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ON CRITICAL AND COCRITICAL RADIUS EDGE-INVARIANT GRAPHS

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Abstract

The concepts of critical and cocritical radius edge-invariant graphs are introduced. We prove that every graph can be embedded as an induced subgraph of a critical or cocritical radius-edge-invariant graph. We show that every cocritical radius-edge-invariant graph of radius $r \geq 15$ must have at least $3r + 2$ vertices.

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1. Introduction

Let $G = (V(G), E(G))$ be an undirected connected graph with no loops or multiple edges. The distance $d_G(u, v)$ (or simply $d(u, v)$) between vertices u and v is the length of a shortest path joining u and v in G. The *eccentricity* $e(v)$ of v is the distance to a farthest vertex from v. The radius $r(G)$ and diameter $d(G)$ are the minimum and maximum eccentricities, respectively. The *center* $C(G)$ and *periphery* $P(G)$ of graph G consist of the sets of vertices of minimum and maximum eccenticity, respectively. Vertices within $C(G)$ are called central vertices, and those within $P(G)$ are peripheral vertices. A graph is $self$-*centered* if $V(G) = C(G)$. The set $N_i(v)$ of all vertices

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at distance i from v will be called *i-th neighbourhood* of v. If $i = 1$ we will simply write $N(v)$. The notions and notations not defined here are used accordingly to the book [2].

For a graph $G - e$ obtained by deleting edge $e \in E(G)$, we have $r(G-e) \geq r(G)$ and $d(G-e) \geq d(G)$. A graph G is *radius-edge-invariant* (*r.e.i.*) if $r(G-e) = r(G)$ for all $e \in E(G)$. If $d(G-e) = d(G)$ for all $e \in E(G)$, then G is *diameter-edge-invariant* (*d.e.i.*). Such graphs were studied in papers [1, 3, 4, 5, 7, 9]. Suppose that G is vertex 2-connected, i.e., any two nonadjacent vertices of G are joined by more than two internally disjoint paths and cannot be separated by the removal of fewer than two vertices.

In [6] Lee and Wang introduced and studied a concept of critical and cocritical d.e.i. graphs as follows.

Definition 1. A vertex 2-connected diameter-edge-invariant graph G is:

(1) *critical* d.e.i. if deletion of any vertex v in $V(G)$ results in a graph $G - v$ which is not d.e.i..

(2) *cocritical* d.e.i. if deletion of any vertex v in $V(G)$ results in a graph $G - v$ which is d.e.i..

It is useless to write similar definition for graphs which are not vertex 2-connected, since then $r(G-v) = \infty$ for some $v \in V(G)$ and thus $r(G-v-e) = r(G-v)$ for all $e \in E(G-v)$. According to the previous definition, we can define special classes of r.e.i. graphs in the following manner.

Definition 2. A vertex 2-connected radius-edge-invariant graph G is:

(1) *critical* r.e.i. if deletion of any vertex v in $V(G)$ results in a graph $G-v$ which is not r.e.i..

(2) *cocritical* r.e.i. if deletion of any vertex v in $V(G)$ results in a graph $G - v$ which is r.e.i..

In this paper we study critical and cocritical r.e.i. graphs. We use graph operations to construct variety of such graphs. We show that every graph can be embedded as an induced subgraph of critical or cocritical r.e.i. graph.

2. Preliminary Results

Dutton *et al.* [3] proved the following important theorem.

Theorem 2.1. *Every self-centered graph on at least three vertices is radius-edge-invariant.*

Walikar *et al.* [9] characterized r.e.i. graphs of radius one as follows.

Theorem 2.2. *A graph of radius one and order n is radius-edge-invariant if and only if G contains at least three vertices of degree $n-1$.*

The following proposition is immediate consequence of the previous theorem.

Proposition 2.3. *A radius-edge-invariant graph G of radius one and order n is cocritical if and only if G contains at least four vertices of degree $n-1$.*

Moreover, an r.e.i. graph G of radius one and order n is critical if and only if removal of any of its vertices decreases the number of vertices of degree $n-2$ below three. But such a number can be decreased only by removing a vertex of degree $n-1$. Since G has at least three vertices of degree $n-1$ we can claim the following observation:

Proposition 2.4. *A radius-edge-invariant graph G of radius one is critical if and only if it is K_3.*

For radius equal two the situation is more complicated. In fact we are unable to characterize even simple r.e.i. graphs of radius two. Moreover, removal of a single vertex can also decrease the radius to one. For example every graph of order n with all vertices of degree $n-2$ is critical r.e.i..

Proposition 2.5. *If G is radius-edge-invariant vertex 2-connected graph and every vertex of G is adjacent to a vertex of degree 2 then it is critical.*

Proof. Consider the graph $G-v$. Since v is adjacent to some u in G, $deg_G(u)=2$, we have $deg_{G-v}(u)=1$ and thus $G-v$ is not r.e.i.. ■

Let G and G' be disjoint graphs and let $u \in V(G')$. We say that a graph H is a substitution of G into G' in place of u, if the vertex set $V(H) = (V(G') - \{u\}) \cup V(G)$ and the edge set $E(H)$ consists of all edges of the graphs $G' - u$ and G and, moreover, every vertex of G is joined to every neighbour of u in G'.

3. The Edge Expansion and Critical Radius-Edge-Invariant Graphs

Let 2-*Gph* be a class of all undirected graphs of the form $\langle H; u, v\rangle$, where $\langle u, v\rangle$ is some arbitrary ordered pair of vertices of H.

Given a directed graph G without loops and a mapping $f : E(G) \to$ 2-*Gph* we construct a new undirected graph (G, f) which is called the *edge expansion* of G by f as follows:

Suppose $ab = e \in E(G)$ and $\langle H; u, v\rangle \in$ 2-*Gph*. If $f(e) = f(ab) = \langle H; u, v\rangle$, then we replace the edge ab in G by the graph H which identifies u with a and v with b. In particular, if $f(e) = \langle H; u, v\rangle$ for all $e \in E(G)$, then we shall use $G[H; u, v]$ to denote the edge expansion of G by f. Lee and Wang [6] constructed by this operation many critical d.e.i. graphs. As we will see, the edge expansion is useful to construct critical r.e.i. graphs as well.

If there is a graph automorphism g of H such that $g(u) = v, g(v) = u$, then the edge expansion results in the same graph independently of an orientation given to G. Thus for such $\langle H; u, v\rangle$ we can also define the edge expansion for undirected G by giving G an arbitrary orientation.

