

## THE SIGNED MATCHINGS IN GRAPHS

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### Abstract

Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . A *signed matching* is a function  $x : E(G) \rightarrow \{-1, 1\}$  satisfying  $\sum_{e \in E_G(v)} x(e) \leq 1$  for every  $v \in V(G)$ , where  $E_G(v) = \{uv \in E(G) \mid u \in V(G)\}$ . The maximum of the values of  $\sum_{e \in E(G)} x(e)$ , taken over all signed matchings  $x$ , is called the *signed matching number* and is denoted by  $\beta'_1(G)$ . In this paper, we study the complexity of the maximum signed matching problem. We show that a maximum signed matching can be found in strongly polynomial-time. We present sharp upper and lower bounds on  $\beta'_1(G)$  for general graphs. We investigate the sum of maximum size of signed matchings and minimum size of signed 1-edge covers. We disprove the existence of an analogue of Gallai's theorem. Exact values of  $\beta'_1(G)$  of several classes of graphs are found.

**Keywords:** signed matching, signed matching number, maximum signed matching, signed edge cover, signed edge cover number, strongly polynomial-time.

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### 1. INTRODUCTION

Structural and algorithmic aspects of covering vertices by edges have been extensively studied in graph theory. A *matching* (*edge cover*) of a graph  $G$  is a set  $C$  of edges of  $G$  such that each vertex of  $G$  is incident to at most (at least) one edge of  $C$ . Let  $b$  be a fixed positive integer. A *simple*

*b*-matching (simple *b*-edge cover) of a graph  $G$  is a set  $C$  of edges of  $G$  such that each vertex of  $G$  is incident to at most (at least)  $b$  edges of  $C$ . The maximum (minimum) size of a simple *b*-matching (simple *b*-edge cover) of  $G$  is called *b*-matching number (*b*-edge cover number), denoted by  $\beta_b(G)$  ( $\rho_b(G)$ ). The (simple) *b*-matching problems have been widely studied in, for instance, [1, 5, 6, 8]. For an excellent survey of results on matchings, edge covers, *b*-matchings and *b*-edge covers, see Schrijver [9].

Let  $G$  be a graph with vertex set  $V(G)$ , edge set  $E(G)$  and maximum degree  $\Delta(G)$ . For a vertex  $v \in V(G)$ , let  $E_G(v) = \{uv \in E(G) : u \in V(G)\}$  denote the set of edges of  $G$  incident to  $v$ . The degree,  $d(v)$ , of  $v$  in  $G$  is  $|E_G(v)|$ . For a vertex  $v \in V(G)$ ,  $v$  is called *odd* (*even*) if  $d(v)$  is odd (even). A *k*-factor of  $G$  is a  $k$ -regular spanning subgraph of  $G$ . In particular,  $F$  is a 1-factor of  $G$  if and only if  $E(F)$  is a *perfect matching* in  $G$ . Let  $f : E(G) \rightarrow \mathbb{R}$  be a real-valued function. For  $X \subseteq E(G)$ , we write  $f(X)$  for  $\sum_{e \in X} f(e)$ . We use  $G \cup H$  to denote the union of two disjoint graphs  $G$  and  $H$ , and we use  $G \cong H$  to denote that  $G$  and  $H$  are isomorphic. We denote by  $\mathbb{N}$  the set of positive integers.

In this paper, we consider a variant of the standard matching problem. A *signed matching* of a graph  $G$  is a function  $x : E(G) \rightarrow \{-1, 1\}$  satisfying  $x(E_G(v)) \leq 1$  for every  $v \in V(G)$ . The maximum of the values of  $x(E(G))$ , taken over all signed matchings  $x$ , is called the *signed matching number* and is denoted by  $\beta'_1(G)$ . A *maximum signed matching* is a signed matching  $x$  satisfying  $x(E(G)) = \beta'_1(G)$ .

Let  $b$  be a fixed positive integer. A *signed b-edge cover* is a function  $x : E(G) \rightarrow \{-1, 1\}$  satisfying  $x(E_G(v)) \geq b$  for every  $v \in V(G)$ . The minimum of the values of  $x(E(G))$ , taken over all signed *b*-edge covers  $x$ , is called the *signed b-edge cover number* and is denoted by  $\rho'_b(G)$ .

The signed *b*-edge cover problem has been studied in [2]. In the special case when  $b = 1$ ,  $\rho'_b$  is the *signed star domination number* investigated in [10, 12]. Other signed edge dominations have been investigated in [4, 7, 11, 13].

