

## THE BONDAGE NUMBER OF GRAPHS: GOOD AND BAD VERTICES

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### Abstract

The *domination number*  $\gamma(G)$  of a graph  $G$  is the minimum number of vertices in a set  $D$  such that every vertex of the graph is either in  $D$  or is adjacent to a member of  $D$ . Any dominating set  $D$  of a graph  $G$  with  $|D| = \gamma(G)$  is called a  $\gamma$ -set of  $G$ . A vertex  $x$  of a graph  $G$  is called: (i)  $\gamma$ -good if  $x$  belongs to some  $\gamma$ -set and (ii)  $\gamma$ -bad if  $x$  belongs to no  $\gamma$ -set. The *bondage number*  $b(G)$  of a nonempty graph  $G$  is the cardinality of a smallest set of edges whose removal from  $G$  results in a graph with domination number greater than  $\gamma(G)$ . In this paper we present new sharp upper bounds for  $b(G)$  in terms of  $\gamma$ -good and  $\gamma$ -bad vertices of  $G$ .

**Keywords:** bondage number,  $\gamma$ -bad/good vertex.

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### 1. INTRODUCTION

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, *et al.* [11]. We denote the vertex set and the edge set of a graph  $G$  by  $V(G)$  and  $E(G)$ , respectively. The subgraph induced by  $S \subseteq V(G)$  is denoted by  $\langle S, G \rangle$ . For a vertex  $x$  of  $G$ ,  $N(x, G)$  denote the set of all neighbors of  $x$  in  $G$ ,  $N[x, G] = N(x, G) \cup \{x\}$  and the degree of  $x$  is  $\deg(x, G) = |N(x, G)|$ . The minimum degree of vertices in  $G$  is denoted by  $\delta(G)$  and the maximum degree by  $\Delta(G)$ . If  $x \in V(G)$  and  $\emptyset \neq Y \subseteq V(G)$

we let  $E(x, Y)$  represents the set of edges of  $G$  of the form  $xy$  where  $y \in Y$ , and let  $e(x, Y) = |E(x, Y)|$ .

A set  $D \subseteq V(G)$  *dominates* a vertex  $v \in V(G)$  if either  $v \in D$  or  $N(v, G) \cap D \neq \emptyset$ . If  $D$  dominates all vertices in a subset  $T$  of  $V(G)$  we say that  $D$  *dominates*  $T$ . When  $D$  dominates  $V(G)$ ,  $D$  is called a *dominating set* of the graph  $G$ . The *domination number*  $\gamma(G)$  of a graph  $G$  is the minimum cardinality taken over all dominating sets of  $G$ . Any dominating set of cardinality  $\gamma(G)$  is called a  $\gamma$ -*set*. A dominating set  $D$  is called an *efficient dominating set* if the distance between any two vertices in  $D$  is at least three. Not all graphs have efficient dominating sets. A vertex  $v$  of a graph  $G$  is *critical* if  $\gamma(G - v) < \gamma(G)$ , and  $G$  is *vertex domination-critical* if each its vertex is critical. We refer to graphs with this property as *vc-graphs*.

Much has been written about the effects on domination related parameters when a graph is modified by deleting an edge. For surveys see [11, Chapter 5] and [12, Chapter 16]. One measure of the stability of the domination number of  $G$  under edge removal is the bondage number defined in [6] (previously called the *domination line-stability* in [2]). The *bondage number*  $b(G)$  of a nonempty graph  $G$  is the cardinality of a smallest set of edges whose removal from  $G$  results in a graph with domination number greater than  $\gamma(G)$ . Since the domination number of every spanning subgraph of a nonempty graph  $G$  is at least as great as  $\gamma(G)$  ([11]), the bondage number of a nonempty graph is well defined. First results on bondage number can be found in a 1983 article of Bauer *et al.* [2].

**Theorem 1.1** (Bauer *et al.* [2]). *If  $G$  is a nontrivial graph, then*

- (i)  $b(G) \leq \deg(u, G) + \deg(v, G) - 1$  for every pair of adjacent vertices  $u$  and  $v$  of  $G$ ;
- (ii) *If there exists a vertex  $v \in V(G)$  for which  $\gamma(G - v) \geq \gamma(G)$ , then  $b(G) \leq \deg(v, G) \leq \Delta(G)$ .*

As a corollary of Theorem 1.1(i) it immediately follows the next theorem.

