(G'_3) < p - 2$. It is also clear that $H'_3 \neq K_{2,2} \cup \overline{K_{1,1}}$ and $H'_3 \neq C_8 \cup 2\overline{K_{1,1}}$. Hence there is a bi-placement,

say f, of H'_3 and G'_3. The function f_* defined by $f_*(v) = f(v)$, for $v \in V(H'_3)$, $f_*(w) = x'$, $f_*(z) = y'$, $f_*(w') = x$ and $f_*(z') = y$ is a bi-placement of H and G.

Subcase 1.3.2. The minimal vertex degree in L' is equal to one.
Let w be such a vertex of L' that $d(w, H) = 1$ and let $z' \in R'$ be the neighbour of w. Note that $H - \{w, z\}$ is a union of a path of odd length and a number of even cycles.

We have $d(z', H) = 2$. Since $p \geq 4$ we may choose $w' \in L'$ such that $d(w', H) = 2$ and $(w', z') \notin E'$. We set $G'_4 = G'_3$, where G'_3 is defined in Subcase 1.3.1, and $H'_4 = H - \{w, z, w', z'\}$. The graphs G'_4 and H'_4 are bi-placeable, by the induction hypothesis. Every bi-placement of H'_4 and G'_4 may be extended to a bi-placement of H and G by mapping the vertex w to x', z to y', w' to x and z' to y.

Case 2. There is no isolated vertex in $V(H)$.
Then the graph H is a sum of two non trivial paths P_1, P_2 and independent cycles.

Subcase 2.1. The paths P_1 and P_2 have length 1.
Let $P_1 = (w, z)$ and $P_2 = (w', z')$, where $w, w' \in L'$ and $z, z' \in R'$, and let $w_1 \in L'$ and $z_1 \in R'$ be two vertices of degree 2 in H.

Subcase 2.1.1. $\delta_L(G) = \delta_R(G) = 0$.
Let $x \in L$ and $y \in R$ be two isolated vertices of G and let $x_1 \in L$ and $y_1 \in R$ by two nonadjacent vertices of G chosen such that the degree sum

$$d(x_1, G) + d(y_1, G) \tag{1}$$

is maximal.

Under the hypothesis of Subcase 2.1.1 we shall prove two claims.

Claim 1. If there is in G a vertex of degree $p - 1$ then G and H are bi-placeable.

***Proof of Claim* 1.** Suppose that $x_0 \in L$ is a vertex of degree $p - 1$ in G. Then $\| G - \{x_0, y\} \| = 2p - 3 - (p - 1) = (p - 1) - 1$. Hence, by Theorem 1, there is a bi-placement f_* of $G - \{x_0, y\}$ and $H - \{w, z\}$ which may be easily extended to a bi-placement of G and H. ∎

Claim 2. If $d(x_1, G) + d(y_1, G) \geq 4$ then G and H are bi-placeable.

***Proof of Claim* 2.** If $G' = G - \{x, y, x_1, y_1\}$ and $H' = H - \{w, z, w_1, z_1\}$ are bi-placeable, then we extend a bi-placement of G' and H' to the bi-placement of G and H mapping $x_1 \mapsto w,\ y_1 \mapsto z,\ x \mapsto w_1,\ y \mapsto z_1$.

So, by the induction hypothesis, G' and H' is one of exceptions (1)–(4) described in the theorem. Note that $H' \neq K_{2,2} \cup \overline{K_{1,1}}$ and $H' \neq C_8 \cup 2K_{1,1}$. So let us suppose that $\Delta(G') = p - 2$. Without loss of generality we may assume that there is a vertex $x' \in L - \{x, x_1\}$, such that $d(x', G') = p - 2$. If $x'y_1 \in E$ then $d(x', G) = p - 1$ and we apply Claim 1. If $x'y_1 \notin E$ then, by the maximality of the sum (1), we have $d(x_1, G) = p - 2$ and the graphs $G'' = G - \{x_1, x', y_1, y\}$ and $H'' = H - \{w, w', z, z'\}$ are bi-placeable, unless $H'' = K_{2,2}$, but then $G = \overline{K_{1,1}} \cup K_{1,1} \cup K_{2,2}$ and $H = 2K_{1,1} \cup K_{2,2}$ are bi-placeable. Any bi-placement of G'' and H'' may be easily extended to a bi-placement of G and H. ■

By Claim 2 we may suppose that $d(x_1, G) + d(y_1, G) < 4$. Consider the following three subcases.

