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ON THE TREE GRAPH OF A CONNECTED GRAPH

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Abstract

Let G be a graph and C be a set of cycles of G. The tree graph of G defined by C, is the graph $T(G,C)$ that has one vertex for each spanning tree of G, in which two trees T and T' are adjacent if their symmetric difference consists of two edges and the unique cycle contained in $T \cup T'$ is an element of C. We give a necessary and sufficient condition for this graph to be connected for the case where every edge of G belongs to at most two cycles in C.

Keywords: tree graph, property Δ^*, property Δ^+.

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Introduction

The *tree graph* of a connected graph G is the graph $T(G)$ whose vertices are the spanning trees of G, in which two trees T and T' are adjacent if $T \cup T'$ contains a unique cycle. It is well-known that the $T(G)$ is always connected.

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Several variations of the tree graph have been studied; see for instance the *adjacency tree graph* studied by Zhang and Chen in [5] and by Heinrich and Liu in [3], and the *leaf exchange tree graph* of Broersma and Li [1] and Harary *et al.* [2].

Let G be a connected graph and C be a set of cycles of G. The *tree graph of* G *defined by* C is the spanning subgraph $T(G,C)$ of $T(G)$ in which two trees T and T' are adjacent if they are adjacent in $T(G)$ and the unique cycle contained in $T \cup T'$ is an element of C.

A *unicycle* U of G is a connected spanning subgraph of G that contains exactly one cycle. In [4], Li *et al.* defined the following property: A cycle σ of G has *property* Δ^* with respect to C if for any unicycle U of G containing σ, there are two cycles $\delta, \gamma \in C$, contained in $U+e$ for some edge e of G, such that $\sigma = \delta\Delta\gamma$. The *closure* $cl_G(C)$ of the set C is the set of cycles obtained from C by successively adding new cycles of G that satisfy property Δ^* until no such cycle remains. A set C is Δ^*-*dense* if $cl_G(C)$ is the set of all cycles of G. Li *et al.* proved that if $T(G,C)$ is connected, then C must span the cycle space of G and that if C is Δ^*-dense, then $T(G,C)$ is connected.

In this article we define a weaker property which we call Δ^+ and show that being Δ^+-dense is also a sufficient condition for $T(G,C)$ to be connected. We also prove that if $T(G,C)$ is connected and every edge of G lies in at most two cycles of C, then C is Δ^+-dense.

1. Δ^+-Dense: A New Sufficient Condition

We denote by $\Gamma(G)$ the cycle space of a graph G. Let σ be a cycle and U be a unicycle of G, we say that U is a σ-*unicycle* if the unique cycle contained in U is σ.

For an integer k, a cycle σ and a σ-unicycle U of G, (σ, U) has *property* Δ_k with respect to C if there exists a set of cycles $X = \{\sigma_0, \ldots, \sigma_k\} \subseteq C$ and a set of edges $Y = \{e_0, \ldots, e_{k-1}\} \subseteq E(G) \backslash E(U)$ such that the following conditions are satisfied:

($\mathcal{P}_1$) X is a basis of $\Gamma(U+Y)$.

($\mathcal{P}_2$) Let $\rho \in X$. For every edge $x \in E(\rho) \backslash E(\sigma)$, there exists a unique $\rho' \in X$ such that $\rho \neq \rho'$ and $x \in E(\rho')$.

($\mathcal{P}_3$) For every edge $x \in E(\sigma)$, there exists a unique $\rho \in X$ such that $x \in E(\rho)$.

We say a cycle σ of G has property Δ^+ with respect to C if for every σ-unicycle U of G, there exists a positive integer k such that (σ, U) has property Δ_k with respect to C.

The closure $cl_G(C)^+$ of the family C is the set of cycles obtained from C by successively adding new cycles of G that satisfy property Δ^+ until no such cycle remains. It is not difficult to prove that this closure is well defined. We say C is Δ^+-*dense* if $cl^+_G(C)$ is the set of all cycles of G.

Remark 1. If C is Δ^*-dense, then C is Δ^+-dense.

Note that σ has property Δ^* with respect to C if and only if (σ, U) has property Δ_1 with respect to C for every σ-unicycle U of G.