**Theorem 1.2** (Fink *et al.* [6]). *If  $G$  is a graph with no isolated vertices, then  $b(G) \leq \delta(G) + \Delta(G) - 1$ .*

An extension of a result in Theorem 1.1 which include distance 2 vertices is the next theorem.

**Theorem 1.3** (Hartnell and Rall [10] and Teschner [17]). *If  $u$  and  $v$  are different vertices of  $G$  such that the distance between them is at most 2, then  $b(G) \leq \deg(u, G) + \deg(v, G) - 1$ .*

A generalization of Theorem 1.2 was found independently by Hartnell and Rall [10] and Teschner [17].

**Theorem 1.4** (Hartnell and Rall [10] and Teschner [17]). *If  $G$  has edge-connectivity  $\lambda(G) \geq 1$ , then  $b(G) \leq \Delta(G) + \lambda(G) - 1$ .*

Hartnell and Rall [9] improved the bound of Theorem 1.1(i) for adjacent vertices.

**Theorem 1.5** (Hartnell and Rall [9]). *For every pair of  $u$  and  $v$  of adjacent vertices of  $G$ ,  $b(G) \leq \deg(u, G) + e(v, V(G) - N[u, G]) = \deg(u, G) + \deg(v, G) - 1 - |N(u, G) \cap N(v, G)|$ .*

In [18], Wang, by careful consideration of the nature of the edges from the neighbors of  $u$  and  $v$ , further refine this bound.

**Theorem 1.6** (Wang [18]). *For each edge  $uv$  of a graph  $G$ , let*  
 $T_1(u, v) = N[u, G] \cap N(v, G)$ ,  
 $T_2(u, v) = \{w : w \in N(v, G) \text{ and } N[w, G] \subseteq N[v, G] - \{u\}\}$ ,  
 $T_3(u, v) = \{w : w \in N(v, G) \text{ and } N[w, G] \subseteq N[x, G] - \{u\}, \text{ where}$   
 $x \in N(u, G) \cap N(v, G)\}$ , *and*  
 $T_4(u, v) = \{w : w \in N(v, G) - (T_1(u, v) \cup T_2(u, v) \cup T_3(u, v))\}$ .  
*Then  $b(G) \leq \min_{u \in V(G), v \in N(u, G)} \{\deg(u, G) + |T_4(u, v)|\}$ .*

The concept of  $\gamma$ -bad/good vertices in graphs was introduced by Fricke *et al.* in [7]. A vertex  $v$  of a graph  $G$  is called:

- (i) [7]  $\gamma$ -good, if  $v$  belongs to some  $\gamma$ -set of  $G$  and
- (ii) [7]  $\gamma$ -bad, if  $v$  belongs to no  $\gamma$ -set of  $G$ .

For a graph  $G$  we define:

$$\begin{aligned} \mathbf{G}(G) &= \{x \in V(G) : x \text{ is } \gamma\text{-good}\}; \\ \mathbf{B}(G) &= \{x \in V(G) : x \text{ is } \gamma\text{-bad}\}; \\ V^-(G) &= \{x \in V(G) : \gamma(G - x) < \gamma(G)\}. \end{aligned}$$

Clearly,  $\{\mathbf{G}(G), \mathbf{B}(G)\}$  is a partition of  $V(G)$ . In this paper we present new sharp upper bounds for  $b(G)$  in terms of  $\gamma$ -good and  $\gamma$ -bad vertices of  $G$ .

## 2. GOOD AND BAD VERTICES

Our main result in this section is the next theorem.

