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ON DISTINGUISHING AND DISTINGUISHING CHROMATIC NUMBERS OF HYPERCUBES

Werner Klöckl

Chair of Applied Mathematics
Montanuniversität Leoben, 8700 Leoben, Austria

e-mail: werner.kloeckl@mu-leoben.at

Abstract

The distinguishing number $D(G)$ of a graph G is the least integer d such that G has a labeling with d colors that is not preserved by any nontrivial automorphism. The restriction to proper labelings leads to the definition of the distinguishing chromatic number $\chi_D(G)$ of G.

Extending these concepts to infinite graphs we prove that $D(Q_{\aleph_0}) = 2$ and $\chi_D(Q_{\aleph_0}) = 3$, where $Q_{\aleph_0}$ denotes the hypercube of countable dimension. We also show that $\chi_D(Q_4) = 4$, thereby completing the investigation of finite hypercubes with respect to χ_D.

Our results extend work on finite graphs by Bogstad and Cowen on the distinguishing number and Choi, Hartke and Kaul on the distinguishing chromatic number.

Keywords: distinguishing number, distinguishing chromatic number, hypercube, weak Cartesian product.

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1. Introduction and Definitions

Given a graph G, its *distinguishing number* $D(G)$ is the least integer d such that G has a d-distinguishing labeling, where a labeling is *d-distinguishing* if it is not invariant under any nontrivial automorphism.

The *distinguishing number* was introduced by Albertson and Collins [2]. There exist numerous results about distinguishing numbers of graphs. For example Bogstad and Cowen [3] determined the distinguishing number of the hypercube Q_n of dimension n. They proved $D(Q_n) = 2$ for $n \geq 4$.

One way of looking at the n-cube is to consider it as K_2^n, the Cartesian product of n factors, all isomorphic to K_2. For the definition and many facts about this product we refer to [8]. In this sense Albertson [1] generalized the result of Bogstad and Cowen to connected, prime graphs G. He proved that $D(G^r) = 2$ for all $r \geq 4$, and, if $|V(G)| \geq 5$, then $D(G^r) = 2$ for all $r \geq 3$. Finally, the distinguishing number of all finite Cartesian powers was determined in [9] by proving that $D(G^k) = 2$ for any connected graph G and any $k \geq 2$, with the following three exceptions: $D(K_2^2) = D(K_2^3) = D(K_3^2) = 3$.

In the following section we generalize the result of Bogstad and Cowen to finite or countably infinite products of K_2's and K_3's, in particular to the infinite hypercube $Q_{\aleph_0}$. Its vertex set consists of all 0-1 sequences with finitely many 1s, where two vertices are adjacent if they differ in only one place. It can also be defined as the weak Cartesian product, see [8], of infinitely many K_2's. The first main result of this paper is that the distinguishing number of the weak Cartesian product of countably many P_i, $P_i \in \{K_2, K_3\}$, is 2.

An interesting variant of the distinguishing number is the *distinguishing chromatic number* $\chi_D(G)$. It is defined as the least integer d such that G has a d-distinguishing labeling which is a proper coloring of G (adjacent vertices have different labels) and was introduced in 2006 by Collins and Trenk [6]. They determined $\chi_D(G)$ of paths and cycles and upper bounds of $\chi_D(G)$ in terms of $\Delta(G)$ for trees and connected graphs in general.

Choi, Hartke and Kaul [5] proved, among other results, that $\chi_D(Q_n) = 3$ for $5 \leq n < \aleph_0$ and $n = 3$. In the third section we complete the investigation of hypercubes with respect to the distinguishing chromatic number. We show that $\chi_D(Q_4) = 4$ and give a proof of $\chi_D(Q_n) = 3$ for $8 \leq n \leq \aleph_0$. In the finite case our coloring has one color that is used only $O(n/2)$ times, whereas both other colors occur $O(2^{n-1})$ times.

