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ON LONG CYCLES THROUGH FOUR PRESCRIBED VERTICES OF A POLYHEDRAL GRAPH

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Abstract

For a 3-connected planar graph G with circumference $c \geq 44$ it is proved that G has a cycle of length at least $\frac{1}{36}c + \frac{20}{3}$ through any four vertices of G.

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1. Introduction and Result

We consider finite, simple, and undirected graphs. For terminology and notation not defined here we refer to [2].

Let G be a planar graph and $S \subseteq V(G)$ be a set of prescribed vertices of G. In this paper we are interested in lower bounds on the length $c_S(G)$ of a longest cycle of G containing S if such a cycle through S in G exists at all.

If $S' \subseteq S \subseteq V(G)$ and if there is a cycle through S in G then $c_{S'}(G) \geq c_S(G)$. The *circumference* $c_\emptyset(G) = c(G)$ is the length of a longest cycle of G.

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*H. Walther passed away in January 2005. The present paper reports partially the last research results obtained by him during the last months before his very sudden and sad death.

In 1963, J.W. Moon and L. Moser [5] proved that for arbitrary $\epsilon > 0$ there is a 3-connected planar graph G such that $c(G) < \epsilon|V(G)|$. Thus, a linear lower bound on $c_S(G)$ should be in terms of $c(G)$ instead in terms of $|V(G)|$.

First consider the case that G is a 2-connected planar graph. We will show that it is possible that $c_S(G)$ is a constant (depending only on $|S|$) and $c(G)$ is arbitrarily large in this case.

For this purpose let G be a subdivision of $K_{2,3}$, i.e., G consists of three pairwise internally disjoint paths P, Q, and R having common end vertices. Furthermore, let S be a set of at least two vertices of G such that $S \subseteq V(P) \cup V(Q)$, both P and Q have an inner vertex in S, and $|V(P) \cup V(Q)| = \max\{|S|, 4\}$. Finally, let R be chosen such that $|V(R)|$ is large. It follows that $c(G) \geq |V(R)| + 1$ and $c_S(G) = |V(P) \cup V(Q)|$.

W.T. Tutte [7, 8] proved that a 4-connected planar graph is hamiltonian, hence, $c_S(G) = c(G)$ for each 4-connected planar graph G and each set $S \subseteq V(G)$.

Now consider the remaining case that G is a 3-connected planar graph.

From results of A.K. Kelmans and M.V. Lomonosov [3] it follows that for any set S of at most five vertices of a 3-connected planar graph G there exists a cycle of G containing S.

Next it is shown that such a result is impossible if $|S| \geq 6$. For this purpose let T be a plane triangulation on $n \geq 5$ vertices. Because $n \geq 5$, T has $2n - 4 \geq n + 1$ faces. Let G be obtained from T by inserting a new vertex into $n+1$ faces of T and connecting it by an edge with each boundary vertex of that face. The graph G is planar and 3-connected, there is no cycle of G containing the set S of the $n + 1 \geq 6$ new vertices of G because S is independent and $|V(G) \setminus S| < |S|$.

Now we consider the case that a 3-connected planar graph G contains a cycle through a set S of at least five prescribed vertices and we will show that it is possible that $c_G(S) = 2|S|$ and $c(G)$ is arbitrarily large.

Proposition 1. *For any two positive integers k and l with $5 \leq k < l$ there is a 3-connected maximal planar graph $G(k,l) = G$ such that G contains a cycle through a certain independent set S of k prescribed vertices of degree 3, $c_S(G) = 2k$, and $c(G) \geq l$.*

Proof. Let H be a 3-connected plane triangulation with $c(H) \geq l$. Furthermore, let f_H be the outer face of H. Consider a plane triangulation T on five vertices and let f be a face of T. The maximal planar graph $G(5,l)$ is obtained by inserting a new vertex of degree three into each face of T

different from f and identifying the boundary of f with the boundary of f_H of H. Let S be the set of the five new vertices of $G(5,l)$. The vertices of S have degree three and are independent in G, a longest cycle containing S has length 10 and $c(G(5,l)) \geq c(H)$.

Let $G(k,l)$ be constructed and consider a vertex $x \in S$ and a longest cycle C through S in $G(k,l)$. Let $\{a,b,c\}$ be the neighbourhood of x and the edges ax and bx belong to C. The graph $G(k+1,l)$ is obtained by inserting two new vertices y and z of degree three into the faces acx and bcx, respectively, and putting $S = (S \setminus \{x\}) \cup \{y,z\}$. Then $G(k+1,l)$ is maximal planar, the set S is an independent set of vertices of degree three in $G(k+1,l)$, $c_S(G(k+1,l)) = 2(k+1)$, and $c(G(k+1,l)) \geq c(G(k,l))$. ■

It remains to consider a 3-connected planar graph G and a set $S \subset V(G)$ with $1 \leq |S| \leq 4$. The following Theorem 2 was proved by A. Saito [6].