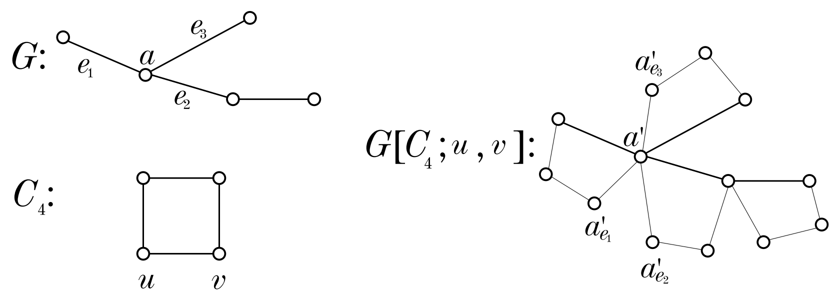

Figure 1

Theorem 3.1. *Let G be a vertex 2-connected radius-edge-invariant graph of radius r. Then $G[C_4; u, v]$, $d_{C_4}(u, v) = 1$ is critical radius-edge-invariant graph of radius $r + 1$ if and only if for every vertex w of G there is a central vertex c of G such that $d_G(w, c) \leq r - 1$.*

Proof. We first introduce some additional notation. Suppose $a \in V(G)$ and a is adjacent to edges $e_1, \dots, e_k$. Then the corresponding vertex $a' \in V(G[C_4; u, v])$ has k more neighbours. Let us mark them $a'_{e_1}, \dots, a'_{e_k}$ (see Figure 1).

It is obvious that for every $b \in V(G)$ we have $e_{G[C_4;u,v]}(b') = e_G(b) + 1$, and for all $b'_{e_i} \in V(G[C_4; u, v]), a \in V(G)$ it is $d_{G[C_4;u,v]}(a', b'_{e_i}) = d_G(a, b)+1$. Moreover, if $c \in V(G)$ is a central vertex of G, then the corresponding $c' \in V(G[C_4; u, v])$ is a central vertex of $G[C_4; u, v]$. Observe that $r(G[C_4; u, v]) = r(G) + 1$.

($\Leftarrow$) We first prove that $G\,[C_4;\, u,\, v]$ is an r.e.i. graph. Obviously $r(G[C_4; u, v] - e \geq r(G[C_4; u, v])$. We will consider the graph $G[C_4; u, v] - e$ and three cases of deleting the edge e.

Case 1. $e = a'_e b'_e; a, b \in V(G)$.
This is the simplest case, since for every vertex $w \in V(G)$ we have $e_{G[C_4;u,v]-e}(w') = e_{G[C_4;u,v]}(w')$. Thus the eccentricity of any central vertex remains unchanged.

Case 2. $e = a'b'; a, b \in V(G)$.
The graph G is r.e.i. and thus in $G[C_4; u, v] - e$ there is at least one vertex c' such that c is a central vertex of G and for all $w' \in G[C_4; u, v], w \in V(G)$ we have $d_{G[C_4;u,v]-e}(c', w') = d_{G-e}(c, w) \leq r(G)$. Since no edge of the form $w'w'_f$, $f \in E(G)$ is missing, we have $d_{G[C_4;u,v]-e}(c', w'_f) \leq r(G) + 1$. Thus $e_{G[C_4;u,v]-e}(c') = e_{G[C_4;u,v]}(c')$ and $r(G[C_4; u, v] - e) = r(G[C_4; u, v])$.

Case 3. $e = a'a'_l$; $l = ab \in E(G)$; $a, b \in V(G)$.
Given assumption, we have a central vertex c of G such that $d_G(c, b) \leq r - 1$. Thus $d_{G[C_4;u,v]-e}(c', a'_l) = d_{G[C_4;u,v]}(a'_l, b') + d_{G[C_4;u,v]}(b', c') = 2 + d_{G[C_4;u,v]}(b', c') \leq r + 1$. For all other vertices $w' \in V(G[C_4; u, v])$ we have $d_{G[C_4;u,v]}(w', c') = d_{G[C_4;u,v]-e}(w', c')$. Thus $e_{G[C_4;u,v]-e}(c') = r + 1 = r(G[C_4; u, v])$. But then $r(G[C_4; u, v] - e) = r(G[C_4; u, v])$ and $G[C_4; u, v]$ is r.e.i..

From Proposition 2.5 it follows that $G[C_4; u, v]$ is critical r.e.i..

($\Rightarrow$) We will prove the reverse course by a contradiction. Suppose that there exists a vertex a such that $d_G(a, c) = r$ for all $c \in C(G)$. Let b be any neighbour of a. Consider the graph $G[C_4; u, v] - b'b'_e$ where $e = ab$.

For any vertex $w \in V(G), e_G(w) > r$ we have $e_{G[C_4;u,v]-b'b'_e}(w') > r + 1$. Obviously the eccentricities of additional vertices obtained by edge expansion are greater by at least one. Now we will inspect the eccentricities of remaining vertices. Suppose $c \in C(G)$. We have $d_{G[C_4;u,v]-b'b'_e}(c',a') = r$ which implies $d_{G[C_4;u,v]-b'b'_e}(c',b'_e) = d_{G[C_4;u,v]-b'b'_e}(c',a') + 2 \geq r+2$. Thus

$$r(G[C_4;u,v] - b'b'_e) = r + 2 > r(G[C_4;u,v]).$$

The graph $G[C_4;u,v]$ is not r.e.i., a contradiction. ■

Theorem 3.2. *For every natural number $r \geq 3$ and every graph G there exists a critical radius-edge-invariant graph H of radius r such that G is an induced subgraph of H.*

Proof. We will obtain the desired graph H in two steps. We first take C_{2r-1} and substitute G into C_{2r-1} in place of some of its vertex. The resulting graph Q is self-centered and thus r.e.i.. It is clear that it also satisfies the condition from Theorem 3.1. Thus $H = Q[C_4;u,v]$ is critical r.e.i. graph if $d_{C_4}(u,v) = 1$. The following example (Figure 2) shows the construction for $r = 3$ and for an arbitrary radius. ■