**Theorem 2.1.** *Let  $G$  be a graph.*

- (i) *If  $V(G) \neq V^-(G)$ , then  $b(G) \leq \min\{\deg(x, G) - (\gamma(G - x) - \gamma(G)) : x \in V(G) - V^-(G)\}$ .*
- (ii) *If  $G$  has a  $\gamma$ -bad vertex, then  $b(G) \leq \min\{|N(x, G) \cap \mathbf{G}(G)| : x \in \mathbf{B}(G)\}$ .*
- (iii) *If  $V_1^-(G) = \{x \in V^-(G) : \deg(x, G) \geq 1\} \neq \emptyset$ , then  $b(G) \leq \min_{x \in V_1^-(G), y \in \mathbf{B}(G-x)} \{\deg(x, G) + |N(y, G) \cap \mathbf{G}(G-x)|\}$ .*

**Proof.** Notice that if  $x \in V(G)$  is isolated then  $x$  is critical and  $\gamma$ -good.

- (i) Let  $x \in V(G)$  with  $\gamma(G - x) = \gamma(G) + p$ ,  $p \geq 0$ . If  $p = 0$ , then  $b(G) \leq \deg(x, G)$  by Theorem 1.1 (ii). Now, we need the following lemma.

**Lemma 2.1.1** ([2]). *If  $v$  is a vertex of a graph  $G$  and  $\gamma(G - v) > \gamma(G)$ , then  $v$  is not an isolate and is in every  $\gamma$ -set of  $G$ .*

We return to the proof of Theorem 2.1. Assume  $p \geq 1$ . By the above lemma, it follows that  $x$  is in every  $\gamma$ -set of  $G$ . Let  $M$  be a  $\gamma$ -set of  $G$ . Then  $Q = (M - \{x\}) \cup N(x, G)$  is a dominating set of  $G - x$  which implies  $\gamma(G) + p = \gamma(G - x) \leq |Q| = \gamma(G) - 1 + \deg(x, G)$ . Hence  $1 \leq p \leq \deg(x, G) - 1$ . Let  $S \subseteq E(x, N(x, G)) = E_x$  and  $|S| = \deg(x, G) - p$ . Then  $\gamma(G - S) \geq \gamma(G - E_x) - p = \gamma(G - x) + 1 - p = \gamma(G) + 1$  which implies  $b(G) \leq |S| = \deg(x, G) + \gamma(G) - \gamma(G - x)$ .

- (ii) **Fact 1.** *Let  $x \in \mathbf{B}(G)$ ,  $y \in \mathbf{G}(G)$ ,  $xy \in E(G)$  and  $\gamma(G - xy) = \gamma(G)$ . Then  $\mathbf{G}(G - xy) \subseteq \mathbf{G}(G)$  and  $\mathbf{B}(G - xy) \supseteq \mathbf{B}(G)$ .*

**Proof.** Every  $\gamma$ -set of  $G - xy$  is a  $\gamma$ -set of  $G$ . □

**Fact 2.** *If  $x \in \mathbf{B}(G)$ , then  $\gamma(G - E(x, \mathbf{G}(G))) > \gamma(G)$ .*

**Proof.** Assume to the contrary, that  $\gamma(G_1) = \gamma(G)$ , where  $G_1 = G - E(x, \mathbf{G}(G))$ . By Fact 1 we have  $\mathbf{B}(G_1) \supseteq \mathbf{B}(G)$  which implies  $N[x, G_1] \subseteq \mathbf{B}(G_1)$ . But this is clearly impossible. □

The result immediately follows by Fact 2.

(iii) Let  $x \in V_1^-(G)$  and  $M$  be a  $\gamma$ -set of  $G - x$ . Then clearly no neighbor of  $x$  is in  $M$  which implies  $\emptyset \neq N(x, G) \subseteq \mathbf{B}(G - x)$ . Since  $\gamma(G - E(x, N(x, G))) = \gamma(G)$  it follows that  $b(G) \leq \deg(x, G) + b(G - x)$ . By (ii),  $b(G - x) \leq |N(y, G) \cap \mathbf{B}(G - x)|$  for any  $y \in \mathbf{B}(G - x)$ . Hence  $b(G) \leq \deg(x, G) + |N(y, G) \cap \mathbf{B}(G - x)|$ . ■