Theorem 2. *Let x, y and z be arbitrary three vertices of a 3-connected planar graph G on at least six vertices. Then $c_{\{x\}}(G) \geq \frac{2}{3}c(G) + 2$, $c_{\{x,y\}}(G) \geq \frac{1}{2}c(G) + 2$, and $c_{\{x,y,z\}}(G) \geq \frac{1}{4}c(G) + 3$.*

Our result is the following Theorem 3.

Theorem 3. *A 3-connected planar graph G with $c(G) \geq 44$ has a cycle of length at least $\frac{1}{36}c(G) + \frac{20}{3}$ through any four of its vertices.*

2. Proof of Theorem 3

For $A, B \subseteq V(G)$ an *$A-B$-path* is a path P from A to B such that $|V(P) \cap A| = |V(P) \cap B| = 1$. A common vertex of A and B is also an $A-B$-path.

A set $S \subseteq V(G)$ *separates* the sets $A, B \subseteq V(G)$ if any $A-B$-path contains a vertex in S. For a set $\mathcal{P}$ of paths put $V(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} V(P)$. A more detailed version of Menger's Theorem (see [1]) is the following

Lemma 1. *Let t be a non-negative integer, G be a graph, $A, B \subseteq V(G)$ such that A and B cannot be separated by a set of at most t vertices. Furthermore, let $\mathcal{Q}$ be a set of t disjoint $A-B$-paths. Then there is a set $\mathcal{R}$ of $t+1$ disjoint $A-B$-paths, such that $A \cap V(\mathcal{Q}) \subset A \cap V(\mathcal{R})$ and $B \cap V(\mathcal{Q}) \subset B \cap V(\mathcal{R})$.*

For a vertex $x \in V(G)$, $N(x)$ denotes the neighbourhood of x in G. A consequence of Lemma 1 (see also [4]) is

Lemma 2. *Let $t < k$ be non-negative integers, G a k-connected graph, $x \in V(G)$, $B \subseteq V(G) \setminus \{x\}$, and $|B| \geq k$. Furthermore, let $\mathcal{Q}$ be a set of t $\{x\} - B$-paths having pairwise only x in common. Then there is a set $\mathcal{R}$ of $t+1$ $\{x\} - B$-paths having pairwise only x in common such that $B \cap V(\mathcal{Q}) \subset B \cap V(\mathcal{R})$.*

***Proof of Lemma* 2.** In case $B \subseteq N(x)$ nothing is to prove. If $B \not\subseteq N(x)$ then $|N(x)| \geq k$, B and $N(x)$ cannot be separated by a set of at most t vertices, and with Lemma 1 we are done. ■

Using Theorem 2, Theorem 3 is a consequence of the following Lemma 3.

Lemma 3. *Let G be a 3-connected planar graph with $c(G) \geq 44$ and $x_1, x_2, x_3, x_4 \in V(G)$. Among all cycles of G containing x_2, x_3, x_4 let C be a longest one. Then there is a cycle D of G which contains x_1, x_2, x_3, x_4 and has length at least $\frac{1}{9}|V(C)| + \frac{19}{3}$.*

***Proof of Lemma* 3.** Note that $|V(C)| \geq \frac{1}{4}c(G) + 3$ by Theorem 2, hence, $|V(C)| \geq 14$.

Consider a fixed orientation ϕ of C. For $a, b \in V(C)$ with $a \neq b$ let $[a, b]$ be the path on C from a to b following ϕ. We write $V[a, b]$ instead of $V([a, b])$.

If $x_1 \in V(C)$ then because $|V(C)| > \frac{1}{9}|V(C)| + \frac{19}{3}$ we are done with $D = C$. Thus we may assume $x_1 \notin V(C)$.

With $B = V(C)$, $x = x_1$, and Lemma 2, let P_1, P_2, P_3 be three $\{x_1\} - V(C)$-paths having only x_1 in common and $V(P_i) \cap V(C) = \{u_i\}$ for $i = 1, 2, 3$. Assume $u_2 \in V[u_1, u_3]$. Because $|V[u_1, u_2]| + |V[u_2, u_3]| + |V[u_3, u_1]| = |V(C)| + 3$ let $|V[u_1, u_2]| \geq \frac{1}{3}|V(C)| + 1$.