Lemma 1. *Let C be a family of cycles of a connected graph G, U be a σ-unicycle of G for some cycle $\sigma \notin C$ and k be a positive integer. If (σ, U) has property Δ_k with respect to C, then every pair of spanning trees of U is connected by a path in $T(G, C)$.*

Proof. By induction on k. The induction basis follows from the proof of Lemma 3.1 in [4]. Nevertheless, we will include a proof here.

Let U be a σ-unicycle of G such that (σ, U) has the property Δ_1 and T and T' be two spanning trees of U. There exists an edge $e \notin E(U)$ and two cycles $\delta, \gamma \in C$ of $U + e$ such that $\sigma = \delta \Delta \gamma$. Let a and b be edges of U such that $T' = (T - a) + b$.

If $a \in E(\delta)\backslash E(\gamma)$ and $b \in E(\gamma)\backslash E(\delta)$, let $Q = (T - a) + e$. It follows that $Q = (T' - b) + e$. Since the unique cycle of G contained in $Q \cup T$ is δ and the unique cycle of G contained in $Q \cup T'$ is γ, we have that Q is adjacent to T and to T' in $T(G, C)$. Thus, we have found the path $\{T, Q, T'\}$ from T to T' in $T(G, C)$.

If $a \in E(\gamma)\backslash E(\delta)$ and $b \in E(\delta)\backslash E(\gamma)$ we can interchange γ with δ in the previous argument.

If $a, b \in E(\delta)\backslash E(\gamma)$ let $c \in E(\gamma)\backslash E(\delta)$ and consider the spanning trees $Q = (T - c) + e$ and $Q' = (T' - c) + e$ of G. Notice that Q is adjacent to T in $T(G, C)$ because the unique cycle contained in $Q \cup T$ is γ and that Q' is adjacent to T' in $T(G, C)$ because the unique cycle contained in $T' \cup Q$ is also γ. Even more, Q is adjacent to Q' in $T(G, C)$ because $Q' = (Q - a) + b$ and the unique cycle contained in $Q \cup Q'$ is δ. Thus, $\{T, Q, Q', T'\}$ is a path connecting T and T' in $T(G, C)$.

Finally, if $a, b \in E(\gamma)\backslash E(\delta)$, we interchange γ with δ in the previous argument.

Suppose now that the result is true for every positive integer less than k. Let U be a σ-unicycle of G such that (σ, U) has the property Δ_k with respect to C. Then there exists a set of cycles $X = \{\sigma_0, \ldots, \sigma_k\} \subseteq C$ and a set of edges $Y = \{e_0, \ldots, e_{k-1}\}$ of G that satisfy $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$.

Let T and T' be spanning trees of U and $a, b \in E(U)$ be such that $T' = (T - a) + b$. Notice that $\sigma_0 \Delta \cdots \Delta \sigma_k = \sigma$ because every edge of σ is in exactly one cycle in X and every edge of a cycle in X that is not an edge of σ is an edge of exactly two cycles in X. Thus, there must exist at least one cycle $\rho \in X$ such that $E(\rho) \cap E(\sigma) \neq \emptyset$. It follows that at least one edge $e \in E(\rho) \cap Y$ is such that $U + e$ contains two cycles δ and γ such that $\delta \Delta \gamma = \sigma$. Without loss of generality we assume $e = e_{k-1}$.

Case 1. $a \in E(\delta) \backslash E(\gamma)$ and $b \in E(\gamma) \backslash E(\delta)$.

Subcase 1.1. $\delta \in X$.
We can rename the cycles in X in such a way that $\delta = \sigma_k$. Let $U' = (U - a) + e_{k-1}$, clearly U' is a γ-unicycle of G. We shall prove that (γ, U') has property Δ_{k-1}. Let $X' = \{\sigma_0, \ldots, \sigma_{k-1}\}$ and $Y' = \{e_0, \ldots, e_{k-2}\}$.

