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PRODUCT ROSY LABELING OF GRAPHS

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Abstract

In this paper we describe a natural extension of the well-known ρ-labeling of graphs (also known as rosy labeling). The labeling, called product rosy labeling, labels vertices with elements of products of additive groups. We illustrate the usefulness of this labeling by presenting a recursive construction of infinite families of trees decomposing complete graphs.

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1. Introduction

Graph labelings of various kinds are being used in many different contexts. One of the most studied applications is the decomposition of complete and complete bipartite graphs into mutually isomorphic subgraphs. In particular, decompositions into isomorphic trees were investigated by many authors.

Let G be a graph with at most n vertices. We say that the complete graph K_n has a *G-decomposition* if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i.

In 1967 A. Rosa [20] introduced some important types of vertex labelings. Graceful labeling (called β-valuation by AR) and rosy labeling (called ρ-valuation by AR) are useful tools for decompositions of complete graphs K_{2n+1} into graphs with n edges. We define rosy labeling in a slightly different manner which suits better our further needs. A *labeling* of a graph G with n edges is an injection from $V(G)$, the vertex set of G, into a subset S of the set $\{0, 1, 2, \dots, 2n\}$ of elements of the additive group Z_{2n+1}. Later we will use a more general definition. Let ρ be the injection. The *length* of an edge e with endvertices x and y is defined as $\ell(e) = \min\{\rho(x)-\rho(y), \rho(y)-\rho(x))\}$. Notice that the subtraction is performed in Z_{2n+1} and hence $1 \leq \ell(e) \leq n$. If the set of all lengths of the n edges is equal to $\{1, 2, \dots, n\}$ and $S \subseteq \{0, 1, \dots, 2n\}$, then ρ is a *rosy labeling*; if $S \subseteq \{0, 1, \dots, n\}$ instead, then ρ is a *graceful labeling*. A graceful labeling ρ is said to be an *α-labeling* if there exists a number ρ_0 with the property that for every edge e in G with endvertices x and y and with $\rho(x) < \rho(y)$ it holds that $\rho(x) \leq \rho_0 < \rho(y)$. Obviously, G must be bipartite to allow an α-labeling. We also say that a graph *is* graceful or rosy rather than that it has a graceful or rosy labeling. For an exhaustive survey of graph labelings, see Gallian [9].

Each graceful labeling is of course also a rosy labeling. One can observe that if a graph G with n edges has a graceful or rosy labeling, then K_{2n+1} can be cyclically decomposed into $2n+1$ copies of G. It is so because K_{2n+1} has exactly $2n+1$ edges of length i for every $i = 1, 2, \dots, n$ and each copy of G contains exactly one edge of each length. The cyclic decomposition is constructed by taking a labeled copy of G, say G_0, and then adding an element $i \in Z_{2n+1}$ to the label of each vertex of G_0 to obtain a copy G_i for $i = 1, 2, \dots, 2n$.

Because of this nice property of rosy labelings, the most studied decompositions of complete graphs into trees deal with trees with n edges decomposing K_{2n+1}. Another direction is based on applications of α-labelings. It was shown in [20] that if G with n edges allows an α-labeling, then there exists a G-decomposition of K_{2nm+1} for every positive integer m. Decompositions into spanning trees (i.e., factorizations) gained some attention only recently. For factorization of complete graphs, see, e.g., [2, 4–8, 12–17], for factorization of complete bipartite graphs, see [3].

Clearly, for decomposition purposes a graceful labeling seemingly offers little advantage over a rosy labeling, because a rosy labeling is less restrictive and therefore often easier to find. It is worth noting that in spite of that there has been significantly more effort in the investigation of graceful graphs than in rosy graphs. This may be due to two reasons. The first reason is the Kotzig-Ringel-Rosa conjecture, which predicts that *all trees are graceful.* (In fact, Ringel [18] only conjectured that for a given tree T on $n+1$ vertices, there is a T-decomposition of K_{2n+1}, while Kotzig later conjectured that there is a *cyclic* decomposition—see [20]. This along with Rosa's discovery of β-labeling and Golomb's introduction of the name "graceful" is probably the complete etymology of the Graceful Tree Conjecture.) This conjecture is indeed very appealing, and apparently very hard. There are too many papers constructing various classes of graceful trees to be listed here. Despite this tremendous effort, the conjecture is far from being solved. The most significant "closed" families of graceful trees are caterpillars [20], trees with at most four vertices of degree 1 [20], trees of diameter at most 4 [21] and at most 5 [11], and all trees with at most 27 vertices [1].