Case 1. $\{x_2, x_3, x_4\} \subseteq V[u_1, u_3]$ or $\{x_2, x_3, x_4\} \subseteq V[u_3, u_2]$.
Then one of the cycles $P_1 \cup P_3 \cup [u_1, u_3]$ and $P_2 \cup P_3 \cup [u_3, u_2]$ contains x_1, x_2, x_3, and x_4 and each of them has length at least $|V[u_1, u_2]| + 2 \geq \frac{1}{3}|V(C)| + 3$. Assume both cycles have length larger than $\frac{1}{3}|V(G)| + 3$. Since $|V(G)|$ is an integer, it means that their length is at least $\frac{1}{3}(|V(G)| + 10)$. Then since $|V(G)| \geq 14$, we have $\frac{1}{3}(|V(G)| + 10) \geq \frac{1}{9}|V(G)| + \frac{19}{3}$.

Assume that the cycle $P_1 \cup P_3 \cup [u_1, u_3]$ has length $\frac{1}{3}|V(C)| + 3$. Then $|V[u_2, u_3]| = 2$, $|V(C)| + 1 = |V[u_1, u_2]| + |V[u_3, u_1]|$, and we may assume that even $|V[u_1, u_2]| > \frac{1}{2}|V(C)|$ in this case. Then both cycles have length greater than $\frac{1}{2}|V(C)| + 2 > \frac{1}{9}|V(C)| + \frac{19}{3}$.

Case 2. $\{x_2, x_3, x_4\} \subseteq V[u_2, u_1]$.
If $\{x_2, x_3, x_4\} \subseteq V[u_2, u_3]$ or $\{x_2, x_3, x_4\} \subseteq V[u_3, u_2]$ then we have Case 1, thus, we may assume $x_2 \in V[u_3, u_1] \setminus \{u_3\}$, $x_3 \in V[u_2, u_3]$, $x_4 \in V[x_3, u_3]$, and $\{x_3, x_4\} \neq \{u_2, u_3\}$. If $x_2 = u_1$ then again we have Case 1, consequently, $x_2 \neq u_1$.

If $|V[u_1, u_2]| < \frac{2}{3}|V(C)| + 1$, then $|V[u_1, u_2]| \leq \frac{1}{3}(2|V(C)| + 2)$ and $|V[u_2, u_1]| \geq \frac{1}{3}|V(C)| + \frac{7}{3}$. Then $|P_1 \cup P_2 \cup V[u_2, u_1]| \geq \frac{1}{3}|V(C)| + \frac{10}{3} \geq \frac{1}{9}|V(C)| + \frac{19}{3}$ for $|V(C)| \geq 14$.

Hence, we may assume $|V[u_1, u_2]| \geq \frac{2}{3}|V(C)| + 1$.
With $t = 2$, $x = x_2$, $B = V[u_1, u_3] \cup V(P_1) \cup V(P_2) \cup V(P_3)$, $\mathcal{Q} = \{[x_2, u_1], [u_3, x_2]\}$, and Lemma 2, consider a set $\mathcal{R} = \{R_1, R_2, R_3\}$ of $\{x_2\} - B$-paths with $V(R_1) \cap B = \{u_1\}$, $V(R_3) \cap B = \{u_3\}$, and $V(R_2) \cap (B \setminus \{u_1, u_3\}) = \{r\}$.

Case 2.1. $r \in V(P_2) \setminus \{x_1, u_2\}$.
In this case the union of $[u_1, u_3], P_1, P_2, P_3, R_1, R_2$, and R_3 form a subdivision of $K_{3,3}$ contradicting the planarity of G.

Case 2.2. $r \in V(P_1) \cup V(P_3) \cup V[u_2, x_3] \setminus \{u_2\} \cup V[x_4, u_3]$.
It is easy to see that there is always a cycle D with $V[u_1, u_2] \cup \{x_1, x_2, x_3, x_4, u_3\} \subseteq V(D)$, hence, $|V(D)| \geq \frac{2}{3}|V(C)| + 5$.

Case 2.3. $r \in V[u_1, u_2] \setminus \{u_2\}$.
For the cycles $C_1 = P_1 \cup P_2 \cup [u_2, u_3] \cup R_3 \cup R_2 \cup [u_1, r]$ and $C_2 = P_1 \cup P_3 \cup [r, u_3] \cup R_2 \cup R_1$ both containing x_1, x_2, x_3, x_4, $|V(C_1)| + |V(C_2)| \geq |V[u_1, u_2]| + 10 \geq \frac{2}{3}|V(C)| + 11$, hence, one of them has length at least $\frac{1}{3}|V(C)| + \frac{11}{2}$.