Subcase 2.1.1.1. $d(x_1, G) + d(y_1, G) = 1$.
Without loss of generality we may suppose that $d(x_1, G) = 1$ and $d(y_1, G) = 0$. By the maximality of the sum (1) we have $d(u, G) \leq 1$ for every $u \in L$ and therefore $2p - 3 = \| G \| \leq p - 1$, contrary to $p \geq 4$.

Subcase 2.1.1.2. $d(x_1, G) + d(y_1, G) = 2$.

• $d(x_1, G) = d(y_1, G) = 1$.
Then the degree of each vertex in L which is not a neighbour of y_1 is 1 at the most. Denote by x_2 the neighbor of y_1. We have $2p - 3 = \| G \| \leq p - 2 + d(x_2, G)$. Hence $d(x_2, G) = p - 1$ and the theorem follows from Claim 1.

• $d(x_1, G) = 0,\ d(y_1, G) = 2$.
Then all the vertices of L which are not the neighbors of y_1 are isolated. Since $\| G \| = 2p - 3$ one of the two neighbors of y_1 has degree at least $p - 1$ and we may apply Claim 1.

Subcase 2.1.1.3. $d(x_1, G) + d(y_1, G) = 3$.

• $d(x_1, G) = 3,\ d(y_1, G) = 0$.

Note that in this subcase we have necessarily $p \geq 5$ (since in R, except of the vertices y and y_1 which are isolated, we have three neighbors of x_1). Let y_2, y_3 and y_4 be the neighbors of x_1. By the maximality of the sum (1) each vertex of R which is not a neighbor of x_1 is isolated. One of the vertices y_2, y_3, y_4 has the degree equal to 3 otherwise $2p-3 \leq 6$ and therefore $p \leq 4$, which is a contradiction. Without loss of generality we may suppose $d(y_2, G) = 3$. Note that now the vertices of L which are not the neighbors of y_2 are isolated in G. Hence $2p-3 = \| G \| \leq 9$ and, in consequence, either $p = 5$ or $p = 6$. If $p = 6$ then $G = K_{3,3} \cup \overline{K_{3,3}}$ and $H = C_8 \cup 2K_{1,1}$ (exceptional couple (4)). If $p = 5$ then $H = C_6 \cup 2K_{1,1}$ and G is one of two graphs G_1, G_2 depicted in Figure 4 (note that in G_2 there are two nonadjacent vertices $u \in L$ and $v \in R$ with degree sum equal to 4).

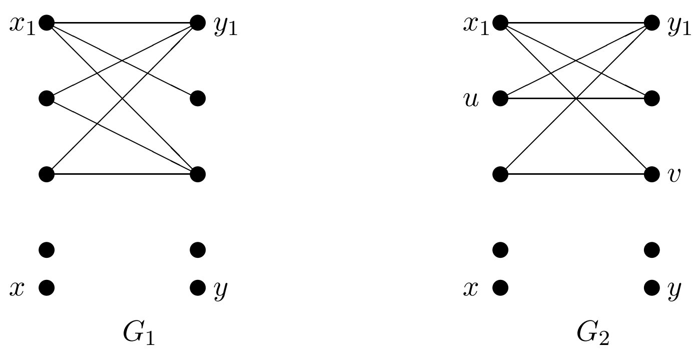


Figure 4. Two bi-placeable graphs with $G = C_6 \cup 2K_{1,1}$.

• $d(x_1, G) = 2$, $d(y_1, G) = 1$ and there is no vertex of degree greater than 2 in G.