As far as the other reason is concerned, we dare to suggest that it is the name of the labeling. While the graceful labeling has an attractive name, this was not true until recently for the rosy labeling, which was only known as the ρ-labeling. We certainly hope that the introduction of its new name will help the labeling to gain more attention.

2. PRODUCT ROSY LABELING

We present here a labeling, which generalizes properties of the rosy labeling introduced by A. Rosa. Using this labeling, we then describe a recursive procedure that produces infinite families of trees that decompose complete graphs. To simplify our notation, we usually identify vertices with their respective labels. We will say "a vertex i" rather than "a vertex x with $\rho(x) = i$".

While the rosy labeling is based on the additive group Z_{2n+1}, we generalize the idea in this note and label graphs by elements of products of odd additive groups, $Z_{2n_1+1} \times Z_{2n_2+1} \times \cdots \times Z_{2n_q+1}$.

For simplicity, we define our labeling just for the product $\mathcal{Z} = Z_{2n+1} \times Z_{2m+1}$. The generalization to a product of q groups is straightforward and left to the reader. We denote by (x, y) a vertex with $x \in Z_{2n+1}, y \in Z_{2m+1}$ and by $[(x, y)(u, v)]$ an edge joining vertices (x, y) and (u, v).

Definition 1. Let $n, m \geq 1$ and G be a graph with $n+m(2n+1) = 2nm+n+m$ edges and the vertex set $V \subseteq Z_{2n+1} \times Z_{2m+1}$. Let $e = [(x_1, y_1)(x_2, y_2)]$ be an edge of G. We define the *dimension* of e as $\dim(e) = \|a, b\|$, where $b = \min(y_1 - y_2, y_2 - y_1)$ and the subtraction is performed in Z_{2m+1}, and

(1) $a = \min(x_1 - x_2, x_2 - x_1)$ if $b = 0$ or

(2) $a = x_1 - x_2$ if $b \neq 0$ and $y_1 = y_2 + b \pmod{2m+1}$ or

(3) $a = x_2 - x_1$ if $b \neq 0$ and $y_2 = y_1 + b \pmod{2m+1}$.

The subtractions $x_1 - x_2$ and $x_2 - x_1$ are performed in Z_{2n+1}.

We say that G has a *product rosy labeling* $\rho^\times$ if the set of dimensions of all edges of G is equal to $\{\|i, 0\| : i = 1, 2, \ldots, n\} \cup \{\|j, k\| : j = 0, 1, \ldots, 2n;$ $k = 1, \ldots, m\}$. (Formally, $\rho^\times$ can be viewed as the mapping from the *unlabeled* graph G into $\mathcal{Z}$.)

Notice that if $b = 0$, then $y_1 = y_2$ and the vertices (x_1, y_1) and (x_2, y_2) belong to the same coset of $\mathcal{Z}/Z_{2m+1}$ and $x_1 \neq x_2$. Therefore, in this case a cannot be equal to zero. At the same time, because here $a = \min(x_1 - x_2, x_2 - x_1)$, it is obvious that $a \leq n$. On the other hand, if the vertices (x_1, y_1) and (x_2, y_2) belong to different cosets of $\mathcal{Z}/Z_{2m+1}$, we always subtract both entries in the same order. It means that either $\dim(e) = \|x_2 - x_1, y_2 - y_1\|$ (when $y_2 - y_1 < y_1 - y_2$), or $\dim(e) = \|x_1 - x_2, y_1 - y_2\|$ (when $y_1 - y_2 < y_2 - y_1$). In this case the difference $x_1 - x_2$ or $x_2 - x_1$ can be any element of Z_{2n+1}.

We also remark that if we allow $m = 0$, then the labeling is equivalent to a rosy labeling on Z_{2n+1}. Therefore, we will further assume that both n and m are greater than zero.

We observed above that the decompositions based on rosy labelings are cyclic. Although it is not true for decompositions arising from product rosy labelings (unless $2n + 1$ and $2m + 1$ are co-prime), one important property of rosy decompositions is carried over to the product rosy decompositions.