Case 2.4. $r \in V[x_3, x_4] \setminus \{x_3, x_4\}$.
With $t = 2$, $x = x_4$, $B = V(P_1) \cup V(P_2) \cup V(P_3) \cup V(R_1) \cup V(R_2) \cup V(R_3) \cup V[u_1, r]$, $\mathcal{Q} = \{[r, x_4], [x_4, u_3]\}$, and Lemma 2, consider a set $\mathcal{S} = \{S_1, S_2, S_3\}$ of $\{x_4\} - B$-paths with $V(S_1) \cap B = \{r\}$, $V(S_3) \cap B = \{u_3\}$, and $V(S_2) \cap (B \setminus \{r, u_3\}) = \{s\}$.

Case 2.4.1. $s \in V(R_1) \setminus \{x_2\} \cup V[u_1, u_2] \setminus \{u_2\} \cup V(P_1) \setminus \{x_1\}$.
It is easy to see that G contains a subdivision of $K_{3,3}$ in this case.

Case 2.4.2. $s \in V(R_2)$.
Then we argue as in Case 2.2.

Case 2.4.3. $s \in V(P_2)\setminus\{u_2\}\cup V(P_3)\cup V(R_3)\cup V[u_2,x_3]\setminus\{u_2\}\cup V[x_3,r]$. It is easy to see that there is always a cycle D with $V[u_1,u_2]\cup\{x_1,x_2,x_3,x_4,u_3\}\subseteq V(D)$, hence, $|V(D)|\geq \frac{2}{3}|V(C)|+5$.

Case 2.4.4. $s = u_2$.
We may assume $x_3 \neq u_2$ because otherwise the cycle $[u_1,u_2]\cup S_2\cup S_1\cup R_2\cup R_3\cup P_3\cup P_1$ has lenght at least $\frac{2}{3}|V(C)|+6$.

With $t=2$, $x=x_3$, $B=V(P_1)\cup V(P_2)\cup V(P_3)\cup V(R_1)\cup V(R_2)\cup V(R_3)\cup V(S_1)\cup V(S_2)\cup V(S_3)\cup V[u_1,u_2]$, $\mathcal{Q}=\{[u_2,x_3],\ [x_3,r]\}$, and Lemma 2, consider a set $\mathcal{T}=\{T_1,T_2,T_3\}$ of $\{x_3\}-B$-paths with $V(T_1)\cap B=\{r\}$, $V(T_2)\cap B=\{u_2\}$, and $V(T_3)\cap(B\setminus\{r,u_2\})=\{q\}$.

Case 2.4.4.1. $q\in V(P_1)\setminus\{u_1\}\cup V(P_2)\cup V(P_3)\cup V(R_3)\setminus\{x_2\}\cup V(S_3)\setminus\{x_4\}$. It is easy to see that G contains a subdivision of $K_{3,3}$ in this case.

Case 2.4.4.2. $q\in V(R_1)\setminus\{u_1\}\cup V(R_2)\cup V(S_1)\cup V(S_2)$.
It is easy to see that there is always a cycle D with $V[u_1,u_2]\cup\{x_1,x_2,x_3,x_4\}\subseteq V(D)$, hence, $|V(D)|\geq\frac{2}{3}|V(C)|+5$.

Case 2.4.4.3. $q\in V[u_1,u_2]\setminus\{u_1\}$.
For the cycles $C_1=P_1\cup P_2\cup S_2\cup S_3\cup R_3\cup R_2\cup T_1\cup T_3\cup[u_1,q]$ and $C_2=P_1\cup P_3\cup S_3\cup S_2\cup[q,u_2]\cup T_3\cup T_1\cup R_2\cup R_1$ both containing x_1,x_2,x_3,x_4, $|V(C_1)|+|V(C_2)|\geq|V[u_1,u_2]|+14\geq\frac{2}{3}|V(C)|+15$, hence, one of them has length at least $\frac{1}{3}|V(C)|+\frac{15}{2}$.

Case 2.4.4.4. $q=u_1$.
The graph obtained from G by removing u_1,u_2 is connected, hence, there is a $(V[u_1,u_2]\setminus\{u_1,u_2\})-(V(P_1)\cup V(P_2)\cup V(P_3)\cup V(R_1)\cup V(R_2)\cup V(R_3)\cup V(S_1)\cup V(S_2)\cup V(S_3)\cup V(T_1)\cup V(T_2)\cup V(T_3))$-path P in G with $u_1,u_2\notin V(P)$. Let $V(P)\cap(V[u_1,u_2]\setminus\{u_1,u_2\})=\{v\}$ and $V(P)\cap(V(P_1)\cup V(P_2)\cup V(P_3)\cup V(R_1)\cup V(R_2)\cup V(R_3)\cup V(S_1)\cup V(S_2)\cup V(S_3)\cup V(T_1)\cup V(T_2)\cup V(T_3))=\{w\}$.