In R there is one isolated vertex (the vertex y), one pendent vertex (the vertex y_1) and all remaining vertices have their degrees equal to 2. Hence $p = 4$ (otherwise there is a vertex $y' \in R$ such that $d(x_1, G) + d(y', G) = 4$ and x_1 and y' are nonadjacent, so Claim 2 is applicable), $G = \overline{K_{1,1}} \cup K_{1,1} \cup C_4$, $H = 2K_{1,1} \cup K_{2,2}$ and G and H are bi-placeable.

Subcase 2.1.2. $\delta_R(G) = 0$ and $\delta_L(G) = 1$.
Let $y \in R$ be an isolated vertex of G, $x_1 \in L$ a vertex of degree 1 and $y_1 \in R$ its neighbor in G. Let $x_2 \in L$ be a vertex not adjacent to y_1 such that the

sum

$$d(x_2, G) + d(y_1, G) \tag{2}$$

is maximum (note, that if $d(y_1, G) = p$ then G and H form an exceptional couple (2)).

Subcase 2.1.2.1. $d(x_2, G) + d(y_1, G) \geq 4$.
Then, by the induction hypothesis, either $G' = G - \{x_1, y, x_2, y_1\}$ and $H' = H - \{w, z, w_1, z_1\}$ are bi-placeable or G' and H' form an exceptional couple (1)–(4).

• If there is a bi-placement of G' and H', then it may be extended to a bi-placement of G and H by mapping $x_2 \mapsto w, y_1 \mapsto z, x_1 \mapsto w_1, y \mapsto z_1$.

• Suppose that $\Delta_L(G') = p - 2$. Let $x_3 \in L - \{x_1, x_2\}$ be a vertex of degree $p - 2$ in G'. Since $\delta_L(G) = 1$ and $(p - 2) + (p - 1) =\| G \|$, we have $d(x_3, G) = p - 2$. Moreover, since x_3 and y_1 are nonadjacent and, by the maximality of the degree sum (2), we have also $d(x_2, G) = p - 2$ and $2p - 3 =\| G \| \geq 2(p - 2) + p - 2$. This gives $p \leq 3$, a contradiction.

• Suppose that there is a vertex $y_2 \in R(G')$ such that $d(y_2, G') = p - 2$. If $d(y_2, G) = p - 1$ then $G - \{x_1, y_2\}$ and $H - \{w, z\}$ are bi-placeable by Theorem 1, and bi-placeability of G and H follows easily. So we may assume that x_2 and y_2 are nonadjacent. Since $d(x_2, G) \geq 1$, we have $\| G - \{x_2, y_2\} \| \leq p - 2$ and, again by Theorem 1, $G - \{x_2, y_2\}$ and $H - \{w, z\}$ are bi-placeable. $x_2 \mapsto w, y_2 \mapsto z$ extands any bi-placement of $G - \{x_2, y_2\}$ and $H - \{w, z\}$ to a bi-placement of G and H.

Note that, since H contains two independent edges, $H' \neq K_{2,2} \cup \overline{K_{1,1}}$. For $p - 2 = 6$ the vertices w_1 and z_1 may be chosen in such a way that $H' \neq C_8 \cup 2K_{1,1}$. Hence G' and H' may be supposed to form neither the exceptional couple (3) nor the exceptional couple (4).

Subcase 2.1.2.2. If $u, v \in L$ and $t \in R$ are such vertices of G that $d(u, G) = 1$, t is the neighbor of u and the vertices v and t are nonadjacent, then

$$d(v, G) + d(t, G) < 4. \tag{3}$$

• If $d(y_1, G) \geq 3$, then either $d(y_1, G) = p$ and G and H form an excluded couple, or there is a vertex $s \in L$ not adjacent to y_1. Since $\delta_L(G) \geq 1$, this contradicts (3).

• Suppose that $d(y_1, G) = 2$ and let x_3 denote the second neighbor of y_1. By (3) we have $d(a, G) \leq 1$ for every $a \in L - \{x_1, x_3\}$. Hence $d(a, G) = 1$ for every $a \in L - \{x_1, x_3\}$ and $2p - 3 = \| G \| = 1 + d(x_3, G) + p - 2 = d(x_3, G) + p - 1$ and therefore $d(x_3, G) = p - 2$.