Note that  $\beta'_1(G \cup H) = \beta'_1(G) + \beta'_1(H)$  and  $\rho'_b(G \cup H) = \rho'_b(G) + \rho'_b(H)$  for two disjoint graphs  $G$  and  $H$ , and  $\beta'_1(G) = |E(G)|$  for the graphs  $G$  with  $\Delta(G) \leq 1$ . Hence, we may assume that all graphs in this paper are connected with maximum degree greater than 1.

In Section 2, we investigate the complexity of the maximum signed matching problem. We show that a maximum signed matching can be found in polynomial time. In Section 3, we investigate the sum of maximum size

of signed matchings and minimum size of signed 1-edge covers. We disprove the existence of an analogue of Gallai's theorem. In Section 4, we present sharp bounds on  $\beta'_1(G)$  for general graphs. In Section 5, we study  $\beta'_1(G)$  for some classes of graphs. Exact values of  $\beta'_1(G)$  for paths, cycles, complete graphs and complete bipartite graphs are found.

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. For all graph-theoretic terminology not defined here, the reader is directed to [3].

## 2. THE COMPLEXITY OF THE MAXIMUM SIGNED MATCHING PROBLEM

An algorithm is said to run in *strongly polynomial time* if the number of elementary arithmetic and other operations is bounded by a fixed polynomial in the size of the input, where any number in the input is counted only for 1. Strongly polynomial time is of relevance only for algorithms that have numbers among their input; otherwise, strongly polynomial time coincides with the more well-known polynomial time. For more background on strongly polynomial time, the reader is referred to [9]. Our main result for this section is the following theorem.

**Theorem 1.** *A maximum signed matching can be found in strongly polynomial time.*

For the proof of Theorem 1, we use the following result from [9].

**Theorem 2.** *Let  $b : V(G) \rightarrow \mathbb{N}$  be an integer-valued function. A maximum  $b$ -matching can be found in strongly polynomial time.*

**Proof of Theorem 1.** We may formulate the maximum signed matching problem to the following.

$$(2.1) \quad \begin{aligned} &\text{Maximize } \sum_{uv \in E(G)} x_{uv} \\ &\sum_{uv \in E_G(u)} x_{uv} \leq 1, \text{ for every } u \in V(G), \\ &x_{uv} \in \{-1, 1\}, \text{ for every } uv \in E(G). \end{aligned}$$

Notice that  $x_{uv} \in \{-1, 1\}$  for each  $uv \in E(G)$ . So, if  $d(u)$  is even for some  $u \in V(G)$ , then  $\sum_{uv \in E_G(u)} x_{uv} \leq 1$  implies that  $\sum_{uv \in E_G(u)} x_{uv} \leq 0$ .

Thus, (2.1) is equivalent to the following:

$$\begin{aligned}
 & \text{Maximize } \sum_{uv \in E(G)} x_{uv} \\
 & \sum_{uv \in E_G(u)} x_{uv} \leq 1, \text{ for every odd } u \in V(G), \\
 & \sum_{uv \in E_G(u)} x_{uv} \leq 0, \text{ for every even } u \in V(G), \\
 & x_{uv} \in \{-1, 1\}, \text{ for every } uv \in E(G).
 \end{aligned}
 \tag{2.2}$$

Now we define  $y_{uv} = \frac{1}{2}(1 + x_{uv})$  for each  $uv \in E(G)$ . It is clear that  $y_{uv} \in \{0, 1\}$  for each  $uv \in E(G)$ . Moreover, (2.2) is equivalent to the following:

$$\begin{aligned}
 & \text{Maximize } -|E(G)| + 2 \sum_{uv \in E(G)} y_{uv} \\
 & \sum_{uv \in E_G(u)} y_{uv} \leq \frac{1}{2}(1 + d(u)), \text{ for every odd } u \in V(G), \\
 & \sum_{uv \in E_G(u)} y_{uv} \leq \frac{1}{2}d(u), \text{ for every even } u \in V(G), \\
 & y_{uv} \in \{0, 1\}, \text{ for every } uv \in E(G).
 \end{aligned}
 \tag{2.3}$$