Case 2.4.4.4.1. $w\in V(P_3)\setminus\{x_1\}\cup V(R_1)\cup V(R_2)\cup V(R_3)\cup V(S_1)\cup V(S_2)\cup V(S_3)\cup V(T_1)\setminus\{x_3\}$.

If $w\in V(R_2)\setminus\{r\}$ it is easy to see that G contains a subdivision of $K_{3,3}$ in this case. If $w=r$, then we obtain a contradiction by reducing this case to Case 2.4.4.3.

Case 2.4.4.4.2. $w \in V(P_1)$.
Let P' be the subpath of P_1 connecting w and x_1. The sum of the lengths of the cycles $P \cup P' \cup P_2 \cup S_2 \cup S_3 \cup R_3 \cup R_2 \cup T_1 \cup T_3 \cup [u_1, v]$ and $P \cup P' \cup P_3 \cup R_3 \cup R_1 \cup T_3 \cup T_1 \cup S_1 \cup S_2 \cup [v, u_2]$ is at least $\frac{2}{3}|V(C)| + 19$, hence, one of them has the desired length.

Case 2.4.4.4.3. $w \in V(P_2)$.
Let P'' be the subpath of P_2 connecting w and x_1. One of the cycles $P \cup P'' \cup P_3 \cup R_3 \cup R_2 \cup S_1 \cup S_2 \cup T_2 \cup T_3 \cup [u_1, v]$ and $P \cup P'' \cup P_1 \cup T_3 \cup T_1 \cup R_2 \cup R_3 \cup S_3 \cup S_2 \cup [v, u_2]$ has the desired length.

Case 2.4.4.4.4. $w \in V(T_2)$.
Let T be the subpath of T_2 connecting w and x_3. One of the cycles $P \cup T \cup T_1 \cup S_1 \cup S_2 \cup P_2 \cup P_3 \cup R_3 \cup R_1 \cup [u_1, v]$ and $P \cup T \cup T_1 \cup S_1 \cup S_3 \cup R_3 \cup R_1 \cup P_1 \cup P_2 \cup [v, u_2]$ has the desired length.

Case 2.4.4.4.5. $w \in V(T_3)$.
Let T' be the subpath of T_3 connecting w and x_3. One of the cycles $P \cup T' \cup T_2 \cup S_2 \cup S_1 \cup R_2 \cup R_3 \cup P_3 \cup P_1 \cup [u_1, v]$ and $P \cup T' \cup T_1 \cup S_1 \cup S_3 \cup R_3 \cup R_1 \cup P_1 \cup P_2 \cup [v, u_2]$ has the desired length.

Case 2.5. $r = u_2$.

Case 2.5.1. $x_3 = u_2$.
We have $x_4 \neq u_3$ (otherwise Case 1). With $t = 2$, $x = x_4$, $B = V[u_1, u_2] \cup V(P_1) \cup V(P_2) \cup V(P_3) \cup V(R_1) \cup V(R_2) \cup V(R_3)$, $\mathcal{Q} = \{[x_4, u_3], [u_2, x_4]\}$, and Lemma 2, consider a set $\mathcal{S} = \{S_1, S_2, S_3\}$ of $\{x_4\} - B$-paths with $V(S_1) \cap (B \setminus \{u_2, u_3\}) = \{s\}$, $V(S_2) \cap B = \{u_2\}$, and $V(S_3) \cap B = \{u_3\}$. Because of planarity $s \in V(P_2) \cup V(P_3) \cup V(R_2) \cup V(R_3)$ and it is easy to see that there is a cycle D with $V[u_1, u_2] \subseteq V(D)$ containing x_1, x_2, x_4, u_3.

Case 2.5.2. $x_3 \neq u_2$.
We remark that possibly $x_4 = u_3$. With $t = 2$, $x = x_3$, $B = V[u_1, u_2] \cup V[x_4, u_3] \cup V(P_1) \cup V(P_2) \cup V(P_3) \cup V(R_1) \cup V(R_2) \cup V(R_3)$, $\mathcal{Q} = \{[x_3, x_4], [u_2, x_3]\}$, and Lemma 2, consider a set $\mathcal{S} = \{S_1, S_2, S_3\}$ of $\{x_3\} - B$-paths with $V(S_1) \cap (B \setminus \{u_2, x_4\}) = \{s\}$, $V(S_2) \cap B = \{u_2\}$, and $V(S_3) \cap B = \{x_4\}$. Because of planarity we have $s \notin V(P_1) \setminus \{x_1\} \cup V(R_1) \setminus \{x_2\} \cup V[u_1, u_2]$.