Let $x \in E(\rho)$, for $\rho \in X'$, then $x \in E(U) \cup Y = (E(U') \cup Y') \cup \{a\}$. Notice that $a \in E(\sigma_k) \cap E(\sigma)$, so the unique cycle in X that contains a is σ_k because σ, U, X and Y satisfy $(\mathcal{P}_3)$. It follows that $x \neq a$, so $x \in E(U') \cup Y'$. Then, every cycle in X' is a cycle of $U' + Y'$. The dimension of $\Gamma(U' + Y')$ is k, and X' is a linear independent set consisting of k cycles of $U' + Y'$, then X' is a basis of $\Gamma(U' + Y')$. Therefore γ, U', X' and Y' satisfy property $(\mathcal{P}_1)$.

Let $\rho \in X'$ and $x \in E(\rho) \backslash E(\gamma)$. If $x \in E(\sigma)$, then $x \in E(\sigma) \backslash E(\gamma)$. Since $\sigma = \gamma \Delta \sigma_k$ and σ, U, X and Y satisfy $(\mathcal{P}_3)$, the unique cycle in X that contains x is σ_k which is impossible because $\rho \neq \sigma_k$ and $\rho \in X$. It follows that $x \notin E(\sigma)$ and $x \notin E(\sigma_k)$ because $\sigma = \sigma_k \Delta \gamma$. Since σ, U, X and Y satisfy $(\mathcal{P}_2)$, there exists a unique $\rho' \in X$ such that $x \in E(\rho')$. Clearly $\rho' \neq \sigma_k$, so $\rho' \in X'$. Therefore γ, U', X' and Y' satisfy $(\mathcal{P}_2)$.

Let $x \in \sigma$. If $x \in E(\gamma)$, there exists a unique cycle $\rho \in X$ such that $x \in E(\rho)$ because σ, U, X and Y satisfy $(\mathcal{P}_3)$. Since $\sigma = \delta \Delta \gamma$, $\rho \neq \sigma_k$. It follows that $\rho \in X'$. If $x \in E(\gamma) \backslash E(\sigma)$, $x \in E(\sigma_k)$. We know that σ, U, X and Y satisfy $(\mathcal{P}_2)$ implying there exists a unique cycle $\rho \in X \backslash \{\sigma_k\} = X'$ containing x. Therefore γ, U', X' and Y' satisfy $(\mathcal{P}_3)$.

Since $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$ are satisfied, (γ, U') has property Δ_{k-1} and the inductive hypothesis applies. Let $Q = (T - a) + e_{k-1}$, since Q and T' are spanning trees of U', by induction Q and T' are joined by a path in $T(G, C)$.

On the other hand, T is adjacent to Q in $T(G,C)$ because δ is the unique cycle of $T \cup Q$. So, T and T' are also joined by a path in $T(G,C)$.

Subcase 1.2. $\gamma \in X$.
This case can be proved in the same way interchanging γ with δ and a with b.

Subcase 1.3. $\delta, \gamma \notin X$.
Since X is a basis of $\Gamma(U+Y)$, by reordering X we can also assume that there exists an integer $1 \leq r \leq k-2$ such that $\sigma_0 \Delta \cdots \Delta \sigma_r = \delta$. Since $\delta \Delta \gamma = \sigma = \sigma_0 \Delta \cdots \Delta \sigma_k$, it follows that $\gamma = \sigma_{r+1} \Delta \cdots \Delta \sigma_k$.

Let $A = \{\sigma_0, \ldots, \sigma_r\}$ and $B = \{\sigma_{r+1}, \ldots, \sigma_k\}$. If an edge $e_i \in Y \backslash \{e_{k-1}\}$ is such that $e_i \in E(\rho)$ for some cycle $\rho \in A$ and $e_i \in E(\rho')$ for some $\rho' \in B$, then $e_i \in E(\delta) \cap E(\gamma)$. Since $E(\delta) \cap E(\gamma) \subseteq E(U + e_{k-1})$, $e_i = e_{k-1}$, which is impossible. Thus, A and B induce a partition of the edges in $Y \backslash \{e_{k-1}\}$ and this partition is not trivial because every $\rho \in X$ contains at least one edge in $Y \backslash \{e_{k-1}\}$.

Then, we can reorder the edges of $Y \backslash \{e_{k-1}\}$ in such a way that there exists an integer $0 \leq t \leq k-2$, such that $A' = \{e_0, \ldots, e_{t-1}\}$ is the set of edges in $Y \backslash \{e_{k-1}\}$ contained in elements of A and $B' = \{e_t, \ldots, e_{k-2}\}$ is the set of edges in $Y \backslash \{e_{k-1}\}$ contained in elements of B.