Let $y_2 \in R$ be a vertex of the maximum degree in R, such that $y_2 \neq y_1$ (since $p \geq 4$ we check at once that such a vertex exists). We have $\| G - \{x_1, x_3, y, y_2\} \| \leq (2p - 3) - p = p - 3$, $\| H - \{w, z, w', z'\} \| \leq 2p - 4$ and, by Theorem 1, there is a bi-placement of $G - \{x_1, x_3, y, y_2\}$ and $H - \{w, z, w', z'\}$ which may be easily extended to a bi-placement of G and H.

• Hence we may suppose that the neighbor of every pendent vertex $u \in L$ is also pendent.

It is clear by (3), that for every $u \in L$ we have $d(u, G) \leq 2$. Since $\delta_L(G) \geq 1$, we have exactly three vertices of degree 1 in $L(G)$ while the remaining $p - 3$ vertices have their degree equal to 2. Let $x_1 \in L$, $y_1 \in R$ be two pendent vertices adjacent in G; $x_2 \in L$ such that $d(x_2, G) = 1$ and $y_3 \in R$ of maximum degree in R (note that $d(y_3, G) \geq 2$). In H we choose the vertices $w, w' \in L'$, $z \in R'$ (each of which has its degree equal to 1) and $z_1 \in R'$ with $d(z_1, H) = 2$. We have $\| G - \{x_1, y_1, x_2, y_3\} \| \leq \| G \| - 4 = 2(p - 2) - 3$ and $\| H - \{w, z, w', z_1\} \| \leq 2(p - 2)$. By the induction hypothesis $G' = G - \{x_1, y_1, x_2, y_3\}$ and $H' = H - \{w, z, w', z_1\}$ are bi-placeable (note that G' and H' are not an excluded couple). Every bi-placement of G' and H' may be extended to a bi-placement of G and H by mapping $x_1 \mapsto w'$, $x_2 \mapsto w$, $y_1 \mapsto z_1$, $y_3 \mapsto z$.

Subcase 2.1.3. There are no isolated vertices in $V(G)$ ($\delta(G) \geq 1$).
Let $x \in L$, $y \in R$ be nonadjacent pendent vertices in $V(G)$, $y_1 \in N(x, G)$, $x_1 \in N(y, G)$ and let $w, w_1 \in L'$, $z, z_1 \in R'$ be such that wz and $w_1 z_1$ are isolated edges in H.

Subcase 2.1.3.1. We can choose vertices x and y in such a way that $(x_1, y_1) \notin E$.

Put $G'_3 = G - \{x, y\}$ and $H'_3 = H - \{w, z_1\}$. Note that $\Delta(G'_3) < p - 1$, otherwise since $\delta(G) \geq 1$ we would have $\| G \| \geq 2(p - 1)$. For $p = 4$ we may choose x and y such that $G'_3 \neq 3\overline{K_{1,1}}$. It is also clear that $H'_3 \neq C_8 \cup 2\overline{K_{1,1}}$. Hence, by the induction hypothesis, there is a bi-placement of G'_3 and H'_3.

• If $f(x_1) \neq w_1$ and $f(y_1) \neq z$ then we extend f to a bi-placing of G and H by mapping $x \mapsto w,\ y \mapsto z_1$.

• If $f(x_1) = w_1$ and $f(y_1) = z$ then f_* defined by: $f_*(v) = f(v)$ for every $v \in V(G'_3) - \{x_1, y_1\}$, $f_*(x) = w, f_*(y_1) = z_1, f_*(y) = z$ and $f_*(x_1) = w_1$ is a desired bi-placement of G and H.

• If $f(x_1) = w_1$ and $f(y_1) = z' \neq z$ then there is a vertex $y' \in R(G'_3)$ such that $f(y') = z$. Define f_* by the formula $f_*(v) = f(v)$ for every $v \in V(G'_3) - \{y_1\}$, $f_*(x) = w, f_*(y_1) = z_1$ and $f_*(y) = z'$.