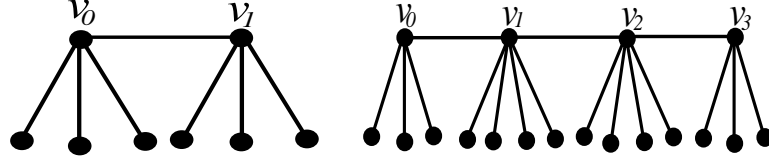
Define

$$b(u) = \begin{cases} \frac{1 + d(u)}{2}, & d(u) \equiv 1 \pmod{2}, \\ \frac{d(u)}{2}, & d(u) \equiv 0 \pmod{2}. \end{cases}$$

By Theorem 2, we know that (2.3) is polynomial-time solvable. ■

### 3. $\rho'_1(G) + \beta'_1(G)$ FOR GENERAL GRAPHS

Let  $k \geq 0$  be any integer. We define a family of trees named  $T_k$  as follows. Let  $v_0, v_1, \dots, v_{k+1}$  be a path with length  $k + 1$ . For each of the vertices  $v_0$  and  $v_{k+1}$ , we add three adjacent leaves. For each of the remaining vertices (if exist) on the path, we add four adjacent leaves. Clearly,  $T_k$  is a tree of order  $5k + 8$ . See Figure 1 for drawings of such trees  $T_0$  and  $T_2$ .

Figure 1. The trees  $T_0$  and  $T_2$ .

The following result (see [9, Theorem 34.1, p. 165, Vol. A]) is a direct analogue of Gallai's theorem, relating maximum-size  $b$ -matchings and minimum-size  $b$ -edge covers.

**Theorem 3.** *Fix  $b$  a positive integer. If  $G$  is a graph of order  $n$  having no isolated vertices, then*

$$\rho_b(G) + \beta_b(G) = bn.$$

Surprisingly, our next theorem exhibits that there is no such analogue of Gallai's theorem relating maximum-size signed matchings and minimum-size signed 1-edge covers.

**Theorem 4.** *Let  $k \geq 0$  be any integer. For the graph  $T_k$  of order  $n = 5k + 8$ ,*

$$\rho'_1(T_k) + \beta'_1(T_k) = 4k + 6.$$

**Proof.** To show that  $\rho'_1(T_k) + \beta'_1(T_k) = 4k + 6$ , we will show  $\rho'_1(T_k) = 3k + 5$  and  $\beta'_1(T_k) = k + 1$  in the following.

Let  $v_0, v_1, \dots, v_{k+1}$  be a path with length  $k + 1$  in  $T_k$ . Notice that  $T_k$  has  $4k + 6$  leaves. Let  $x$  be a signed 1-edge cover of  $T_k$  such that  $x(E(T_k)) = \rho'_1(T_k)$ . By the definition of  $\rho'_1(G)$ , we must assign 1 to every leaf of  $T_k$ . To minimize  $x(E(T_k))$ , we should assign -1 to each edge  $v_i v_{i+1}$  where  $0 \leq i \leq k$ . Hence,  $\rho'_1(T_k) = 3k + 5$ .

Now we prove  $\beta'_1(T_k) = k + 1$ . Let  $y$  be a signed matching of  $T_k$ . Notice that the vertex  $v_i$  has an even degree for every  $0 \leq i \leq k + 1$ . Hence,  $y(E_{T_k}(v_i)) \leq 0$  for every  $0 \leq i \leq k + 1$ . By simple calculations, we know that

$$\begin{aligned}
y(E(T_k)) &= \sum_{i=0}^{k+1} y(E_{T_k}(v_i)) - \sum_{i=0}^k y(v_i v_{i+1}) \\
&\leq - \sum_{i=0}^k y(v_i v_{i+1}) \leq k+1.
\end{aligned}$$

Hence,  $\beta'_1(T_k) \leq k+1$ . To show that  $\beta'_1(T_k) \geq k+1$ , it suffices to produce a signed matching  $y'$  of  $T_k$  such that  $y'(E(T_k)) = k+1$ . We assign  $y'(v_i v_{i+1}) = -1$  for  $0 \leq i \leq k$ . For each  $v_i$ ,  $0 \leq i \leq k+1$ , note that  $v_i$  is even, it is possible to assign -1 and 1 appropriately so that  $y'(E_{T_k}(v_i)) = 0$ . It is not hard to verify that  $y'$  is a signed matching of  $T_k$  satisfying  $y'(E(T_k)) = k+1$ , which completes the proof. ■

It would be interesting to determine sharp lower and upper bounds on  $\rho'_1(G) + \beta'_1(G)$  for general graphs  $G$ . We leave this as an open problem.

#### 4. LOWER AND UPPER BOUNDS

In this section, we present sharp lower and upper bounds on the signed matching number of general graphs.