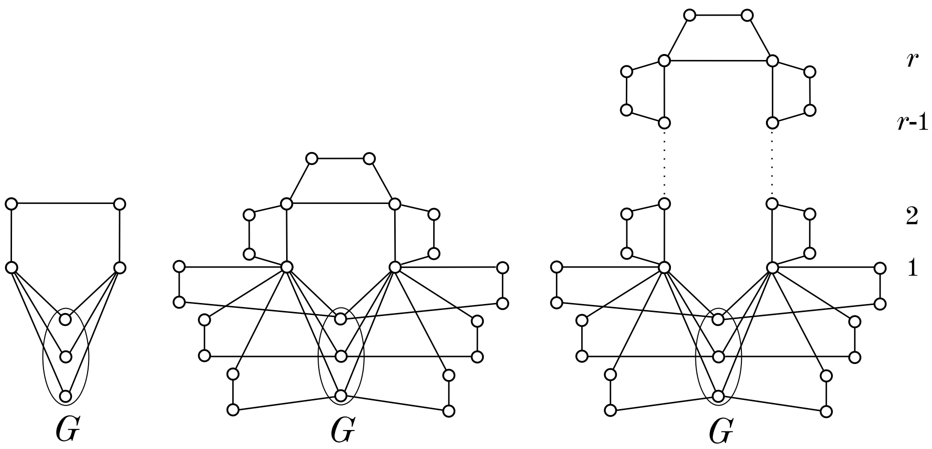

Figure 2

Because of the previous theorem, we cannot obtain a forbidden subgraph characterization for critical radius-edge-invariant graphs of radius greater than two. For radius equal to two the situation remains unclear. Little more complicated construction shows that Q does not need to be necessarily self-centered and thus there are many possibilities for the values of radius and diameter of H.

Theorem 3.3. *Let r, d be two natural numbers such that $5 \leq r+1 < d \leq 2r-1$. Then for any graph G there exists a critical radius-edge-invariant graph H such that $r(H) = r, d(H) = d, V(G) \subseteq C(H)$ and G is an induced subgraph of H.*

Proof. Consider the graph Q on Figure 3. We first show that Q is r.e.i. of radius $r-1$ and diameter $d-2$ and that Q contains at least one central vertex $c, d(c, v) \leq r-2$ for every $v \in V(Q)$. The demanded result then follows from Theorem 3.1.

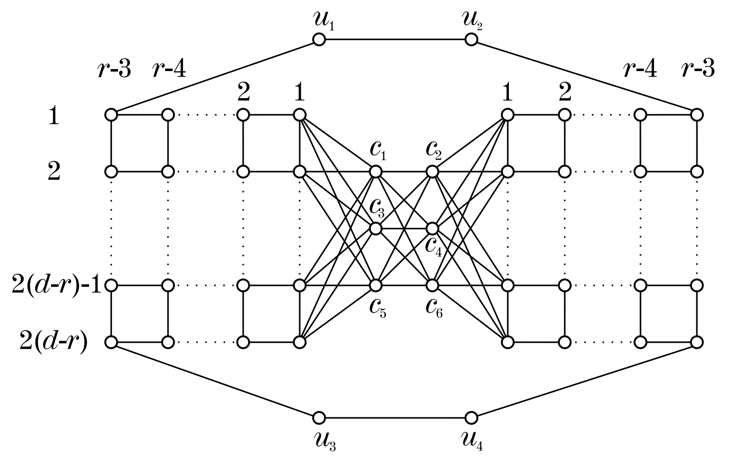

Figure 3

First suppose that $d \neq 2r-3, d \neq 2r-2$. Observe that $C(Q) = \{c_1, c_2, \ldots, c_6\}$. We have $d(c_i, u_j) = r-2$ if i, j are both odd or even and $d(c_i, u_j) = r-1$ otherwise. For any other $v \in V(Q), v \neq c_i, v \neq u_j$ there is $d(c_i, v) \leq r-2$ for all c_i. Moreover, for every vertex $w \in V(Q)$ we have at least two central vertices c_i, c_j such that $d(c_i, w) \leq r-1$, $d(c_j, w) \leq r-1$ and there are two geodesics c_i-w, c_j-w which are edge disjoint. Thus Q is an r.e.i. graph of radius $r-1$.

Now we show that $d(Q) = d-2$. We need to prove that $e(v) \leq d-2$ for all $v \in V(G)$ and find two vertices $a, b \in V(Q)$ such that $d(a, b) = d-2$. We have $e(c_i) = r-1$. Consider any other vertex $x, x \neq u_j$ and arbitrary vertex y. We are going to show that $d(x, y) \leq d-2$.

If both x and y lie on the left (right) side of the center of Q, then they lie in a cycle x-y-c_i-x. We can form such a cycle having the length no greater than $2(d-r)-1+(r-3)+(r-3) = 2d-7$. But then $d(x, y) \leq d-4$.

If x and y lie in the distinct parts then they belong to two cycles of the form x-c_i-y-u_2-u_1-x and x-c_i-y-u_4-u_3-x. We can form such cycles having summary length not exceeding $2[2(d-r)-1]+2[2(r-3)+1]+3+3=4d-6$. Thus x and y lie in at least one cycle of length not exceeding $2d-3$ which implies $d(x,y) \leq d-2$.

At last if $x = u_i$ then every $y \neq u_i$ lies in a cycle of the form x-c_i-y-x of length at most $3+2(r-3)+1+2(d-r)-1 = 2d-3$ and thus $d(x,y) \leq d-2$. If $x = u_i$, $y = u_j$, u_i and u_j are not adjacent, then $d(x,y) = \min\{2(d-r)-1+3, 2(r-3)+3\}$. But either $d-2 \geq 2(d-r)-1+3 \Leftrightarrow 2r-4 \geq d$ or $d-2 \geq 2(r-3)+3 \Leftrightarrow d \geq 2r-1$. Thus $e(u_i) \leq d-2$ and $d(Q) \leq d-2$.

To obtain two vertices a, b such that $d(a,b) = d-2$ it is sufficient to take the vertex a in row 1 and column 1 and the vertex in row $2(d-r)-1$ and column $d-1$ if $d \leq 2r-5$ and u_1, u_4, otherwise.

If G is K_1, K_2 or $\overline{K_2}$ then it is already contained in the center of Q. Otherwise we can substitute G in place of any c_i and the resulting graph Q' is still r.e.i. of radius $r-1$ and diameter $d-2$. The demanded critical r.e.i. graph H of radius r and diameter d can now be obtained as $H = Q'[C_4; u, v]$.

If $d = 2r-3$ or $d = 2r-2$ we simply take $d-4$ rows of vertices instead of $2(d-r)$ rows in Q. It is fairly easy to see that we obtain an r.e.i. graph of radius $r-1$ and diameter $d-2$ as well. ■

4. Cocritical Radius-Edge-Invariant Graphs

We first introduce a general construction of graphs which was shown to be very useful for construction of d.e.i. and cocritical d.e.i. graphs (see [5, 6]). We will show that it is applicable for construction of critical r.e.i. graphs as well.