Let $U_1 = (U-b) + e_{k-1}$ and $U_2 = (U-a) + e_{k-1}$. Notice that U_1 is a δ-unicycle and U_2 is a γ-unicycle. We will show that (δ, U_1) and (γ, U_2) have properties Δ_r and Δ_{k-r+1}, respectively.

It is not difficult to see that no cycle of A contains b and that no cycle of B contains a. That means that every cycle in A is contained in $U_1 + A'$ and every cycle in B is contained in $U_2 + B'$. It follows that A is a linear independent set of $\Gamma(U_1 + A')$ and B is a linear independent set of $\Gamma(U_2 + B')$, so $|A| \leq |A'|+1$ and $|B| \leq |B'|+1$. That is, $r+1 \leq t+1$ and $(k+1)-(r+1) \leq ((k-1)-t)+1$. Thus, $t \leq r \leq t$; this implies $r = t$.

Since the dimension of $\Gamma(U_1 + A')$ is $t+1$ and A is a linear independent set of $\Gamma(U_1 + A')$ with $t+1$ cycles, A is a basis of $\Gamma(U_1 + A')$. Therefore δ, U_1, A and A' satisfy $(\mathcal{P}_1)$. Analogously γ, U_2, B and B' satisfy $(\mathcal{P}_1)$.

Let $\rho \in A$ and $x \in E(\rho) \backslash E(\delta)$. If $x \in E(\sigma) \cap ((E(\rho) \backslash E(\delta))$, there exists a unique cycle in X that contains x, because σ, U, X and Y satisfy $(\mathcal{P}_3)$. Then, ρ is the unique cycle of X that contains x and therefore also the unique cycle of A that contains x. Thus, x is an edge of $\Delta_{\rho \in A} \rho = \delta$ which is impossible. It follows that $x \notin E(\sigma)$. Since σ, U, X and Y satisfy $(\mathcal{P}_2)$, there exists a unique $\rho' \in X$ such that $\rho' \neq \rho$ and $x \in E(\rho')$. It is clear that if $\rho' \in B$, then $x \in E(\gamma) \cap E(\delta)$ which is impossible. Thus, $\rho' \in A$.

Therefore δ, U_1, A and A' satisfy $(\mathcal{P}_2)$. Analogously γ, U_2, B and B' satisfy $(\mathcal{P}_2)$.

Let $x \in E(\delta)$. If $x \in E(\sigma)$, then there exists a unique cycle $\rho \in X$ such that $x \in E(\rho)$ because σ, U, X and Y satisfy $(\mathcal{P}_3)$. Since $x \in E(\delta)$, $\rho \in A$. If $x \notin E(\sigma)$, then $x \in E(\gamma) \cap E(\delta)$. Since $\delta = \Delta_{\rho \in A}\rho$, $x \in E(\rho)$ for some $\rho \in A$ and since $\gamma = \Delta_{\rho \in B}\rho$, $x \in E(\rho')$ for some $\rho' \in B$. There are at most two cycles in X containing x because (σ, U) has property Δ_k with respect to C. Thus, if $x \in E(\delta)$, there exists a unique cycle $\rho \in A$ such that $x \in E(\rho)$. Therefore δ, U_1, A and A' satisfy $(\mathcal{P}_3)$. Analogously γ, U_2, B and B' satisfy $(\mathcal{P}_3)$.

We can now apply the inductive hypothesis to (δ, U_1) and to (γ, U_2).

Let $R = (T - a) + e_{k-1} = (T' - b) + e_{k-1}$. Since R is a spanning tree of both U_1 and U_2, there exists a path in $T(G, C)$ from T to R and a path from R to T'. Therefore T and T' can also be joined by a path in $T(G, C)$.

Case 2. $a \in E(\gamma) \backslash E(\delta)$ and $b \in E(\delta) \backslash E(\gamma)$.
This case can be proved as Case 1 by interchanging a with b.