Subcase 2.1.3.2. For each choice of vertices x and y we have $(x_1, y_1) \in E$. If $d(x_1, G) = p$ or $d(y_1, G) = p$ then G and H are exceptional and the theorem is proved. So assume that $d(x_1, G) \leq p - 1$ and $d(y_1, G) \leq p - 1$. Note that $G'_3 = G - \{x, y\}$ and $H'_3 = H - \{w, z_1\}$ is not an exceptional couple of graphs hence, by induction hypothesis, there is a bi-placement of G'_3 and H'_3. If $f(x_1) \neq w_1$ and $f(y_1) \neq z$ we extend f to a bi-placement of G and H easily.

So, we suppose that $f(x_1) = w_1$ or $f(y_1) = z$. Without loss of generality we may assume that $f(x_1) = w_1$. Then there is a vertex $y_2 \in R - N(x_1, G)$ and a vertex $z_2 \in R(H'_3)$ such that $f(y_2) = z_2$. We map $y \mapsto z_2$, $y_2 \mapsto z_1$ and

• if $f(y_1) \neq z$ then $x \mapsto w$,

• if $f(y_1) = z$ then choose $x_2 \in L - N(y_1, G)$. Let $w_2 = f(x_2)$. Map $x \mapsto w_2,\ x_2 \mapsto w$.

Subcase 2.2. $|P_1| \geq 3$ or $|P_2| \geq 3$.

Subcase 2.2.1. There is an isolated vertex, say y, in $V(G)$.
Without loss of generality we may assume that $y \in R$. Let $x \in L$ and $d(x, G) = \Delta_L(G)$. There is a pendent vertex $w \in L'$ such that, if $z \in N(w, H)$ then $d(z, H) = 2$. If the graphs $G' = \{x, y\}$ and $H' = \{w, z\}$ are bi-placeable, then there is also a bi-placement of G and H. Note also, that the the couple G' and H' is neither exception (3) nor (4) of the theorem. Hence, by the induction hypothesis, $\Delta(G') = p-1$. Note that since $\Delta_L(G) = d(x, G)$ we have $\Delta(G') = \Delta_R(G')$, otherwise $\| G \| \geq 2(p-1)$, a contradiction.

Let $y_1 \in R(G')$ be a vertex of degree $p-1$ in G'. If $d(y_1, G) = p$ then G and H is an exceptional couple of graphs. For $d(y_1, G) = p-1$ define $G'' = G - \{x, x_1, y, y_1\}$ where $x_1 \in L(G)$ is a pendent vertex of G and $H'' = H - \{w_1, w_2, z_1, z_2\}$, where $w_1, w_2 \in L(H)$, $z_1, z_2 \in R(H)$, z_1 is pendent, w_1 is the neighbor of z_1, z_2 is a neighbor of w_1 if $d(w_1, G) = 2$, otherwise z_2 is any vertex of $R(G) - \{z_1\}$, and w_2 is any vertex of $L(G) - \{w_1\}$. We have $\| G'' \| \leq 2p-3-(p-1+2) < p-3$ and $\| H'' \| < 2(p-2)$ hence, by Theorem 1, G'' and H'' are bi-placeable. The mappings $x \mapsto w_1, x_1 \mapsto w_2, y_1 \mapsto z_1, y \mapsto z_2$ extend any bi-placement of G'' and H'' to a bi-placement of G and H.

Subcase 2.2.2. There is no isolated vertex in $V(G)$.
There are pendent vertices $x \in L$ and $y \in R$ such that $(x, y) \notin E$. Let y_1 be the neighbor of x and x_1 the neighbor of y in G.

It is easily seen that in H there are pendent vertices $w \in L'$ and $z \in R'$, such that their respective neighbors $z' \in R'$ and $w' \in L'$ have their degrees equal to 2. Note that the couple of graphs $G' = G - \{x, y\}$ and $H' = H - \{w, z'\}$ is not exceptional. Hence, by induction hypothesis, G' and H' are bi-placeable.

Let w_1 be the second neighbor of z' in H ($w_1 \neq w$). If there is a bi-placement f of G' and H' such that $f(x_1) \neq w_1$ then f may be extended by the mapping $x \to w$, $y \to z'$ to a bi-placement of G and H. Therefore we may assume that $f(x_1) = w_1$.