Case 2.5.2.1. $s \in V(P_2) \cup V(P_3) \setminus \{u_3\} \cup V(R_2) \cup V(R_3) \setminus \{u_3\}$.
It is easy to see that there is always a cycle D with $V[u_1, u_2] \cup \{x_1, x_2, x_3, x_4, u_3\} \subseteq V(D)$, hence, $|V(D)| \geq \frac{2}{3}|V(C)| + 5$.

Case 2.5.2.2. $s \in V[x_4, u_3] \setminus \{x_4\}$.
With $t = 2$, $x = x_4$, $B = V[s, u_3] \cup V[u_1, u_2] \cup V(P_1) \cup V(P_2) \cup V(P_3) \cup V(R_1) \cup V(R_2) \cup V(R_3) \cup V(S_1) \cup V(S_2)$, $\mathcal{Q} = \{S_3, [x_4, s]\}$, and Lemma 2, consider a set $\mathcal{T} = \{T_1, T_2, T_3\}$ of $\{x_4\} - B$-paths with $V(T_1) \cap (B \setminus \{s, x_3\}) = \{q\}$, $V(T_2) \cap B = \{s\}$, and $V(T_3) \cap B = \{x_3\}$. Because of planarity $q \notin V(P_1) \setminus \{x_1\} \cup V(R_1) \setminus \{x_2\} \cup V[u_1, u_2] \setminus \{u_2\}$.

Case 2.5.2.2.1. $q \in V(P_2) \setminus \{u_2\} \cup V(P_3) \setminus \{u_3\} \cup V(R_2) \setminus \{u_2\} \cup V(R_3) \setminus \{u_3\}$. It is easy to see that there is always a cycle D with $V[u_1, u_2] \cup \{x_1, x_2, x_3, x_4\} \subseteq V(D)$, hence, $|V(D)| \geq \frac{2}{3}|V(C)| + 5$.

Case 2.5.2.2.2. $q \in V[s, u_3] \setminus \{s\}$.
With $t = 2$, $A = V(S_1) \cup V(T_1) \cup V(T_2) \cup V(T_3) \cup V[s, q]$, $B = V[u_1, u_2] \cup V(P_1) \cup V(P_2) \cup V(P_3) \cup V(R_1) \cup V(R_2) \cup V(R_3)$, $\mathcal{Q} = \{S_2, [q, u_3]\}$, and Lemma 1, consider a set $\mathcal{U} = \{U_1, U_2, U_3\}$ of disjoint $A - B$-paths with $x_3, q \in A \cap V(\mathcal{U})$ and $u_2, u_3 \in B \cap V(\mathcal{U})$. Note that in case $q = u_3$ one of the paths of $\mathcal{U}$ consists of that single vertex. Because of planarity for $u \in B \cap V(\mathcal{U}) \setminus \{u_2, u_3\}$ and $u' \in A \cap V(\mathcal{U}) \setminus \{x_3, q\}$ we have $u \notin V(P_1) \setminus \{x_1\} \cup V(R_1) \setminus \{x_2\} \cup V[u_1, u_2]$ and $u' \notin V(T_2) \setminus \{x_4, s\}$.

Consider the subgraph $H = S_1 \cup T_1 \cup T_2 \cup T_3 \cup [s, q]$ of G. It can be seen easily that there is a path P of H connecting x_3 and u' with $x_4 \in V(P)$ and $q \notin V(P)$ and that there is a path Q of H connecting q and $x \in \{x_3, u'\}$ with $x_3, x_4 \in V(Q)$. Note that the property $q \notin V(P)$ will be used in case $q = u_3$. Consider the cycle $D' = P_1 \cup P_3 \cup R_3 \cup R_2 \cup [u_1, u_2]$. If $u \in V(P_3)$ then the cycle D obtained from D' by replacing the subpath of D' between u and u_3 not containing x_1 by the union of the two paths of $\mathcal{U}$ containing u and u_3 and P or Q has all desired properties. The case $u \in V(R_2) \cup V(R_3)$ can be handled similarly. If $u \in V(P_2)$ then consider the cycle $D'' = P_2 \cup P_3 \cup R_3 \cup R_1 \cup [u_1, u_2]$ instead of D'.

Case 2.5.2.2.3. $q \in V(S_1) \cup V(S_2)$.
This case can be handled similarly as case 2.5.2.2.2.