2. Finite and Countable Products of K_2 and K_3

We start with a formal definition of the Cartesian product of possibly infinitely many factors. To this end let I be an index set and G_i, $i \in I$, be a family of graphs. Then the *Cartesian product*

$$G = \prod_{i \in I} G_i$$

is defined on the set x of all functions $x : i \mapsto x_i,\ x_i \in V(G_i)$, where two vertices $x,\ y$ are adjacent if there exists a $k \in I$ such that $x_k y_k \in E(G_k)$ and $x_i = y_i$ for $i \in I \setminus \{k\}$.

For products of infinitely many nontrivial graphs G_i, we note the first fundamental difference to the finite case. If we have only finitely many factors, then the product is connected if and only if the factors are. If we have infinitely many nontrivial factors, there are vertices that differ in infinitely many coordinates x_i. One cannot connect them by paths of finite length, since the endpoints of every edge differ in just one coordinate. Therefore such products are disconnected and we call the components of G *weak Cartesian products*. To identify a component, it suffices to know a single vertex of it. Thus the weak Cartesian product

$$G = \prod_{i \in I}^{a} G_i$$

is the connected component of $G = \prod_{i \in I} G_i$ containing the vertex a. Since we consider (only) countably infinite products, we can identify vertices with sequences, for example: The vertex $x : \mathbb{N} \to \bigcup_{i \in \mathbb{N}} V(G_i),\ i \mapsto x_i \in V(G_i)$ can be identified with the sequence $(x_1, x_2, \ldots)$.

The goal of this section is to prove $D(H) = 2$, where H is the weak Cartesian product $\prod_{i \in \mathbb{N}}^{v_0} P_i$ with $V(K_i) = \{0, 1, \ldots, i-1\}$, $P_i \in \{K_2, K_3\}$ and $v_0 = (0, 0, \ldots)$. Note that the infinite hypercube $Q_{\aleph_0}$ is a special case of the graph H. We begin with the labeling that was used by Bogstad and Cowen to show that $D(K_2^n) = 2$ for $n > 3$, because variants of this labeling will be used for the new results.

Theorem 2.1 (Bogstad and Cowen [3])**.** *$D(K_2^n) = 2$ for $n > 3$.*

Proof. Given $n \in \mathbb{N},\ n > 3$. We represent the vertices of K_2^n by all $0-1$ vectors of length n. Denote the vertex all of whose coordinates are zero by v_0 and the vertices whose first i coordinates are 1 and all the others zero by v_i $(i = 1, 2, \ldots, n)$. Clearly $v_0 v_1 v_2 \ldots v_n$ is a path P of length n that is isometrically embedded in K_2^n.

(a) We color all vertices of P and $v = (1, 0, 0, \ldots, 0, 1)$ white, the others black, and claim that this is a distinguishing coloring. The only white vertex with three white neighbors is v_1, thus it is fixed by any color preserving automorphism α. The vertices v, v_0 and v_n are the only white ones which have exactly one white neighbor. From $n > 3$ we conclude that v_n has the largest distance to v_1 among them. Hence the vertices $v_1, v_2, \ldots, v_n$ are fixed by α. But then v_0 is fixed as the antipode of v_n and also v as the only remaining white vertex.

(b) Consider two different vertices x, y of the hypercube that are not on the path P. Suppose they differ in coordinate i: $x(i) = 1 \neq 0 = y(i)$. If they have different distance to v_i, x cannot be mapped on y by α. If they have equal distance to v_i, we know that $d(x, v_{i-1}) = d(x, v_i) + 1 = d(y, v_i) + 1 = d(y, v_{i-1}) + 2$, which means that x and y have different distance to v_{i-1}. Therefore we know again that x cannot be mapped on y by α. Since x and y were arbitrarily chosen, all vertices of Q_n are fixed by α. ■

The main additional idea of the following corollary is that two fixed vertices in a triangle also fix the third vertex in the triangle. Using this fact, we can generalize the result of Bogstad and Cowen to arbitrary finite Cartesian products of K_2's and K_3's with more than three factors.