We shall prove that $d(x_1, G) = p-2$ and for every $v \in L - \{x_1\}$ $d(v, G) = 1$ (unless G and H are bi-placeable). It is clear that $d(x_1, G) \leq p-2$, since there is no isolated vertex in L and $\sum_{v \in L} d(v, G) = 2p-3$. Moreover, if $d(x_1, G) = p-2$ then all remaining vertices of L are pendent.

Suppose that $d(x_1, G) \leq p-3$. Then there is a vertex $y_2 \in R$ such that $y_2 \neq y_1$, $x_1 y_2 \notin E(G)$ and $f(x_1) f(y_2) \notin E(H)$ (we remember that w_1 has in H at most two neighbors). Let z'' denote the vertex $f(y_2)$ and define $f_* : V \to V'$ by the following formulas: $f_*(v) = f(v)$ for $v \neq x, y, y_2$, $f_*(x) = w$, $f_*(y_2) = z'$ and $f_*(y) = z''$. f_* is a bi-placement of G and H.

In the exactly the same way we prove that either G and H are bi-placeable, or $d(y_1, G) = p-2$.

Observe now that either

• x_1 and y_1 are adjacent and G is the union of two independent edges and two stars $K_{1,p-3}$ and $K_{p-3,1}$ with adjacent centers (see Figure 5a) or else

• x_1 and y_1 are nonadjacent and G is the union of two stars $K_{1,p-2}$, $K_{p-2,1}$ and an isolated edge (see Figure 5(b)).

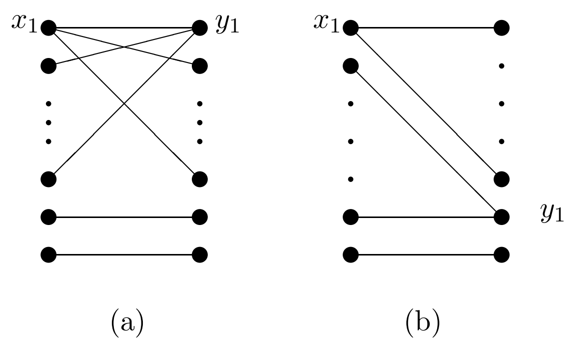


Figure 5

To finish the proof one may verify easily that then G and H (which is a union of two non-trivial paths and some cycles) are bi-placeable. ■

REFERENCES

[1] M. Aigner and S. Brandt, *Embedding arbitrary graphs of maximum degree two*, J. London Math. Soc. (2) **48** (1993) 39–51.

[2] D. Amar, I. Fournier and A. Germa, *Covering the vertices of a graph by cycles of prescribed length*, J. Graph Theory **13** (1989) 323–330.

[3] B. Bollobás, Extremal Graph Theory (Academic Press, London, 1978).

[4] P.A. Catlin, *Subgraphs of graphs*, I, Discrete Math. **10** (1974) 225–233.

[5] K. Corrádi and A. Hajnal, *On the maximal number of independent circuits in a graph*, Acta. Math. Acad. Sci. Hungar. **14** (1963) 423–439.

[6] M. El-Zahar, *On circuits in graphs*, Discrete Math. **50** (1984) 227–230.

[7] J.-L. Fouquet and A.P. Wojda, *Mutual placement of bipartite grahps*, Discrete Math. **121** (1993) 85–92.

[8] L. Lesniak, *Independent cycles in graphs*, J. Comb. Math. Comb. Comput. **17** (1995) 55–63.

[9] B. Orchel, *Placing bipartite graphs of small size* I, Folia Scientarum Universitatis Technicae Resoviensis **118** (1993) 51–58.

[10] H. Wang, *On the maximum number of independent cycles in a bipartite graph*, J. Combin. Theory (B) **67** (1996) 152–164.

[11] M. Woźniak, Packing of graphs (Dissertationes Mathematicae CCCLXII, Warszawa, 1997).

[12] H.P. Yap, *Packing of graphs — a survey*, Discrete Math. **72** (1988) 395–404.

Received 23 May 2008
Accepted 14 July 2008