Definition 2. Let $\mathcal{H} = \{H_0, H_1, \ldots, H_{2p}\}$ be an H-decomposition of K_{2p+1} defined by isomorphisms $\varphi_i(H_0) = H_i$ for $i = 0, 1, \ldots, 2p$. If for a vertex $x \in H$ the mapping $\Phi(x) : \{\varphi_i(x) : i = 0, 1, \ldots, 2p\} \to V(K_{2p+1})$ is a bijection, we say that x is a *bijective* vertex in $\mathcal{H}$. If all vertices of H are bijective in $\mathcal{H}$, then we say that $\mathcal{H}$ is a *bijective decomposition* of K_{2p+1}.

As an example of a graph that allows both bijective and non-bijective decomposition, we use a caterpillar R with four edges. There are two such caterpillars, P_4, and R with adjacent vertices x_0 and x_1 of degree 2 and 3, respectively, and three vertices of degree 1: y_0 adjacent to x_0 and y_1, y_2 adjacent to x_1. Obviously, K_9 has a bijective R-decomposition, since R is graceful, and the decomposition is in fact cyclic. Let us now denote the vertices of K_9 by pairs (i, j), where $i, j \in \{0, 1, 2\}$. Let the nine copies of R be R_{rs}, where also $r, s \in \{0, 1, 2\}$, and let $R_{rs} = \psi_{rs}(R)$. Set $\psi_{rs}(x_l) = (r, s + l)$ and $\psi_{rs}(y_t) = (r + 1, t)$, where the addition is performed modulo 3. One can check that this collection gives an R-decomposition of K_9 in which, e.g., the vertex $(1,0)$ is an image of y_0 in precisely three copies of R. Namely, in R_{00}, R_{01}, R_{02}. Therefore, the decomposition is *not* bijective.

We now prove that a graph G with $2nm+n+m$ edges and a product rosy labeling decomposes the complete graph K_{2p+1} with $2p+1 = (2n+1)(2m+1)$ vertices.

Theorem 3. *Let $n, m \geq 1$ and G be a graph with $p = 2nm + n + m$ edges that allows a product rosy labeling. Then there exists a G-decomposition of the complete graph K_{2p+1} with $2p+1 = (2n+1)(2m+1)$ vertices. Moreover, this decomposition is bijective.*

Proof. Let $G_{00} \cong G$. Define mappings $\varphi'_{cd} : \mathcal{Z} \to \mathcal{Z}$ for $c = 0, 1, \ldots, 2n$; $d = 0, 1, \ldots, 2m$ by $\varphi'_{cd}(x, y) = (x + c, y + d)$. Obviously, the induced mappings $\varphi_{cd} : V(G_{00}) \to V(G_{cd})$ are graph isomorphisms. Our goal is to show that the family $\mathcal{G} = \{G_{cd} : c = 0, 1, \ldots, 2n;\ d = 0, 1, \ldots, 2m\}$ is a G-decomposition of K_{2p+1}.

First we want to prove that the isomorphisms are edge-dimension-preserving. Suppose that $\dim[(x, y)(u, v)] = \|a, b\|$ where, without loss of generality (WLOG), $a = u - x$, $b = v - y$. Then $[\varphi(x, y)\varphi(u, v)] = [(x + c, y + d)(u + c, v + d)]$ and since $u + c - (x + c) = u - x = a$ and $v + d - (y + d) = v - d = b$, we have $\dim[\varphi(x, y)\varphi(u, v)] = \|a, b\|$.

Now we show that an arbitrary edge $e' = [(x', y')(u', v')]$ with dimension $\|a, b\|$ belongs to at least one image of G_{00}. We can suppose WLOG that

$$a = u' - x' \quad \text{and} \quad b = v' - y'. \tag{1}$$

Because G_{00} has a product rosy labeling, there is exactly one edge of dimension $\|a, b\|$ in G_{00}, say $e = [(x, y)(u, v)]$. There indeed exist $c \in Z_{2n+1}$ and $d \in Z_{2m+1}$ such that

$$x' = x + c \quad \text{and} \quad y' = y + d. \tag{2}$$

We want to show that then $u' = u+c, v' = v+d$ and hence $e' = [(x', y')(u', v')]$ belongs to G_{cd}. For if $\dim(e) = \|a, b\|$, then WLOG

$$a = u - x \quad \text{and} \quad b = v - y. \tag{3}$$

Combining (1) and (3), we get

$$u' - x' = u - x \quad \text{and} \quad v' - y' = v - y, \tag{4}$$

which along with (2) immediately yields

$$u' - u = x' - x = c \text{ and } v' - v = y' - y = d,$$

which we wanted to show. Thus $e' = [(x', y')(u', v')] = [(x + c, y + d)(u + c, v + d)]$ and $e' \in G_{cd}$.