Case 3. $x_2 \in V[u_1, u_2] \setminus \{u_1, u_2\}$, $x_3 \in V[u_2, u_3] \setminus \{u_2, u_3\}$, $x_4 \in V[u_3, u_1] \setminus \{u_1, u_3\}$.

Again let $|V[u_1, u_2]| \geq \frac{1}{3}|V(C)| + 1$.

With $t = 2$, $x = x_4$, $B = V[u_1, u_3] \cup V(P_1) \cup V(P_2) \cup V(P_3)$, $\mathcal{Q} = \{[x_4, u_1], [u_3, x_4]\}$, and Lemma 2, consider a set $\mathcal{R} = \{R_1, R_2, R_3\}$ of $\{x_4\} - B$-paths with $V(R_1) \cap B = \{u_1\}$, $V(R_3) \cap B = \{u_3\}$, and $V(R_2) \cap (B \setminus \{u_1, u_3\}) = \{r\}$. Because of planarity $r \notin V(P_2) \setminus \{x_1, u_2\}$.

Case 3.1. $r \in V(P_1) \cup V(P_3) \cup V[u_2, u_3] \setminus \{u_2\}$.
It is easy to see that there is always a cycle D with $V[u_1, u_2] \cup \{x_1, x_3, x_4, u_3\} \subseteq V(D)$, hence, $|V(D)| \geq \frac{1}{3}|V(C)| + 5$.

Case 3.2. $r \in V[u_1, u_2]$.
With $t = 2$, $x = x_3$, $B = V[u_1, u_2] \cup V(P_1) \cup V(P_2) \cup V(P_3) \cup V(R_1) \cup V(R_2) \cup V(R_3)$, $\mathcal{Q} = \{[x_3, u_3], [u_2, x_3]\}$, and Lemma 2, consider a set $\mathcal{S} = \{S_1, S_2, S_3\}$ of $\{x_3\} - B$-paths with $V(S_1) \cap (B \setminus \{u_2, u_3\}) = \{s\}$, $V(S_2) \cap B = \{u_2\}$, and $V(S_3) \cap B = \{u_3\}$. Because of planarity $s \notin V(P_1) \setminus \{x_1\} \cup V(R_1) \setminus \{x_4\}$.

Case 3.2.1. $s \in V(P_2) \cup V(P_3) \cup V(R_2) \setminus \{r\} \cup V(R_3)$.
This case can be handled similarly as Case 3.1.

Case 3.2.2. $s \in V[u_1, u_2]$.
Because of planarity $s \in V[r, u_2]$.

Case 3.2.2.1. $r, s \in V[u_1, x_2], r \neq s$.
One of the cycles $[r, u_2] \cup S_2 \cup S_3 \cup P_3 \cup P_1 \cup R_1 \cup R_2$ and $[s, u_2] \cup P_2 \cup P_1 \cup [u_1, r] \cup R_2 \cup R_3 \cup S_3 \cup S_1$ has the desired length.

Case 3.2.2.2. $r, s \in V[x_2, u_2], r \neq s$.
This case can be handled similarly as Case 3.2.2.1.

Case 3.2.2.3. $r \in V[u_1, x_2], s \in V[x_2, u_2]$.
One of the cycles $[r, u_2] \cup S_2 \cup S_3 \cup P_3 \cup P_1 \cup R_1 \cup R_2$ and $[u_1, s] \cup S_1 \cup S_2 \cup P_2 \cup P_3 \cup R_3 \cup R_1$ has the desired length.

Case 3.2.2.4. $r = s \in V[u_1, x_2] \setminus \{u_1, x_2\}$.

Case 3.2.2.4.1. $|V[r, u_2]| \geq \frac{1}{9}|V(C)| + \frac{4}{3}$.
The cycle $[r, u_2] \cup S_2 \cup S_3 \cup P_3 \cup P_1 \cup R_1 \cup R_2$ has the length at least $\frac{1}{9}|V(C)| + 5 + \frac{4}{3}$.