Consider a finite connected graph I. Let $\{G_i : i \in V(I)\}$ be a class of graphs indexed by a finite set $V(I)$. The Sabidussi sum $S^+(\{G_i : i \in V(I)\})$ (or simply S^+) of $\{G_i : i \in V(I)\}$ is a graph defined as follows:

$$V(S^+(\{G_i : i \in V(I)\})) = \bigcup \{V(G_i) : i \in V(I)\},$$

$$E(S^+(\{G_i : i \in V(I)\})) =$$

$$= \bigcup \{E(G_i) : i \in V(I)\} \cup \{xy : x \in V(G_i), y \in V(G_j), ij \in E(I)\}.$$

Sabidussi sum is sometimes called *X-join*. One can show that for $d(I) \geq 2$ we have $d(S^+(\bigcup\{G_i : i \in V(I)\})) = d(I)$.

Theorem 4.1. *Let p, q be any two nonnegative integers, I be a connected graph with at least three vertices and let $\{G_i : i \in V(I)\}$ be a class of graphs, every with at least $p+q+1$ vertices. Then for any two vertices $v_i \in G_i, v_j \in G_j, d_I(i,j) > 1$ and for any other p vertices $u_1, \ldots, u_p$ and q edges $e_1, \ldots, e_q$ of the graph $S^+(\{G_i : i \in V(I)\})$ we have*

$$d_{S^+}(v_i, v_j) = d_{S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q}(v_i, v_j).$$

If v_i and v_j belong to the same G_i then

$$d_{S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q}(v_i, v_j) \le 2,$$

and if $d_I(i,j) = 1$ then

$$d_{S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q}(v_i, v_j) \le 3.$$

Moreover, if $d_I(i,j) = 1$ and $deg_{G_i}(v_i) + deg_{G_j}(v_j) \ge p + q$ then

$$d_{S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q}(v_i, v_j) \le 2.$$

Proof. *Case* 1. Suppose $v_i \in G_i, v_j \in G_j$ are two vertices such that $d_I(i,j) > 1$. Observe that $d_{S^+}(v_i, v_j) = d_I(i,j)$. Since every G_k has at least $p+q+1$ vertices, we have at least $p+q+1$ edge and vertex disjoint v_i-v_j geodesics in S^+. But then we have at least one geodesic in $S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q$ of the same length.

Case 2. Consider the case when $v_i, v_j \in G_i$. Since I is connected, we have at least one vertex $k \in I$ adjacent to i. But then we have at least $p+q+1$ edge and vertex disjoint paths of length two in S^+, all of the form v_i-v_{k_a}-v_j where $v_{k_a} \in G_k, a = 1, \ldots, p+q+1$. Thus there exists at least one v_i-v_{k_a}-v_j path in $S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q$ of length two.

Case 3. Let $v_i \in G_i, v_j \in G_j$ be two vertices such that $d_I(i,j) = 1$. Since I is connected, we have at least one vertex $k \in I$ adjacent either to i or j. Without loss of generality assume that $ki \in E(I)$. Then $v_i v_j \in E(S^+)$ and we have $p+q$ additional vertex and edge disjoint paths of length three of the form v_j-v_{i_a}-v_{k_b}-v_i where $v_{i_a} \in G_i$, $v_{i_a} \ne v_i$, $v_{k_b} \in G_k, a = 1, \ldots, p+q$, $b = 1, \ldots, p+q$. Thus in $S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q$ we have at least one v_i-v_j path of length at most three.

If $deg_{G_i}(v_i)+deg_{G_j}(v_j) \geq p+q$, then we have together at least $p+q$ paths of the form v_i-v_{i_a}-$v_j, v_{i_a} \in G_i$ or v_i-v_{j_b}-$v_j, v_{j_b} \in G_j$. Thus again exists at least one v_i-v_j path of length at most two in $S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q$. ■

Corollary 4.2. *Let p, q be any two nonnegative integers, let r, d be two positive integers such that $2 \leq r \leq d, 2 < d \leq 2r$ and let I be a graph of radius r and diameter d. Let moreover, $\{G_i : i \in V(I)\}$ be a class of graphs with at least $p + q + 1$ vertices. Then for any p vertices $u_1, \ldots, u_p$ and q edges $e_1, \ldots, e_q$ of the graph $S^+(\{G_i : i \in V(I)\})$ we have*

$$r(S^+) = r(S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q),$$

and

$$d(S^+) = d(S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q).$$

Proof. Suppose c is a central vertex of I. We first show, that there is a vertex $v_{c_j} \in G_c$, $v_{c_j} \in S^+ - u_1 - \cdots - u_p$, such that $e_{S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q}(v_{c_j}) \leq r$. Since G_c has at least $p+q+1$ vertices, we can take a vertex v_{c_j} not adjacent to any edge $e_1, \ldots, e_q$. Thus for all $x \in S^+ - u_1 - \cdots - u_p$, $d_{S^+}(v_{c_j}, x) = 1$ we have $d_{S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q}(v_{c_j}, x) = 1$. For all other vertices $x' \in S^+ - u_1 - \cdots - u_p$, $d_{S^+}(v_{c_j}, x') > 1$ we have $d_{S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q}(v_{c_j}, x') = d_{S^+}(v_{c_j}, x')$ according to Theorem 4.1. Thus eccentricity of v_{c_j} does not exceed $r(S^+)$.

Since every G_i has at least $p + q + 1$ vertices, eccentricity of any vertex of S^+ cannot be decreased by removing of p vertices. Thus $r(S^+) = r(S^+ - u_1 - \cdots - u_p - e_1 - \cdots - e_q)$.

Since $d(G) \geq 3$ the second part of this theorem is immediate consequence of Theorem 4.1. ■

If we take $p = q = 1$ we have the following observation:

Corollary 4.3. *Let r, d be two positive integers such that $2 \leq r \leq d, 2 < d \leq 2r$ and let I be a graph of radius r and diameter d. Let moreover, $\{G_i : i \in V(I)\}$ be a class of graphs with at least 3 vertices. Then $S^+(\{G_i : i \in V(I)\})$ is cocritical radius-edge-invariant and cocritical diameter-edge-invariant graph of radius r and diameter d.*

Corollary 4.4. *Let r, d be two positive integers such that $2 \leq r \leq d, 2 < d \leq 2r$. Every graph G can be induced in cocritical radius-edge-invariant and cocritical diameter-edge-invariant graph of radius r and diameter d.*

Proof. Suppose I is an arbitrary graph of radius r and diameter $d > 2$. Consider the Sabidussi sum $S^+(\{G_i : i \in V(I)\})$ where $|V(G_i)| \geq 3$ and G is an induced subgraph of G_k for some $k \in V(I)$ (For example we can take $G_k = G \cup K_2$ for all k and for arbitrary G.). According to Corollary 4.3 the graph S^+ is cocritical r.e.i., cocritical d.e.i. and obviously G is an induced subgraph of S^+. ■

Our last goal will be to prove the lower bound for the number of vertices for cocritical r.e.i. graphs. We first give several lemmas, which can be found useful anyway as they describe some structural properties of cocritical r.e.i. graphs.