Corollary 2.2. *$D(\prod_{i \in S} P_i) = 2$ for $P_i \in \{K_2, K_3\}$ if S is a finite set with $|S| > 3$.*

Proof. $H = \prod_{i \in S} P_i$, $|S| = n$. The vertex set of H be the set of all vectors of length n with entries 0, 1 or 2 in the coordinates i with $P_i = K_3$ and entries 0 or 1 in the coordinates j with $P_j = K_2$. The vertices v_0, v_1,…,v_n and v and the path P be defined as in the proof of Theorem 2.1.

We color all vertices of P and $v = (0, 1, 0, 0, \ldots.)$ white, the others black. Then each single vertex of P is fixed by any color preserving automorphism α by the same arguments as in part (a) of the last proof.

Furthermore we define the vertex u_{i_0} for every index i_0 with $P_{i_0} = K_3$ as follows: u_{i_0} is the vertex with $i_0 - 1$ entries 1 in the first $i_0 - 1$ coordinates, 2 in the i_0-th and 0 in the other coordinates. u_{i_0} is fixed, because it is the only common neighbor of v_{i_0-1} and v_{i_0}.

Consider two different vertices x, y of the given product that are not on the path P. Suppose they differ in coordinate i. W.l.o.g. we assume $x(i) = 2 \neq 0 = y(i)$. If they have different distance to u_i, x cannot be mapped on y by α. If they have equal distance to u_i, we know that $d(x, v_{i-1}) = d(x, u_i) + 1 = d(y, u_i) + 1 = d(y, v_{i-1}) + 2$, which means that x and y have different distance to v_{i-1}. Therefore we infer that x cannot be mapped onto y by α. Since x and y were arbitrarily chosen, all vertices of the product are fixed by α. ■

We now present the main result of the section. It states that the distinguishing number of the weak Cartesian product of K_2's and K_3's is 2. The proof extends the preceding ideas.

Theorem 2.3. $D(\prod_{i\in\mathbb{N}}^{v_0} P_i) = 2$ *for* $V(K_i) = \{0, 1, \ldots, i{-}1\}$, $P_i \in \{K_2, K_3\}$ *and* $v_0 = (0, 0, \ldots)$.

Proof. Given $H = \prod_{i\in\mathbb{N}}^{v_0} P_i$ as in the statement. The vertex set of H is the set of all sequences with finitely many entries different from 0, where the entries in the coordinates i with $P_i = K_3$ are from the set $\{0, 1, 2\}$ and the other entries are in $\{0, 1\}$. Let the vertices v_1, v_2, ... be defined as in the proof of Theorem 2.1 and P be the one-sided infinite path $v_0v_1v_2\ldots$. We color all vertices of P white, the others black, and claim that this is a distinguishing coloring.

Every color-preserving automorphism α of H stabilizes P. Since v_0 is the only vertex of degree 1 in P, considered as a subgraph of H, it is fixed by α. But then v_1, as the only neighbor of v_0 in P, is also fixed. In general, each vertex v_i $(i > 0)$ is the only white vertex of distance i to v_0. Thus every v_i must be fixed.

The proof is completed analogously to the proof of Corollary 2.2. ■

3. The Distinguishing Chromatic Number

At the beginning of this section we determine the only not-yet-known distinguishing chromatic number of a finite hypercube, namely $\chi_D(Q_4)$. In the first part of the proof of Theorem 3.1 we show that there is no chromatic 3-distinguishing coloring of Q_4. Unfortunately we have to consider many cases. In the second part we simply define a proper 4-coloring which turns out to be also 4-distinguishing.

Theorem 3.1. *The distinguishing chromatic number of the hypercube of dimension* 4 *is* 4.