Because there are exactly $(2n + 1)(2m + 1)$ graphs G_{cd} defined by the above isomorphisms, it now follows by a simple counting argument that e' belongs to exactly one such image. Therefore, the family $\mathcal{G}$ forms the desired decomposition.

The fact that all $2p+1$ images of any $(x, y) \in G_{00}$ are mutually distinct vertices of K_{2p+1} should be obvious by similar arguments as above. ■

The following theorem is a natural generalization of Theorem 3. The proof is a straightforward modification of the previous one and is left to the reader.

Theorem 4. *Let $2p + 1 = (2n_1 + 1)(2n_2 + 1) \dots (2n_q + 1)$ with $n_i \geq 1$ and G be a graph with p edges that allows a product rosy labeling. Then there exists a G-decomposition of the complete graph K_{2p+1}. Moreover, this decomposition is bijective.*

3. EXAMPLES

We illustrate the usefulness of the product labeling by two simple examples.

Example 5. Let H_1 be a graph with n_1 edges and a rosy labeling. Pick any $n_2 \geq 1$, and glue the central vertices of n_2 stars $S_2^i \cong K_{1,2n_1+1}$ for $i = 1, 2, \dots, n_2$ to some vertices of H_1. Notice that we can glue more than one star to the same vertex of H_1. The resulting graph, H_2, allows a product rosy labeling. Each original label y of H_1 is replaced by label $(y, 0)$. The vertices of degree one in each star S_2^i for $i = 1, 2, \dots, n_2$ will receive labels (j, i), $i = 0, 1, \dots, 2n_1$ for some j. Therefore, their dimensions are $\|j, i\|$, $j = 0, 1, \dots, 2n_1; i = 1, 2, \dots, n_2$. It is easy to check that the resulting graph has a product rosy labeling.

Because this labeling is bijective, we can now construct a sequence $H_1, H_2, \dots, H_k, \dots, H_m$ of graphs with product rosy labelings. In each step, to obtain a graph H_{k+1} we glue to the graph H_k with p_k edges n_{k+1} stars $S_{k+1}^i \cong K_{1,2p_k+1}$.

In the following example we present a method of constructing trees with product rosy labelings from smaller trees with rosy labelings.

Example 6. Let T and R be trees with n and m edges and rosy labelings τ and ρ, respectively. Pick a vertex $t \in T$ and label T such that $\tau(t) = 0$. Similarly, select a vertex $r \in R$, label R such that $\rho(r) = 0$, and view R as a rooted tree with the root 0. Glue R and T together at the vertices labeled 0.

Now we assign labels to the vertices of the amalgamated tree as follows. Each vertex of T receives label $(y, 0)$, where y is the original label of that vertex in τ. More precisely, for every $t \in T$ let $\rho^{\times}(t) = (\tau(t), 0)$. On the other hand, a vertex of R with label x in ρ will receive label $(0, x)$ in $\rho^{\times}$ or, strictly speaking, for every $r \in R$ let $\rho^{\times}(r) = (0, \rho(r))$. Finally, we add some new edges. For every edge labeled $[(0, x_1)(0, x_2)]$, where x_1 is the parent and x_2 is the child in the rooted tree R, we add $2n$ new edges $[(0, x_1)(k, x_2)]$ for $k = 1, 2, \ldots, 2n$. One can check that the resulting tree has a product rosy labeling. As the verification is straightforward, we are leaving it to the reader.

We believe that this labeling can be useful for further attempts to prove the original Ringel conjecture. While the decompositions based on product rosy labeling in general are not cyclic, there may be large classes of graphs obtained by amalgamating graceful and bigraceful graphs (defined by Ringel, Llado, and Serra—see, e.g., [19]) that would allow product rosy labeling and thus decompose complete graphs. Although we currently have no particular results in this direction, we certainly hope that our method will become a useful tool in the future.

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