Lemma 4.5. *Every vertex v of a cocritical radius-edge invariant graph G has $deg(v) > 2$.*

Proof. Since G is vertex 2-connected and $|V(G)| > 2$, $deg(v) > 1$ for all $v \in V(G)$. Suppose $N(v) = \{u, w\}$. This implies $deg_{G-u}(v) = 1$ and thus $G - u$ is not r.e.i., a contradiction. ■

Lemma 4.6. *Let G be a cocritical radius-edge-invariant graph with central vertex c and radius r. If $|N_i(c)| = 2$ for $1 < i < r$, then $|N_{i-1}(c)| > 3$.*

Proof. We will prove this lemma by a contradiction. It is obvious that $|N_i(c)| > 1$. Otherwise $G - N_i(c)$ is not connected. Suppose $N_i(c) = \{a, b\}$ and $N_{i-1}(c) = \{u, v, w\}$. The case when $|N_{i-1}(c)| = 2$ can be handled analogously. Both graphs $G - a$ and $G - b$ are edge 2-connected and thus a and b are both adjacent to at least two vertices of $N_{i-1}(c)$ (see Figure 4). Hence at least one vertex of $N_{i-1}(c)$ is adjacent to both a and b.

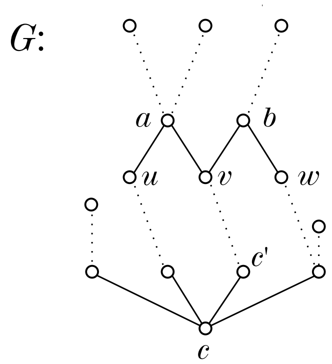

Figure 4

Let $av, bv \in E(G)$. Suppose c' is a vertex such that $c' \in N(c)$, $d(c', v) = i-2$. Observe that $e(c') = r-1$. But then $r(G) \leq e(c') < r$, a contradiction. ■

Lemma 4.7. *Let G be a cocritical radius-edge-invariant graph of radius $r \geq 6$ with central vertex c. If $N_i(c) = \{a, b\}$ for some $2 < i \leq r-3$, then a and b are adjacent to a distinct pairs of vertices of $N_{i+1}(c)$. Moreover, a and b are not adjacent together and not adjacent to any common vertex w.*

Proof. Suppose $N_i(c) = \{a, b\}$ for some $2 < i \leq r-3$. As we already know, both a and b are adjacent to at least two vertices in $N_{i-1}(c)$. Consider the graph $G - a$. This graph is edge 2-connected and thus we have at least 2 edge disjoint paths from $N_{i-1}(c)$ to $N_{i+1}(c)$. But then b must be adjacent to at least two vertices in $N_{i+1}(c)$. Condition for a can be proved analogously.

Now we show that a and b are not adjacent to a common vertex. It follows from the proof of Lemma 4.6 that if such vertex w exists, then $w \notin N_{i-1}(c)$. Let $w \in N_{i+1}(c)$ and let c' be a vertex of the c-w geodesic such that $d(c, c') = 3$. For any $z \in V(G)$ we have either $d(c', z) \leq d(c', c) + d(c, z) \leq 3 + (i-1) < r$ (when $d(c, z) < i$) or $d(c', z) \leq d(c', w) + d(w, z) \leq r-1$ (when $d(c, z) \geq i$). Thus $e(c') = r-1$, a contradiction. Similar arguments can be used to prove that $ab \notin E(G)$. ■

Lemma 4.8. *Let G be a cocritical radius-edge-invariant graph of radius $r \geq 3$. Then $|N_r(c)| + |N_{r-1}(c)| + |N_{r-2}(c)| \geq 8$ for every central vertex c of G.*

Proof. It is clear that $|N_{r-1}(c)| > 1$ and since every vertex has degree at least three $|N_r(c)| + |N_{r-1}(c)| \geq 4$. According to Lemma 4.6 if $|N_{r-1}(c)| = 2$, then $|N_{r-2}(c)| \geq 4$. Thus if $|N_r(c)| \geq 3$, then the result is obvious.

We need to show that none of the following configurations is possible. In all cases we will find $c' \in V(G)$ such that $e_G(c') = r-1$, or prove that G is not cocritical.

Case 1. $|N_r(c)| = 1, |N_{r-1}(c)| = 3, |N_{r-2}(c)| = 3$.
Suppose v is a unique vertex such that $d(v, c) = r$, $N_{r-1}(c) = \{u_1, u_2, u_3\}$ and $N_{r-2}(c) = \{w_1, w_2, w_3\}$. Recall that $deg(u_i) \geq 3$. Thus there are at least six edges joining vertices of $N_{r-1}(c)$ together or joining vertices of $N_{r-1}(c)$ to those in $N_{r-2}(c)$. We have either (without any loss of generality)

(a) w_1 adjacent to all vertices of $N_{r-1}(c)$,

(b) u_1 adjacent to all other vertices of $N_{r-1}(c)$,

(c) w_1 adjacent to u_1, u_2 and u_2 adjacent to u_3, or

(d) every u_i adjacent to a distinct pair of $\{w_1, w_2, w_3\}$ (see Figure 5).