Case 3.2.2.4.2. $|V[r, u_2]| < \frac{1}{9}|V(C)| + \frac{4}{3}$.
We have $|V[u_1, r]| > \frac{2}{9}|V(C)| - \frac{1}{3}$. With $t = 2$, $x = x_2$, $B = V[u_1, r] \cup V(P_1) \cup V(P_2) \cup V(P_3) \cup V(R_1) \cup V(R_2) \cup V(R_3) \cup V(S_1) \cup V(S_2) \cup V(S_3)$, $\mathcal{Q} = \{[r, x_2], [x_2, u_2]\}$, and Lemma 2, consider a set $\mathcal{T} = \{T_1, T_2, T_3\}$ of $\{x_2\} - B$-paths with $V(T_1) \cap (B \setminus \{u_2, r\}) = \{q\}$, $V(T_2) \cap B = \{u_2\}$, and $V(T_3) \cap B = \{r\}$. Because of planarity $q \notin V(P_3) \setminus \{x_1\} \cup V(R_1) \setminus \{u_1\} \cup V(R_2) \cup V(R_3) \cup V(S_3) \setminus \{x_3\}$.

Case 3.2.2.4.2.1. $q \in V(P_1) \setminus \{u_1\} \cup V(P_2) \cup V(S_1) \cup V(S_2)$.
It is easy to see that there is always a cycle D with $V[u_1, r] \cup \{x_1, x_2, x_3, x_4, u_3\} \subseteq V(D)$, hence, $|V(D)| > \frac{2}{9}|V(C)| + 5 - \frac{1}{3}$. Because $|V(D)|$ is an integer, we obtain $|V(D)| \geq \frac{2}{9}|V(C)| + \frac{43}{9} \geq \frac{1}{9}|V(C)| + \frac{19}{3}$. Note that $|V(C)| \geq 14$ is needed here.

Case 3.2.2.4.2.2. $q \in V[u_1, r] \setminus \{u_1\}$.
Consider the cycles $[q, r] \cup S_1 \cup S_3 \cup R_3 \cup R_1 \cup P_1 \cup P_2 \cup T_2 \cup T_1$ and $[u_1, q] \cup T_1 \cup T_2 \cup P_2 \cup P_3 \cup S_3 \cup S_1 \cup R_2 \cup R_1$. The sum of their length is at least $|V[u_1, r]| + 13 > \frac{2}{9}|V(C)| + 13 - \frac{1}{3}$. Hence, one of them has the desired length.

Case 3.2.2.4.2.3. $q = u_1$.
We have $|V[u_1, r]| > \frac{2}{9}|V(C)| - \frac{1}{3} \geq \frac{25}{9} > 2$. Then since $|V[u_1, r]|$ is an integer, we have $|V[u_1, r]| \geq 3$ and $V[u_1, r] \setminus \{u_1, r\} \neq \emptyset$. The graph G' obtained by removing $\{u_1, r\}$ is still connected. Hence, there is a $V[u_1, r] - (V(P_1) \cup V(P_2) \cup V(P_3) \cup V(R_1) \cup V(R_2) \cup V(R_3) \cup V(S_1) \cup V(S_2) \cup V(S_3) \cup V(T_1) \cup V(T_2) \cup V(T_3))$-path P connecting $h \in V[u_1, r] \setminus \{u_1, r\}$ and a certain vertex g in G'. Again consider the graph G. Because of planarity $g \in V(R_1) \cup V(R_2) \cup V(T_1) \cup V(T_3)$. The cases $g \in V(R_2)$ and $g \in V(T_1)$ can be handled similarly as the cases 3.2.2.1 and 3.2.2.4.2.2, respectively.

Case 3.2.2.4.2.3.1. $g \in V(R_1)$.
Let Q be the subpath of R_1 connecting g and x_4. Consider the cycles $Q \cup P \cup [q, h] \cup T_1 \cup T_3 \cup S_1 \cup S_2 \cup P_2 \cup P_3 \cup R_3$ and $Q \cup P \cup [h, r] \cup S_1 \cup S_2 \cup T_2 \cup T_1 \cup P_1 \cup P_3 \cup R_3$. The sum of their length is at least $|V[u_1, r]| + 13 > \frac{2}{9}|V(C)| + 13 - \frac{1}{3}$. Hence, one of them has the desired length.

Case 3.2.2.4.2.3.2. $g \in V(T_3)$.
Let Q be the subpath of T_3 connecting g and x_2. Consider the cycles $Q \cup P \cup [u_1, h] \cup P_1 \cup P_3 \cup R_3 \cup R_2 \cup S_1 \cup S_2 \cup T_2$ and $Q \cup P \cup [h, r] \cup S_1 \cup$

$S_2 \cup P_2 \cup P_3 \cup R_3 \cup R_1 \cup T_1$. The sum of their length is at least $|V[u_1, r]| + 13 > \frac{2}{9}|V(C)| + 13 - \frac{1}{3}$. Hence, one of them has the desired length.

Case 3.2.2.5. $r = s \in V[x_2, u_2] \setminus \{x_2, u_2\}$.
This case can be handled similarly as Case 3.2.2.4. ∎

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