Proof. We label the vertices of Q_4 with the subsets of the set $\{1, 2, 3, 4\}$ in such a way that adjacent vertices have labels that differ in exactly one of the elements 1, 2, 3, 4. For example, the vertices $\{1, 2\}$ and $\{1, 2, 3\}$ are adjacent, but not $\{1, 2, 3\}$ and $\{1, 2, 4\}$, because they can be distinguished by 3 and 4, see Figure 1.

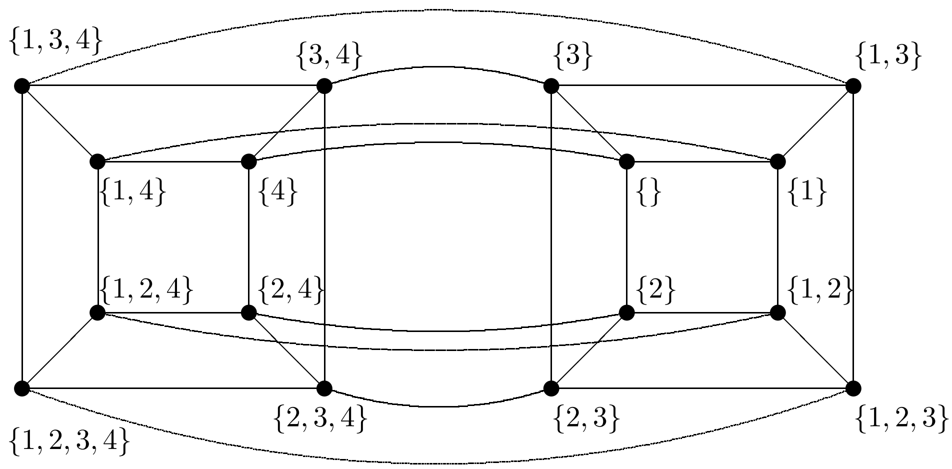

Figure 1. The labeling of Q_4

The distance between two vertices in Q_4 is the cardinality of the symmetric difference of their labels. Thus $\{\}$ and $\{1,2,3,4\}$ are antipodal vertices just as $\{1,3\}$ and $\{2,4\}$. The set of vertices of distance i $(0 \leq i \leq 4)$ from $\{\}$ constitutes level i and is denoted L_i.

It is nice to see that the interchange of two digits, for example 2 and 3, in each label defines an automorphism on Q_4. Such automorphisms are denoted by $\alpha_{(ij)}$ $(1 \leq i < j \leq 4)$, where the digits i and j are interchanged. Similarly $\alpha_{(ij)(kl)}$ denotes the product of $\alpha_{(ij)}$ and $\alpha_{(kl)}$. All these automorphisms preserve all L_i.

It is useful to see that $V_1 \cup V_2$ is the bipartition of Q_4, where $V_1 = \bigcup_{i \in \{1,3\}} L_i$ and $V_2 = \bigcup_{i \in \{0,2,4\}} L_i$. Further, we sometimes need that the union of two neighborhoods of two vertices in V_2 (it also holds for V_1) covers at least six vertices. This is clear, because by symmetry we can assume that $\{\}$ is one of the vertices and the second vertex covers at least two vertices of L_3.

Claim. The union of the neighborhoods of three vertices in V_1 or V_2, respectively, covers at least seven vertices in V_2 or V_1, respectively.

Proof of the Claim. By symmetry we can assume that $\{\}$ is one of the three vertices. If the the antipode $\{1,2,3,4\}$ is one of the two other vertices, all eight vertices in V_1 are covered by the neighborhoods of these three vertices. If the two other vertices are both in L_2, they cover at least three (additional) vertices in L_3, because two vertices in L_2 have at most one common neighbor in L_3.

We now show that there is no chromatic distinguishing coloring on three colors.

Suppose there is a chromatic distinguishing coloring on three colors, say white, black and green.