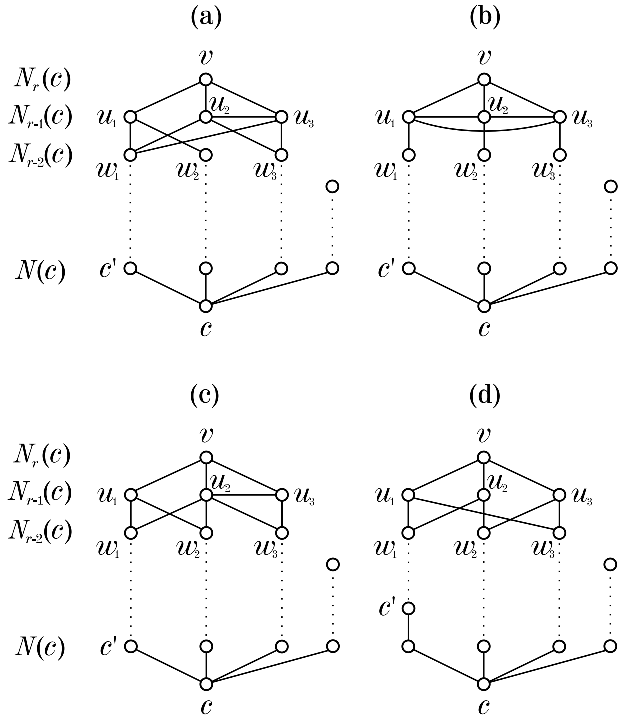

Figure 5

The vertex c' can be taken as the second vertex on the c-w_1 geodesic in the first three cases and as the third vertex on the c-w_1 geodesic otherwise. For all $x \in V(G)$ we have $\min\{d(c, c') + d(c, x), d(c', w_1) + d(w_1, x)\} \leq r - 1$. Thus $r(G) \leq e(c') \leq r - 1$, a contradiction.

Case 2. $|N_r(c)| = 1, |N_{r-1}(c)| = 4, |N_{r-2}(c)| = 2$.
We will mark $N_r(c) = \{v\}$, $N_{r-1}(c) = \{u_1, u_2, u_3, u_4\}$ and $N_{r-2}(c) = \{w_1, w_2\}$.

First suppose that w_1 is adjacent to at least three vertices of $N_{r-1}(c)$, namely u_1, u_2, u_3. Since $deg(u_4) \geq 3$, u_4 is either adjacent to w_1 or to some vertex of the set $\{u_1, u_2, u_3\}$ (see Figure 6 (a)). But then it is sufficient to take c' as the second vertex on the c-w_1 geodesic.

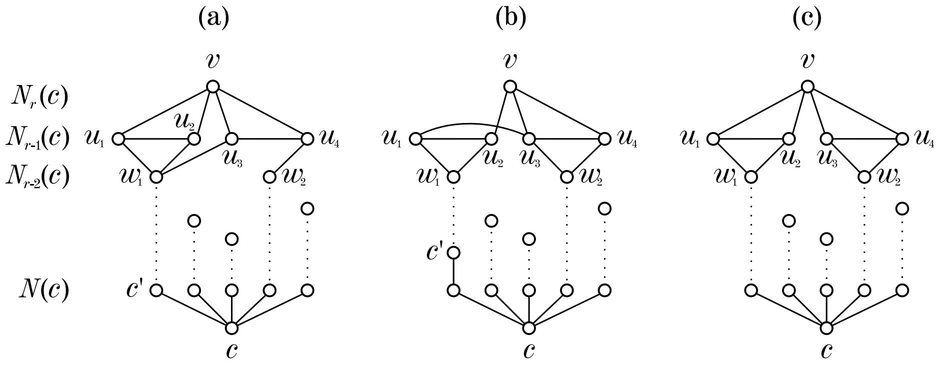

Figure 6

Now suppose $|N(w_1) \cap N_{r-1}(c)| = |N(w_2) \cap N_{r-1}(c)| = 2$, $w_1u_1, w_1u_2, w_2u_3, w_2u_4 \in E(G)$. We have either

(b) $deg(v) = 3$, or

(c) $deg(v) = 4$.

Let $N(v) = \{u_2, u_3, u_4\}, w_1u_1 \in E(G)$. We have $deg(u_1) \geq 3$ and thus u_1 is adjacent to u_3 (u_4) giving $d(w_1, w_2) \leq 3$. The vertex c' can be taken as the third vertex on the c-w_1 geodesic.

At last let $deg(v) = 4$. If any vertex of $\{u_1, u_2\}$ is adjacent to any vertex of $\{u_3, u_4\}$, we can use the same arguments as in the previous case. Otherwise we get the configuration shown in Figure 6(c). Observe that $r(G-v) = r-1$. Since for any central vertex c'' of $G-v$ we have $e_{G-v}(c'') = r-1$, it follows that $d_G(c'', v) = r$. Otherwise $e_G(c'') = r-1$. Thus $d_{G-v}(c'', u_i) = r-1$ and $d_{G-v}(c'', w_j) = r-2$. But then for example $e_{G-v-u_1w_1}(c'') = r$. The graph $G-v$ is not r.e.i., a contradiction.

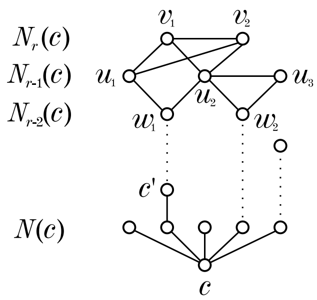

Figure 7

Case 3. $|N_r(c)| = 2, |N_{r-1}(c)| = 3, |N_{r-2}(c)| = 2$.
Let $N_r(c) = \{v_1, v_2\}$, $N_{r-1}(c) = \{u_1, u_2, u_3\}$ and $N_{r-2}(c) = \{w_1, w_2\}$. Since $deg(v_1) \geq 3$ and $deg(v_2) \geq 3$, we have at least one vertex of $N_{r-1}(c)$ adjacent to both v_1 and v_2. Let us mark this vertex as u_1. $G - w_1$ and $G - w_2$ are both r.e.i. and thus w_1 and w_2 are both adjacent to at least two vertices of $N_{r-1}(c)$. Every u_i has degree at least three and thus is adjacent to at least one vertex of $N_r(c)$ or $N_{r-1}(c)$. This implies $d(w_1, x) \leq 3$ for all $x \in N_{r-2}(c) \cup N_{r-1}(c) \cup N_r(c)$. It is sufficient to take a vertex c' as the third vertex on the c-u_1 geodesic. ■

Lemma 4.9. *Let G be a cocritical radius-edge-invariant graph with central vertex c and radius $r \geq 5$. If $|N_{r-3}(c)| = 2$, then $|N_r(c)| + |N_{r-1}(c)| + |N_{r-2}(c)| \geq 9$.*

Proof. According to Lemma 4.7 if $|N_{r-3}(c)| = 2$, then $|N_{r-2}(c)| \geq 4$. Since $|N_r(c)| + |N_{r-1}(c)| \geq 4$ it is sufficient to show that there is no cocritical r.e.i. graph having $|N_{r-3}(c)| = 2, |N_{r-2}(c)| = 4, |N_{r-1}(c)| = 3, |N_r(c)| = 1$ and no cocritical r.e.i. graph having $|N_{r-3}(c)| = 2, |N_{r-2}(c)| = 4, |N_{r-1}(c)| = 2, |N_r(c)| = 2$.