**Lemma 2.2.** *Under the notation of Theorem 1.6, if  $u$  is critical, then  $(T_1(u, v) - \{u\}) \cup T_2(u, v) \cup T_3(u, v) \subseteq N(v, G) \cap \mathbf{B}(G - u)$ .*

**Proof.** From definitions  $T_1(u, v) \cup T_2(u, v) \cup T_3(u, v) \subseteq N(v, G)$ . By the proof of Theorem 2.1 (iii),  $N(u, G) \subseteq \mathbf{B}(G - u)$ . Since  $T_1(u, v) - \{u\} \subseteq N(u, G)$  we have  $T_1(u, v) - \{u\} \subseteq \mathbf{B}(G - u)$ . Observe that if  $H$  is a graph,  $z \in \mathbf{B}(H)$ ,  $y \in V(H)$  and  $N[y, H] \subseteq N[z, H]$  then clearly  $y \in \mathbf{B}(H)$ . From this fact and  $N(u, G) \subseteq \mathbf{B}(G - u)$  it immediately follows that  $T_2(u, v) \cup T_3(u, v) \subseteq \mathbf{B}(G - u)$ . ■

By Lemma 2.2, if  $u$  is a critical vertex of a graph  $G$ , then

$$\deg(u, G) + \min_{v \in N(u, G)} \{|T_4(u, v)|\} \geq \deg(u, G) + \min_{v \in N(u, G)} \{|N(v, G) \cap \mathbf{B}(G - u)|\} \geq \deg(u, G) + \min_{v \in \mathbf{B}(G - u)} \{|N(v, G) \cap \mathbf{B}(G - u)|\}.$$

Hence Theorem 1.6 (and clearly Theorems 1.1, 1.2 and 1.5) can be seen to follow from Theorem 2.1. Any graph  $G$  with  $b(G)$  achieving the upper bound of some of Theorems 1.1, 1.2, 1.5 and 1.6 can be used to show that the bound of Theorem 2.1 is sharp. For such examples see [5, 6, 9, 14, 18].

**Example 2.3.** Let  $t \geq 2$  be an integer. Let  $H_1, H_2, \dots, H_{t+1}$  be mutually vertex-disjoint graphs such that  $H_{t+1}$  is isomorphic to  $K_{t+3}$  and  $H_i$  is isomorphic to  $K_{t+3} - e$  for  $i = 1, 2, \dots, t$ . Let  $x_{t+1} \in V(H_{t+1})$  and  $x_{i1}, x_{i2} \in V(H_i)$ ,  $x_{i1}x_{i2} \notin E(H_i)$  for  $i = 1, 2, \dots, t$ . Define a graph  $R_t$  as follows:

$$V(R_t) = \{u, v\} \cup (\cup_{k=1}^{t+1} V(H_k)) \text{ and}$$

$$E(R_t) = (\cup_{k=1}^{t+1} E(H_k)) \cup (\cup_{i=1}^t \{ux_{i1}, ux_{i2}\}) \cup \{ux_{t+1}, uv\}.$$

Observe that  $\gamma(R_t) = t+2$ ,  $\mathbf{G}(R_t) = V(R_t)$ ,  $\deg(u, R_t) = 2t+2$ ,  $\deg(x_{t+1}, R_t) = t+3$ ,  $\deg(v, R_t) = 1$ ,  $\lambda(R_t) = 1$  and for each  $y \in V(R_t - \{v, u, x_{t+1}\})$ ,  $\deg(y, R_t) = t+2$ . Moreover,  $v$  is a critical vertex and if  $y \in V(R_t) - \{v\}$  then  $\gamma(R_t - y) = \gamma(R_t)$ . Hence each of the bounds stated in Theorems 1.1–1.6 is greater than or equals  $t+2$ .

Consider the graph  $R_t - uv$ . Clearly  $\gamma(R_t - uv) = \gamma(R_t)$  and  $\mathbf{B}(R_t - uv) = \mathbf{B}(R_t - v) = \{u\} \cup V(H_{t+1} - x_{t+1}) \cup (\cup_{k=1}^t \{x_{k1}, x_{k2}\})$ . Therefore  $N(u, R_t) \cap$

$\mathbf{G}(R_t - v) = \{x_{t+1}\}$  which implies that the upper bound stated in Theorem 2.1 (iii) is equals to  $\deg(v, R_t) + |\{x_{t+1}\}| = 2$ . Clearly  $b(R_t) = 2$  and hence this bound is sharp for  $R_t$ .