At first we wish to show that there is no three-coloring of Q_4, where both parts of the above bipartition consist of three colors: Assume the coloring has this property. No part of the partition can include more than four vertices of one color, because otherwise there would be no place for a vertex of this color in the other part. Clearly there must be one color, say green, with three or four vertices in V_1, but this implies that V_2 contains at most one green vertex. Hence we can assume without loss of generality that there are four white and three black vertices in V_2. Thus there can be at most one white and one black vertex in V_1, contrary to the fact that V_1 consists of eight vertices.

Since it is not possible that one part consists of vertices of three colors and all vertices in the other part have the same color, we always can assume that one part has exactly two colors, say V_2 and that it is colored white and green. Now we just have to check the cases (a), where V_1 is monochromatic and (b), where V_1 is two- or three-chromatic.

Case (a) All vertices in V_1 are black.
For symmetry reasons it is sufficient to consider the cases $1 \leq g_2 \leq 4$, where g_i denotes the number of green vertices in V_i ($i \in \{1, 2\}$).

Subcase (i) $g_2 = 1$.
We can assume $\{\}$ is the green vertex in L_2. Each $\alpha_{(ij)}$ works then.

Subcase (ii) $g_2 = 2$.
If the green vertices are not antipodal we can assume that $\{\}$ and $\{i, j\}$ are green. $\alpha_{(ij)}$ does the job. Otherwise we can assume that $\{\}$ and $\{1, 2, 3, 4\}$ are green. Each $\alpha_{(ij)}$ works in this case.

Subcase (iii) $g_2 = 3$.
If no two of the three green vertices are antipodal, we can assume $\{\}$, $\{1, 2\}$ and $\{1, 3\}$ are green. $\alpha_{(23)}$ does the job.

If there is an antipodal green pair, we can assume $\{\}$, $\{1, 2\}$ and $\{1, 2, 3, 4\}$ are green. $\alpha_{(12)}$ works then.

Subcase (iv) $g_2 = 4$.
If no two of the four green vertices are antipodal, we can assume $\{\}$, $\{1,2\}$, $\{1,3\}$ and $\{1,4\}$ are green. $\alpha_{(23)}$ does the job.

If there is one antipodal green pair, we can assume $\{\}$, $\{1,2\}$, $\{1,3\}$ and $\{1,2,3,4\}$ are green. $\alpha_{(23)}$ preserves the labeling.

If there are two antipodal green pairs, we can assume $\{\}$, $\{1,2\}$, $\{3,4\}$ and $\{1,2,3,4\}$ are green. $\alpha_{(12)}$ works then.

Case (b) Assume $\{\}$ to be green.
We consider the subcases $g_2 = 1, 2, 3, 4$.

Subcase (i) $g_2 = 1$.
If $g_1 = 0$, all vertices in V_1 must be black, which was considered in Case (a).

Hence, $1 \leq g_1 \leq 4$:

The green vertices of V_1 must be in L_3. If $g_1 < 3$ we can interchange two white vertices of L_3, otherwise two green vertices, where all colors and levels are preserved.

In detail: If $g_1 = 1$, we can assume that $\{1,2,3\}$ is green and $\alpha_{(12)}$ is color preserving. If $g_1 = 2$, we can assume by using level preserving automorphisms that $\{1,2,3\}$ and $\{1,2,4\}$ are green, thus $\alpha_{(12)}$ is color preserving again. If $g_1 = 3$, we can assume that $\{1,2,3\}$, $\{1,2,4\}$ and $\{1,3,4\}$ are green and $\alpha_{(34)}$ is color preserving in this case. If all vertices in L_3 are green, any level preserving automorphism works.

Subcase (ii) $g_2 = 2$.
Then $0 \leq g_1 \leq 2$. $g_1 = 0$ was considered in Case (a). If $g_1 > 0$, the second green vertex of V_2 must be in level 2 and we can assume that it is $\{1,2\}$. The green vertices of V_1 are in L_3 and in any case we can find some color preserving automorphism analogously to subcase (i).