**Theorem 5.** *Let  $G$  be a graph of order  $n$  with  $k$  odd vertices. Then*

$$\beta'_1(G) \leq k/2.$$

*The bound is sharp by Theorem 9.*

**Proof.** Let  $S \subseteq V(G)$  be the set of odd vertices. By the fact that every graph has even number of odd vertices, so  $|S| = k$  is even.

Taking any signed matching  $x$  of  $G$ , by the definition, we have that for every  $v \in S$ ,  $x(E_G(v)) \leq 1$ , and otherwise  $x(E_G(v)) \leq 0$ . Thus,  $\sum_{v \in V(G)} x(E_G(v)) \leq k$ , i.e.,  $2x(E(G)) \leq k$ . The proof is now complete. ■

**Theorem 6.** *For any graph  $G$  of order  $n$ ,*

$$\beta'_1(G) \geq -1.$$

*The bound is sharp.*

**Proof.** To start our proof, we construct a graph  $H$  from  $G$  as follows. If  $G$  is eulerian, then  $H = G$ . If not, as every graph has even number of odd vertices (say  $G$  has  $k$  odd vertices), then  $H$  is obtained by adding  $\frac{k}{2}$  new vertices  $w_1, \dots, w_{\frac{k}{2}}$  to  $G$ , and joining each  $w_i$  to two distinct odd vertices of  $G$ , so that for all  $i \neq j$  the vertices  $w_i$  and  $w_j$  have distinct neighbours. It is clear that  $H$  has no odd vertices and each  $w_i$  ( $1 \leq i \leq \frac{k}{2}$ ) has degree 2. By Theorem 4.1 on page 94 of [3],  $H$  is eulerian. Let  $C$  be an eulerian circuit of  $H$ . We start with an edge  $e \in E(G)$  and assign values  $-1$  and  $+1$  alternately along  $C$ . This defines a function  $x : E(H) \rightarrow \{-1, 1\}$ .

It is not hard to see that for every  $v \in V(H)$ ,

$$(4.1) \quad x(E_H(v)) \leq 0,$$

and

$$(4.2) \quad x(E(G)) = x(E(H)) \geq -1.$$

Let  $x'$  be the restriction of  $x$  on  $E(G)$ , i.e.,  $x'(e) = x(e)$  for every  $e \in E(G)$ . By (4.1) and (4.2), we have that for every  $v \in V(G)$ ,  $x'(E_G(v)) \leq 1$ , and  $x'(E(G)) \geq -1$ . Hence,  $x'$  is a signed matching of  $G$  satisfying that  $x'(E(G)) \geq -1$ . So,  $\beta'_1(G) \geq -1$ . As  $\beta'_1(C_{2k+1}) = -1$  for any positive integer  $k$ , the bound is sharp. ■

The following result follows immediately from Theorems 5 and 6.

**Corollary 7.** *Let  $G$  be eulerian of order  $n$  and size  $m$ . Then*

$$\beta'_1(G) = \frac{1}{2}((-1)^m - 1).$$

## 5. EXACT VALUES FOR CLASSES OF GRAPHS

The following theorem is an easy exercise and is left to the reader.

**Theorem 8.** *Let  $n \geq 3$  be an integer. We have*

$$(1) \quad \beta'_1(C_n) = \begin{cases} -1, & n \equiv 1 \pmod{2}, \\ 0, & n \equiv 0 \pmod{2}. \end{cases}$$

$$(2) \quad \beta'_1(P_n) = \begin{cases} 0, & n \equiv 1 \pmod{2}, \\ 1, & n \equiv 0 \pmod{2}. \end{cases}$$

Our next theorems provide the exact values for complete graphs and complete bipartite graphs.

**Theorem 9.** *Let  $n \geq 2$  be an integer. We have the following*

$$\beta'_1(K_n) = \begin{cases} n/2, & n \equiv 0, 2 \pmod{4}, \\ 0, & n \equiv 1 \pmod{4}, \\ -1, & n \equiv 3 \pmod{4}. \end{cases}$$

**Proof.** We consider three cases.

*Case 1.*  $n = 2k$  for some positive integer  $k$ .