Case 3. $a, b \in E(\delta) \backslash E(\gamma)$.
Consider an edge $c \in E(\gamma) \backslash E(\delta)$ and let $Q = (T - a) + c = (T' - b) + c$. Notice that Q is a spanning tree of U. Applying Case 1 to T and Q and to Q and T', we have that T and Q are joined by a path in $T(G, C)$ and also there exists a path between Q and T' in $T(G, C)$. It follows that there exists a path from T to T' in $T(G, C)$.

Case 4. $a, b \in E(\gamma) \backslash E(\delta)$.
The proof of this case is analogous to that of Case 3 by interchanging δ with γ. ∎

Theorem 1. *Let C be a set of cycles of a connected graph G. The graph $T(G, C)$ is connected if and only if $T(G, cl_G^+(C))$ is connected.*

Proof. If $T(G, C)$ is connected then $T(G, cl_G^+(C))$ is connected because $C \subseteq cl_G^+(C)$.

Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the sequence of cycles added to C while obtaining $cl_G^+(C)$ and let $C_0 = C$ and $C_i = C \cup \{\sigma_1, \sigma_2, \ldots, \sigma_i\}$ for $1 \leq i \leq n$. Let $1 \leq i \leq n$. Since σ_i has property Δ^+ with respect to C_{i-1}, for every σ_i-unicycle U of G there exists a positive integer k such that (σ_i, U) has property Δ_k with respect to C_{i-1}.

Suppose that $T(G, C_i)$ is connected and let T and T' be two spanning trees of G adjacent in $T(G, C_i)$. If T and T' are not adjacent in $T(G, C_{i-1})$, then $T \cup T'$ is a σ_i-unicycle. By Lemma 1, there exists a TT'-path in $T(G, C_{i-1})$. Therefore if $T(G, C_i)$ is connected, then $T(G, C_{i-1})$ is connected. Clearly this implies that if $T(G, cl_G^+(C))$ is connected, then $T(G, C)$ is connected. ■

Corollary 1. *If C is Δ^+-dense, then $T(G, C)$ is connected.*

Proof. If C is Δ^+-dense, then $T(G, cl_G^+(C)) = T(G)$ which is always connected. By Theorem 1, $T(G, C)$ is connected. ■

2. Main Result

Let G be a connected graph and C be a family of cycles of G such that $T(G, C)$ is connected. For any spanning trees T and T' of G, let $d_C(T, T')$ denote the distance between T and T' in $T(G, C)$. For any edge x of G, we denote by A_x the set of spanning trees of G containing x. For a spanning tree T of G such that $x \notin E(T)$, the distance $d_C(T, A_x)$ from T to A_x is the minimum distance $d_C(T, R)$ with $R \in A_x$.

Lemma 2. *Let C be a family of cycles of a connected graph G such that every edge of G is in at most two cycles of C. Let U be a σ-unicycle of G for some cycle $\sigma \notin C$. If $T(G, C)$ is connected, then (σ, U) has property Δ_k for some positive integer k.*

Proof. Let T be a spanning tree of U, and let x be the unique edge of U not in T. Since $T(G, C)$ is connected, there exists a spanning tree T_x such that the path between T and T_x has length $d = d_C(T, A_x)$. Since $\sigma \notin C$, $d \geq 2$. We shall prove by induction on d that (σ, U) has property Δ_k for some integer k.

Suppose $d = 2$ and let $S \notin A_x$ be such that $\{T, S, T_x\}$ is a path in $T(G, C)$. Since T is adjacent to S in $T(G, C)$, there exist $a \in E(T) \backslash E(S)$, $e \in E(S) \backslash E(T)$ and $\delta \in C$ such that $S = (T - a) + e$ and $S \cup T$ is a δ-unicycle of G. On the other hand, since S is adjacent to T_x in $T(G, C)$, there exist $b \in E(S) \backslash E(T_x)$ and $\gamma \in C$ such that $T_x = (S - b) + x$ and $T_x \cup S$ is a γ-unicycle of G.

Notice that $x \in E(\gamma) \backslash E(\delta)$ since $S \notin A_x$. Thus $\delta \neq \gamma$ and $U + e$ contains three different cycles: δ, γ and σ. Therefore $\sigma = \delta \Delta \gamma$. We conclude that (σ, U) has property Δ_1.