Let us consider the first case. Suppose $N_r(c) = \{v\}$, $N_{r-1}(c) = \{u_1, u_2, u_3\}$, $N_{r-2}(c) = \{w_1, w_2, w_3, w_4\}$, $N_{r-3}(c) = \{z_1, z_2\}$. According to Lemma 4.7 every vertex of $N_{r-3}(c)$ is adjacent to two distinct vertices of $N_{r-2}(c)$. Let z_1w_1, z_1w_2, z_2w_3, $z_2w_4 \in E(G)$. Similarly, at least two vertices of $N_{r-1}(c)$ are adjacent to either $\{w_1, w_2\}$ or to $\{w_3, w_4\}$ (see Figure 8). Let $c' \in N(c), c'' \in N_2(c)$ be two vertices such that $d(c', z_1) = r - 4, d(c'', z_1) = r - 5$ and let u_1, u_2 be adjacent to w_1 or w_2. If u_3 is adjacent to w_1, w_2, u_1 or u_2 too, then $e(c') = r - 1$, a contradiction. Otherwise u_3 is adjacent to both w_3 and w_4. Now there is either some edge joining $\{w_1, w_2\}$ and $\{w_3, w_4\}$ giving $e(c'') = r - 1$ or $G - z_2 - vu_3$ is not connected. In both cases we obtain a contradiction.

At last suppose that $N_{r-3}(c) = \{z_1, z_2\}$, $N_{r-2}(c) = \{w_1, w_2, w_3, w_4\}$, $N_{r-1}(c) = \{u_1, u_2\}$ and $N_r(c) = \{v_1, v_2\}$. Since every vertex of G has degree at least three, v_1 and v_2 are joined together and adjacent to both u_1 and u_2. Let $c' \in N(c)$ be a vertex such that $d(c', z_1) = r - 4$. If u_2 is adjacent to successors of both z_1 and z_2, then $e(c') = r - 1$, a contradiction. The same holds for u_1. Otherwise we have $e_{G-u_2}(c') = r - 1 = r(G - u_2)$. For every central vertex c'' of $G - u_2$ we have $d_G(c'', u_2) = r$, $d_G(c'', w_3) = d_G(c'', w_4) = d_G(c'', v_1) = d_G(c'', v_2) = r - 1$ and $d_G(c'', z_2) = d_G(c'', u_1) = r - 2$. But then for example $e_{G-u_2-v_2u_1}(c'') = r$ for every central vertex of $G - u_2$. $G - u_2$ is not r.e.i., a contradiction.

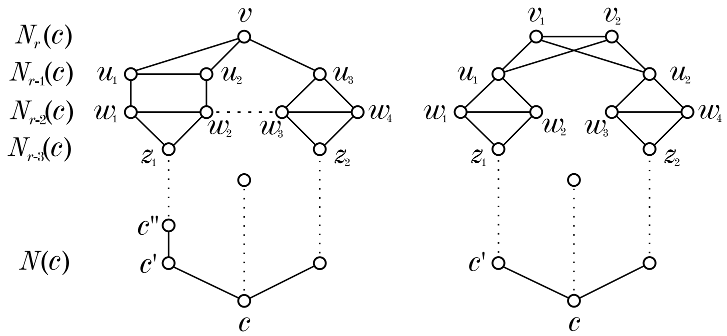

Figure 8

■

Lemma 4.10. *Let G be a cocritical radius-edge-invariant graph with central vertex c and radius $r \geq 7$. Suppose i is a natural number such that $2 \leq i \leq r - 5$. If $|N_i(c)| = |N_{i+1}(c)| = |N_{i+2}(c)| = 3$, then there are three vertices $v_1 \in N_i(c), v_2 \in N_{i+1}(c), v_3 \in N_{i+2}(c)$ such that $G - v_i - v_j$ is not connected for every pair $\{i, j\} \subset \{1, 2, 3\}$.*

Proof. Let $N_i(c) = \{x_1, x_2, x_3\}$, $N_{i+1}(c) = \{y_1, y_2, y_3\}$ and $N_{i+2}(c) = \{z_1, z_2, z_3\}$. We first prove that a subgraph H of G induced by $N_i(c) \cup N_{i+1}(c) \cup N_{i+2}(c)$ is not connected.

We will prove this by a contradiction. Suppose that given subgraph is connected. Observe that no x_i is adjacent to all y_j. If all x_i are adjacent to two vertices in $N_{i+1}(c)$ or if any x_i is adjacent to a single vertex of $N_{i+1}(c)$, then the graph $H - x_i$ remains connected for some x_i. It is well known (see [8]) that every graph with n vertices and radius r has $\Delta(G) \leq n - 2r + 2$. Since at least one $y_j \in H - x_i$ has degree at least three, we have $r(H - x_i) \leq 3$. Let c'' be a central vertex of H and let c' be a vertex of G such that c' lies on the c''-c geodesic and $c' \in N_4(c)$ if $d(c, c'') \geq 4$ and $c' = c''$ otherwise (see Figure 9). We have $e(c') < r$, a contradiction.

Now suppose that H is not connected and has two distinct sets A, B of vertices such that no vertex of the set A is adjacent to a vertex belonging to B. Moreover, let $N_i(c) = N_i^A(c) \cup N_i^B(c)$, $N_{i+1}(c) = N_{i+1}^A(c) \cup N_{i+1}^B(c)$, $N_{i+2}(c) = N_{i+2}^A(c) \cup N_{i+2}^B(c)$, $A = N_i^A(c) \cup N_{i+1}^A(c) \cup N_{i+2}^A(c)$ and $B = N_i^B(c) \cup N_{i+1}^B(c) \cup N_{i+2}^B(c)$.

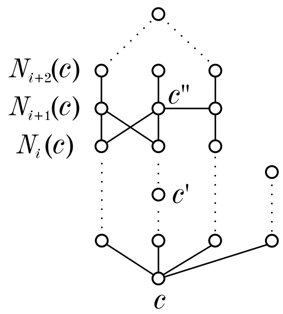

Figure 9

It is obvious that three sets of $N_i^A(c), N_{i+1}^A(c), N_{i+2}^A(c)$, $N_i^B(c)$, $N_{i+1}^B(c)$, $N_{i+2}^B(c)$ have at most one vertex. Moreover, it is not possible that either $|N_i^A(c)| = |N_{i+1}^A(c)| = |N_{i+2}^A(c)| = 1$ or $|N_i^B(c)| = |N_{i+1}^B(c)| = |N_{i+2}^B(c)| = 1$. Also if some $|N_j^A(c)| = |\{o\}| = 1$ ($|N_j^B(c)| = |\{o\}| = 1$) then, since $G - o$ is r.e.i. and thus 2 edge-connected, it cannot hold that $|N_k^B(c)| = |N_{k+1}^B(c)| = 1$ ($|N_k^A(c)| = |N_{k+1}^A(c)| = 1$) for any two successive levels of $i, i+1, i+2$. Thus the only possible configuration is the following: $|N_i^A(c)| = 1$, $|N_{i+1}^A(c)| = 2$, $|N_{i+2}^A(c)| = 1$ and $|N_i^B(c)| = 2$, $|N_{i+1}^B(c)| = 1$, $|N_{i+2}^B(c)| = 2$ (see Figure 10).