By Theorem 9.19 on page 273 of [3],  $K_n$  is 1-factorable. Note that  $K_n$  can be decomposed into  $(2k - 1)$  1-factors. So, we can assign  $x(e) = 1$  for each edge  $e$  of  $k$  1-factors and assign  $x(e') = -1$  for each edge  $e'$  of the remaining  $k - 1$  1-factors. It is straightforward to verify that  $x$  is a signed matching of  $K_n$  and  $x(E(K_n)) = k = \frac{n}{2}$ . Hence,  $\beta'_1(K_n) \geq \frac{n}{2}$ . Note that  $\beta'_1(K_n) \leq \frac{n}{2}$  by Theorem 5. Thus,  $\beta'_1(K_n) = \frac{n}{2}$ .

*Case 2.*  $n = 4k + 1$  for some positive integer  $k$ .

It is not hard to see that  $K_n$  is Eulerian with size  $m = 2k(4k + 1)$ . It follows from Corollary 7 that  $\beta'_1(K_n) = 0$ .

*Case 3.*  $n = 4k + 3$  for some positive integer  $k$ .

Similar to Case 2,  $K_n$  is Eulerian with size  $m = (2k + 1)(4k + 3)$ . It follows from Corollary 7 that  $\beta'_1(K_n) = -1$ . ■

For the graph  $K_{1,t}$ , one can show that

$$\beta'_1(K_{1,t}) = \begin{cases} 0, & t \equiv 0 \pmod{2}, \\ 1, & t \equiv 1 \pmod{2}. \end{cases}$$

In general, we prove the following theorem.



**Theorem 10.** For positive integers  $s \geq 2$  and  $t \geq 2$ ,

$$\beta'_1(K_{s,t}) = \begin{cases} 0, & st \equiv 0 \pmod{2}, \\ \min\{s, t\}, & st \equiv 1 \pmod{2}. \end{cases}$$

**Proof.** Let  $S = \{u_1, \dots, u_s\}$  and  $T = \{v_1, \dots, v_t\}$  be the partite sets of  $K_{s,t}$ . We discuss two cases.

*Case 1.*  $st$  is even.

Without loss of generality we assume that  $s$  is even. Let  $x$  be any signed matching of  $K_{s,t}$ . By the definition, we have

$$x(E_{K_{s,t}}(v)) \leq 0$$

for every  $v \in T$ . Hence,  $x(E(K_{s,t})) \leq 0$  implying that  $\beta'_1(K_{s,t}) \leq 0$ .

To show that  $\beta'_1(K_{s,t}) = 0$ , it suffices to show that there exists a signed matching  $x$  such that  $x(E(K_{s,t})) = 0$ . In fact,  $x$  is such a signed matching if  $x(u_i v_j) = (-1)^{i+j}$  for all  $1 \leq i \leq s$  and  $1 \leq j \leq t$ .

*Case 2.* Both  $s$  and  $t$  are odd.

Observe that  $\beta'_1(K_{s,t}) \leq \min\{s, t\}$ . To show the other direction, it suffices to show that there exists a signed matching  $x$  such that  $x(E(K_{s,t})) = \min\{s, t\}$ .

*Case 2.1.*  $s = t$ .

By Theorem 9.18 on page 272 of [3], the complete bipartite graph  $K_{s,s}$  is 1-factorable. In this case  $K_{s,s}$  can be decomposed into  $s$  1-factors. Assigning 1 to each edge of  $\frac{s+1}{2}$  1-factors and -1 to each edge of the remaining  $\frac{s-1}{2}$  1-factors, we produce a signed matching  $x$  such that  $x(E(K_{s,t})) = s$ .

*Case 2.2.*  $s < t$ .

Let  $G$  and  $H$  be the subgraphs induced by the edge sets  $\bigcup_{i=1}^s \bigcup_{j=1}^s \{u_i v_j\}$  and  $\bigcup_{i=1}^s \bigcup_{j=s+1}^t \{u_i v_j\}$ , respectively.

Clearly,  $E(K_{s,t}) = E(G) \cup E(H)$ ,  $G \cong K_{s,s}$  and  $H \cong K_{s,t-s}$ . As we have proven in Case 2.1, there exists a signed matching  $x_G$  of  $G$  such that  $x_G(E(G)) = s$ . Now we can obtain a signed matching  $x$  of  $K_{s,t}$  by setting  $x(e) = x_G(e)$  for every  $e \in E(G)$  and  $x(u_i v_j) = (-1)^{i+j}$  for all  $1 \leq i \leq s$  and  $s+1 \leq j \leq t$ . By simple calculations, we have  $x(E(K_{s,t})) = s$ .

*Case 2.3.*  $s > t$ .

The proof is similar to that of Case 2.2, so we omit it. ■

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