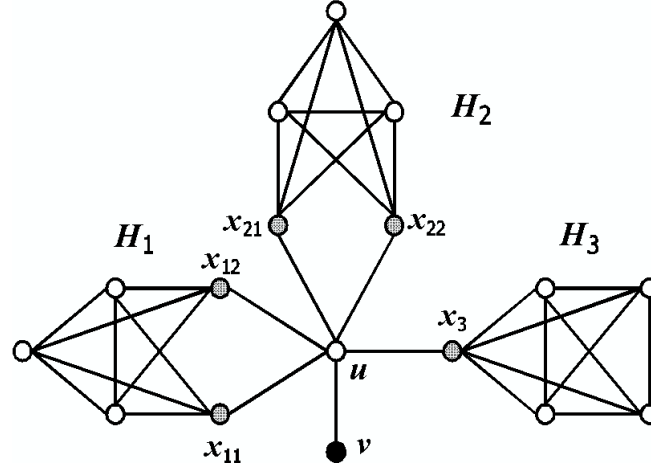


Figure 1. The graph  $R_2$ .

By Example 2.3, it immediately follows:

**Remark 2.4.** For every integer  $t \geq 2$ , the difference between any upper bound stated in Theorems 1.1–1.6 and the upper bound of Theorem 2.1(iii), provided  $G = R_t$ , is greater than or equals  $t$ .

### 3. VC-GRAPHS

The concept of vc-graphs plays an important role in the study of the bondage number. For instance, it immediately follows from Theorem 1.1(ii) that if  $b(G) > \Delta(G)$  then  $G$  is vc-graph. The bondage number of a vc-graphs is examined in [15]. If  $G$  is a vc-graph then  $|V(G)| \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1$ . In this section we give an upper bound for the bondage number of such vc-graphs. We need the following results.

**Theorem 3.1.** *Let  $G$  be a vc-graph.*

- (i) [3] *Then  $|V(G)| \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1$ .*

- (ii) [8] If  $|V(G)| = (\Delta(G) + 1)(\gamma(G) - 1) + 1$ , then  $G$  is regular.

**Theorem 3.2** [1]. *Let  $G$  be a graph.*

- (i) *If  $G$  has vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ , then  $G$  has an efficient dominating set if and only if some subcollection of  $\{N[v_1, G], N[v_2, G], \dots, N[v_n, G]\}$  partitions  $V(G)$ .*
- (ii) *If  $G$  has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number of  $G$ .*

**Lemma 3.3.** *Let  $x$  and  $v$  be different critical vertices of a graph  $G$ . Let  $v$  belong to some efficient dominating set of  $G - x$  and let  $G - v$  have an efficient dominating set. Then  $x$  belongs to all efficient dominating sets of  $G - v$  and  $v$  belongs to all efficient dominating sets of  $G - x$ .*

**Proof.** Let  $M$  be an arbitrary efficient dominating set of  $G - v$ ,  $Q$  be an efficient dominating set of  $G - x$  and  $v \in Q$ . Hence the closed neighborhoods of each two different vertices of  $Q$  are vertex disjoint and each vertex of  $Q - \{v\}$  dominates a unique vertex of  $M$ , by Theorem 3.2. Since  $|M| = \gamma(G - v) = \gamma(G) - 1 = \gamma(G - x) = |Q|$ , there exists exactly one vertex in  $M$ , say  $w$ , which is not dominated by  $Q - v$ . If  $w \neq x$  then  $w$  must be dominated by  $v$ , which is impossible because  $|M| = \gamma(G - v) < \gamma(G)$  implies that  $M$  does not dominate  $v$  in  $G$ . Therefore  $x$  belongs to all efficient dominating sets of  $G - v$ . By symmetry,  $v$  belongs to all efficient dominating sets of  $G - x$ . ■