We proceed by induction assuming $d \geq 3$ and that (γ, V) has property Δ_s for some positive integer s for each cycle $\gamma \notin C$ and each γ-unicycle V of G whenever there exists a spanning tree R of V such that $R + z = V$ for some edge $z \in E(V)\backslash E(R)$ and $d_C(R, A_z) < d$.

Let $\{T = T_0, T_1, \ldots, T_n = T_d\}$ be a TT_x-path of length d in $T(G, C)$. Since T and T_1 are adjacent in $T(G, C)$, there exists $a \in E(T)\backslash E(T_1)$, $b \in E(T_1)\backslash E(T)$ and a cycle $\delta \in C$ such that $T_1 = (T - a) + b$ and $T \cup T_1$ is a δ-unicycle of G. Let γ be the unique cycle in $V = T_1 + x$.

Since $x \notin E(T_1)$, $d_C(T_1, A_x) = d-1$. By the inductive hypothesis, (γ, V) has property Δ_s for some integer s. Therefore there exists a set of cycles $X_1 = \{\gamma_0, \ldots, \gamma_s\} \subseteq C$ and a set of edges $Y_1 = \{e_0, \ldots, e_{s-1}\} \subseteq E(G)\backslash E(V)$ such that γ, V, X_1 and Y_1 satisfy $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$.

Case 1. $\gamma = \sigma$.
If $b \notin E(\rho)$ for any $\rho \in X_1$ then $E(\rho) \subseteq (E(V)\backslash\{b\}) \cup E(Y_1) = E(U) \cup E(Y_1)$. So (σ, U) has the property Δ_s because σ, U, X_1 and Y_1 satisfy $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$.

Then, we can assume $b \in E(\rho)$ for some $\rho \in X_1$. Clearly $b \notin E(\sigma)$ because $b \notin E(U)$ and U contains σ. Since γ, V, X_1 and Y_1 satisfy $(\mathcal{P}_2)$ there exists $\rho' \in X_1$ such that $b \in E(\rho')$. Since b is also an edge of δ and b is in at most two cycles of C, without loss of generality we can assume $\rho = \delta$. Since $a \in E(\delta)\backslash E(V)$, $a = e_t$ for some $0 \leq t \leq s-1$. Then, it is not difficult to prove that σ, U, X and Y satisfy $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$, where $X = X_1$ and $Y = (Y_1\backslash\{a\}) \cup \{b\}$. This also implies that (σ, U) has property Δ_s.

Case 2. $\rho \neq \sigma$.
In this case $U + b$ contains exactly three cycles σ, δ, γ. Since $b \notin E(\sigma)$, $b \in E(\delta) \cap E(\gamma)$ and $\delta \Delta \gamma = \sigma$.

Subcase 2.1. $a \notin E(\rho)$ for any $\rho \in X$.
Let $X = X_1 \cup \{\delta\}$ and $Y = Y_1 \cup \{a\}$. Since δ is the unique cycle in X containing a, then X is a linear independent set of $\Gamma(G)$. Note that every cycle in X is a cycle of $U + Y$ and $\Gamma(U + Y)$ has dimension $s + 1$, then X is a basis of $\Gamma(U + Y)$.

Let $x \in E(\rho)\backslash E(\sigma)$ for some $\rho \in X$. If $x \notin E(\gamma)$, then there exists $\rho' \in X_1$ such that $x \in E(\rho')$ because γ, V, X_1 and Y_1 satisfy $(\mathcal{P}_2)$. If $x \in E(\gamma)$ then $x \in E(\delta)$ because $\sigma = \delta\Delta\gamma$. It follows that if $x \in E(\rho)$, then x lies in at least two cycles of X. Since every edge is in at most two cycles of C, x belongs to exactly two cycles of X. In other words, if $x \in E(\rho)$ for some $\rho \in X'$, then there exists a unique cycle $\rho' \in X$ such that $x \in E(\rho')$.

Let $x \in E(\sigma)$. If $x \notin E(\delta)$, then $x \in E(\gamma)$ because $\sigma = \delta\Delta\gamma$. Since γ, V, X_1, and Y_1 satisfy $(\mathcal{P}_3)$, there exists a unique cycle $\rho \in X_1$ such that $x \in E(\rho)$. Thus there exists a unique cycle $\rho \in X$ such that $x \in E(\rho)$ because $x \notin E(\delta)$.