Subcase (iii) $g_2 = 3$.
If $\{1,2,3,4\}$ is green, $g_1 = 0$, which was considered in Case (a). If there are two green vertices in L_2, two things are possible: They can have distance two as $\{1,2\}$ and $\{1,3\}$. In this case $\{2,3,4\}$ can be green, too, but then $\alpha_{(23)}$ is color preserving.

They can be antipodal as $\{1,2\}$ and $\{3,4\}$, but then $g_1 = 0$.

Subcase (iv) $g_2 = 4$.
If there is an antipodal pair of green vertices in V_2, there must be also a white antipodal pair in V_2, but then all vertices in V_1 are black, which was considered in Case (a). If there is no antipodal pair of green vertices in V_2,

we can assume that $\{1,2\}$, $\{1,3\}$ and $\{1,4\}$ are green. In this case $\{2,3,4\}$ can be green and $\{1\}$ can be white. All other vertices of L_1 are in any case black. Thus $\alpha_{(34)}$ works.

Now we know $\chi_D(Q_4) > 3$. To show that $\chi_D(Q_4) = 4$ we define a 4-coloring, see Figure 2, and show that it is distinguishing.

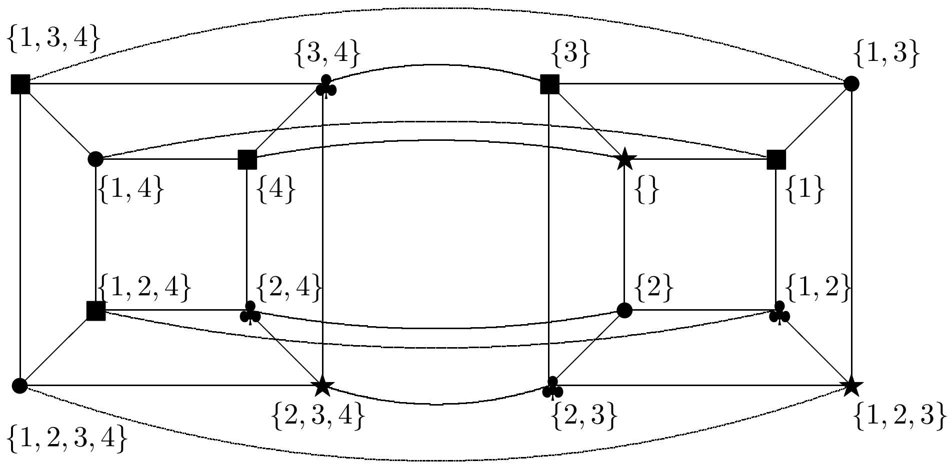

Figure 2. $\chi_D(Q_4) \le 4$

$\{\}$ is the only ★ vertex with no ♣ neighbor, $\{1,2,3\}$ is the only ★ vertex with exactly two ♣ neighbors and $\{2,3,4\}$ is the only ★ vertex with three ♣ neighbors. Hence the ★ vertices are fixed. Their antipodal vertices $\{4\}$, $\{1\}$ and $\{1,2,3,4\}$ are fixed, too. $\{2\}$ is the only • neighbor of $\{\}$, thus it is fixed as $\{1,3,4\}$, its antipode. $\{\}$ fixed implies: Neighbors of $\{\}$ must be mapped on neighbors of $\{\}$. $\{1\}$, $\{2\}$ and $\{4\}$ fixed implies $\{3\}$ and its antipode $\{1,2,4\}$ are fixed. Different vertices have different neighborhoods and the neighborhoods of the vertices in level two consist of vertices in level one and three, which are already fixed. From this we conclude that all vertices in level two are fixed, too. ■

The next theorem pertains to finite and infinite graphs. For finite graphs it is an immediate consequence of a theorem of Choi, Hartke and Kaul [5]. Our proof works for finite and countably infinite hypercubes. For finite dimension n it uses only $O(n/2)$ vertices of one color and $O(2^{n-1})$ vertices of the others. A set of all vertices of one color will be called a *color class* henceforth.