**Theorem 3.4.** *Let  $G$  be a  $vc$ -graph with  $(\Delta(G) + 1)(\gamma(G) - 1) + 1$  vertices. Then for every vertex  $x \in V(G)$ ,  $G - x$  has exactly one  $\gamma$ -set and the unique  $\gamma$ -set of  $G - x$  is efficient dominating.*

**Proof.** Let  $x$  be an arbitrary vertex of  $G$  and  $M$  an arbitrary  $\gamma$ -set of  $G - x$ . Since  $|M| = \gamma(G) - 1$  and  $\Delta(G - x) \leq \Delta(G)$ , it follows that  $|V(G - x)| \leq |M|(\Delta(G - x) + 1) \leq (\gamma(G) - 1)(\Delta(G) + 1) = |V(G - x)|$ . Hence each element of  $M$  dominates exactly  $\Delta(G) + 1$  vertices in  $G - x$  and the closed neighborhoods of all vertices of  $M$  form a partition of  $V(G - x)$ . Now Theorem 3.2 implies that  $M$  is an efficient dominating set of  $G - x$ . Hence for each  $u \in V(G)$ , all  $\gamma$ -sets of  $G - u$  are efficient dominating. Now if  $v$  is a  $\gamma$ -good vertex of  $G - x$  then by Lemma 3.3,  $v$  belongs to all efficient dominating sets of  $G - x$ . Hence  $G - x$  has exactly one  $\gamma$ -set. ■

**Lemma 3.5** ([17]). *If  $G$  is a nontrivial graph with a unique minimum dominating set, then  $b(G) = 1$ .*

**Lemma 3.6.** *Let  $G$  be a graph,  $x \in V^-(G)$ ,  $\deg(x, G) \geq 1$  and let  $G - x$  have exactly one  $\gamma$ -set. Then  $b(G) \leq \deg(x, G) + 1$ .*

**Proof.** It follows by Lemma 3.5 that  $b(G - x) = 1$ . Hence  $b(G) \leq e(x, N(x, G)) + b(G - x) = \deg(x, G) + 1$ . ■

We now state and prove the principal result of this section.

**Theorem 3.7** (Teschner [15] when  $\gamma(G) = 3$ ). *If  $G$  is a nontrivial vc-graph with  $(\Delta(G) + 1)(\gamma(G) - 1) + 1$  vertices, then  $b(G) \leq \Delta(G) + 1 = \delta(G) + 1$ .*

**Proof.** Let  $x \in V(G)$ . By Theorem 3.1 (ii),  $\deg(x, G) = \Delta(G) = \delta(G)$  and by Theorem 3.4,  $G - x$  has exactly one  $\gamma$ -set. The result immediately follows by Lemma 3.6. ■

#### 4. OPEN PROBLEMS

**Conjecture 4.1** (Teschner [15]). *For any vc-graph  $G$ ,  $b(G) \leq 1.5\Delta(G)$ .*

Teschner [15] has shown that Conjecture 4.1 is true when  $\gamma(G) \leq 3$ . Note that if  $G = K_t \times K_t$  for a positive integer  $t \geq 2$ , then  $b(G) = 1.5\Delta(G)$  as was found independently by Hartnell and Rall [9] and Teschner [16].

**Conjecture 4.2** (Hailong Liu and Liang Sun [13]). *For any positive integer  $r$ , there exists a vc-graph  $G$  such that  $b(G) \geq \Delta(G) + k(G) + r$  where  $k(G)$  is the vertex connectivity of  $G$ .*

Motivated by Theorem 2.1(iii) and Theorem 3.6 we state the following:

**Conjecture 4.3.** *For every connected nontrivial vc-graph  $G$ ,*  

$$\min_{x \in V(G), y \in \mathbf{B}(G-x)} \{\deg(x, G) + |N(y, G) \cap \mathbf{G}(G-x)|\} \leq 1.5\Delta(G).$$

**Conjecture 4.4.** *If  $G$  is a vc-graph with  $(\Delta(G) + 1)(\gamma(G) - 1) + 1$  vertices then  $b(G) = \Delta(G) + 1$ .*



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