Now suppose that $x \in E(\delta)$, then $x \notin E(\gamma)$ because $\sigma = \sigma\Delta\gamma$. If $x \in E(\rho)$ for some $\rho \in X_1$, then there exists $\rho' \in X_1$ such that $x \in E(\rho')$ because γ, V, X_1 and Y_1 satisfy $(\mathcal{P}_2)$. Then x is an edge of δ, ρ and ρ' which is impossible because x can not be an edge of three cycles of C. Thus, the unique cycle in X containing x is δ.

Therefore σ, U, X and Y satisfy $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$. This implies that (σ, U) has property Δ_{s+1}.

Subcase 2.2. $a \in E(\rho)$ for some $\rho \in X$.
In this case we can suppose w.l.o.g that $a = e_{s-1}$. By property Δ_s of (γ, V), the edge a is in exactly two cycles of X_1. Since $a \in E(\delta)$ and a is in at most two cycles of C, then we can assume $\delta = \gamma_s$. Let $X = X_1 \backslash \{\gamma_s\}$ and $Y = Y_1 \backslash \{e_{s-1}\}$.

Since $b \in E(\delta) \backslash E(\sigma)$ and $\sigma = \delta\Delta\gamma$, then $b \in E(\delta) \cap E(\gamma)$. Thus, there exists a unique cycle $\rho \in X_1$ containing b because γ, V, X_1 and Y_1 satisfy $(\mathcal{P}_3)$. Therefore, $\delta = \rho$. It follows that $E(\rho) \subseteq E(V + Y_1) = (E(U) \cup E(Y)) \backslash \{b\}$ for every $\rho \in X$. Then, it is clear that X is a basis of $\Gamma(U + Y)$.

Let $x \in E(\rho) \backslash E(\sigma)$ for some $\rho \in X$. Since every edge in $E(\gamma) \backslash E(\sigma)$ is an edge of δ, it follows that $E(\gamma) \cap E(\rho) = \emptyset$ because γ, V, X_1 and Y_1 satisfy $(\mathcal{P}_3)$. Thus, $x \notin E(\gamma)$. It follows that there exists a unique $\rho' \in X_1$ such that $x \in E(\rho')$ because V, X_1 and Y_1 satisfy $(\mathcal{P}_2)$. If $\rho' = \delta$, then $x \in E(\delta) \backslash (E(\sigma) \cup E(\gamma))$ which is impossible because $\sigma = \delta\Delta\gamma$. Therefore $\rho' \in X$.

Let $x \in E(\sigma)$. If $x \in E(\sigma) \cap E(\gamma)$, there exists a unique $\rho \in X_1$ such that $x \in E(\rho)$ because γ, V, X_1 and Y_1 satisfy $(\mathcal{P}_3)$. Clearly $\rho \neq \delta$ because $E(\sigma) \cap E(\gamma) \cap E(\delta) = \emptyset$. If $x \in E(\sigma) \backslash E(\gamma) = E(\sigma) \cap E(\delta)$. Since $\delta \in X_1$ and γ, V, X_1 and Y_1 satisfy $(\mathcal{P}_2)$, there exists a unique $\rho' \in X_1$, $\rho' \neq \delta$ such that $x \in E(\rho')$. Thus, we have proved that there exists a unique $\rho \in X$ such that $x \in E(\rho)$.

Therefore σ, U, X and Y satisfy $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$, which, in this case, implies that (σ, U) has property Δ_{s-1}. ∎

Theorem 2. *Let C be a family of cycles of G such that every edge of G lies in at most two cycles of C. The graph $T(G,C)$ is connected if and only if every cycle $\sigma \notin C$ has property Δ^+ with respect to C.*

Proof. If every cycle $\sigma \notin C$ has property Δ^+ with respect to C, then C is Δ^+-dense. By Corollary 1, $T(G,C)$ is connected.

Conversely, let σ be a cycle of G not in C. By Lemma 2, if $T(G,C)$ is connected, then for each σ-unicycle of G, (σ, U) has property Δ_k for some integer k. This implies that σ has property Δ^+ with respect to C.

■

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