Theorem 3.2. *The distinguishing chromatic number of the hypercube Q_n with $8 \leq n \leq \aleph_0$ is three. In the finite case our labeling has one color class of size $O(n/2)$, whereas the other two have size $O(2^{n-1})$.*

Proof. (a) n is finite.
We label the vertices with the subsets of $\{1, 2, \ldots, n\}$. The vertices v_i $(0 \leq i \leq n)$ are defined as $\{1, 2, \ldots, i\}$ and $v_0, v_1, \ldots, v_n$ be the path P. The idea is to fix this path as in Lemma 2.1. When we have done this, we are ready, because the rest is analogous to part (b) of the proof of Lemma 2.1.

Let V_1 be the set of vertices in Q_n with odd distance to v_0, V_2 the set of those with even distance to v_0 and L_i the set of vertices with distance i to v_0. Clearly $V_1 \cup V_2$ is the bipartition of Q_n. The vertices v'_i $(i \in \mathbb{N})$ are defined as $\{1, 2, \ldots, i-1, i+1\}$.

We color all vertices of P that are in V_2 green $(O(n/2))$, the remaining vertices of V_2 black. Next we color the vertices p and q green, where p is defined as $\{2, 4, 6\}$ and q as $\{4, 6, 8\}$ if $n < 10$. For bigger n we set $q = \{6, 8, 10, \ldots, 2 * [n/2]\}$ if $[n/2]$ is odd and $q = \{8, 10, \ldots, 2 * [n/2]\}$ if $[n/2]$ is even. This ensures that both, p and q, are in V_1. The other vertices of V_1 are colored white. Neither p nor q has a green neighbor in V_2, so we have a chromatic three-coloring.

The vertex v_0 must be mapped onto itself by any color preserving automorphism α, because v_0 and $v_{2*[n/2]}$ are the only green vertices in V_2 that have distance two to exactly one green vertex and there is no green vertex x in V_1 with $d(x, v_{2*[n/2]}) = d(p, v_0) = 3$. But then it is not hard to see that all green vertices of P are fixed by α. Since $d(p, v_2) < d(q, v_2)$, p and q are also fixed.

For odd i we know that v_i and v'_i are the only common neighbors of v_{i-1} and v_{i+1}, hence α maps $\{v_i, v'_i\}$ onto itself. The vertices v_i and v'_i have different distance to at least one of the fixed vertices p or q, thus they are fixed by α.

(b) $n = \aleph_0$.
The vertex set of $Q_{\aleph_0}$ can be considered to be the set of all finite subsets of $\mathbb{N}$. The vertices v_i, v'_j and the vertex sets V_1, V_2 be defined as in (a), the one-sided infinite path $v_0 v_1 v_2 \ldots$ will be called P.

We color all vertices of P that are in V_2 green, the other vertices of V_2 black. In V_1 we color the vertices $\{2, 4, 6\}$, $\{8, 10, 12, 14, 16\}$, $\{18, 20, 22, 24, 26, 28, 30\}$, $\ldots$ green, the remaining vertices white.

It is not hard to see that no two green vertices are adjacent. Since $V_1 \cup V_2$ is the bipartition of $Q_{\aleph_0}$, this is a chromatic three-coloring. The vertex v_0 is the only green vertex to which only one green vertex has distance two, hence it is fixed by any color preserving automorphism α and therefore all green vertices of P. The green vertices of V_1 have pairwise different distance to v_0, thus they are also fixed by α.

The white vertices v_i of P (those with odd index) are fixed, because v_i and v_i' have different distance to one green vertex in V_1 and they are the only common neighbors of v_{i-1} and v_{i+1}, the remaining vertices of G are fixed by the same arguments as in the proof of Lemma 2.1. ■

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