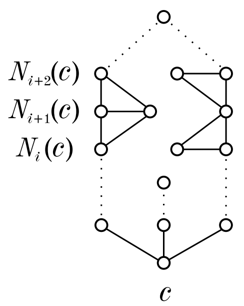

Figure 10

■

Lemma 4.11. *Let G be a cocritical radius-edge-invariant graph of radius $r \geq 4$. If for some $v \in V(G)$ there is $d(G-v) \geq 2r-1$, then $|V(G)| \geq 3r+2$.*

Proof. Suppose u, w are two peripheral vertices of $G - v$ such that $d_{G-v}(u, w) = d(G - v)$. We have $u \in N_{d(G-v)}(w)$ and $w \in N_{d(G-v)}(u)$. $G - v$ is r.e.i. and thus $G - v - e$ is connected for every $e \in E(G - v)$. Hence $|N_{G-v}(u)| > 1$ and if $|N_i(u)| = 1, 1 < i < d(G - v)$, then $|N_{i+1}(u)| > 1$,

$|N_{i-1}(u)| > 1$ and the single vertex in $N_i(u)$ is adjacent to at least two vertices in $N_{i+1}(u)$ and to at least two vertices in $N_{i-1}(u)$. We will now distinguish the following cases depending on the value of $d(G-v)$.

Case 1. $d(G-v) \geq 2r$.
If $d(G-v) \geq 2r$ then $G-v$ has $2r+1$ distinct sets $\{u\}, N(u), N_2(u), \ldots, N_{2r}(u)$. At most $r+1$ of them contains only one vertex. Thus if $d(G-v) \geq 2r$, then $|V(G)| \geq 1 + |V(G-v)| \geq 1 + 2r + 1 + r = 3r + 2$.

Case 2. $d(G-v) = 2r-1$.
If $d(G-v) = 2r-1$ then $G-v$ has $2r$ distinct sets $\{u\}, N(u), N_2(u), \ldots, N_{2r-1}(u)$ and at most r of them contains only one vertex. Thus $|V(G)| = 1 + |V(G-v)| \geq 1 + 2r + r = 3r + 1$. It is sufficient to show that it is not possible to obtain a cocritical r.e.i. graph of radius r having $3r+1$ vertices. We will prove this by a contradiction.

Suppose such a graph G exists and $e_{G-v}(u) = 2r-1$ for some $u, v \in V(G)$. Since no sucessive pair $N_i(u), N_{i+1}(u)$, $1 < i < r$ has only two vertices together and $|V(G)| = 3r+1$, we have either $|N(u)| = 2, |N_2(u)| = 1, |N_3(u)| = 2, \ldots, |N_i(u)| = 1, |N_{i+1}(u)| = 2, |N_{i+2}(u)| = 2, |N_{i+3}(u)| = 1, \ldots, |N_{2r-2}(u)| = 2, |N_{2r-1}(u)| = 1$ or $|N(u)| = 2, |N_2(u)| = 1, |N_3(u)| = 2, \ldots, |N_{2k}(u)| = 1, |N_{2k+1}(u)| = 2, \ldots, |N_{2r-2}(u)| = 1, |N_{2r-1}(u)| = 2$.

(**A**): Suppose $|N_{i+1}(u)| = |N_{i+2}(u)| = 2$, $1 < i < 2r-4$. We denote by $N(u) = \{u_1^1, u_1^2\}, N_2(u) = \{u_2\}, \ldots, N_{2r-2}(u) = \{u_{2r-2}^1, u_{2r-2}^2\}$ and $N_{2r-1}(u) = \{u_{2r-1}\}$ (see Figure 11).

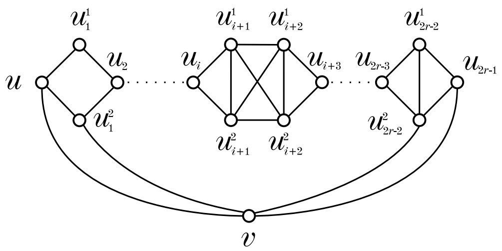

Figure 11

Since G has no cutvertices, $G - u_2$ and $G - u_{2r-3}$ are edge 2-connected, v is adjacent to at least two vertices of the set $\{u, u_1^1, u_1^2\}$ and to at least two vertices of the set $\{u_{2r-2}^1, u_{2r-2}^2, u_{2r-1}\}$. But then $e_G(v) = r-1$, a contradiction.

If $i = 1$ then $N(u) = \{u_1^1, u_1^2\}, N_2(u) = \{u_2^1, u_2^2\}, N_3(u) = \{u_3\}$ or if $i = 2r-4$ then $N_{2r-4}(u) = \{u_{2r-4}\}, N_{2r-3}(u) = \{u_{2r-3}^1, u_{2r-3}^2\}$, $N_{2r-2}(u) = \{u_{2r-2}^1, u_{2r-2}^2\}$, $N_{2r-1}(u) = \{u_{2r-1}\}$. This can be handled analogously and we left the details for the reader.

(**B**): We denote by $N_{2k+1}(u) = \{u_{2k+1}^1, u_{2k+1}^2\}$ and $N_{2k}(u) = \{u_{2k}\}$ (see Figure 12).

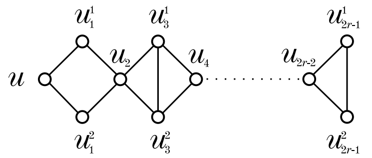

Figure 12

Since G has no cutvertices, $G - u_2$ and $G - u_{2r-2}$ are edge 2-connected, v is adjacent to at least two vertices of the set $\{u, u_1^1, u_1^2\}$ and to both vertices u_{2r-1}^1, u_{2r-1}^2. Moreover, v is not adjacent to any vertex of $N_2(u) \cup N_3(u) \cup \cdots \cup N_{2r-2}(u)$. Otherwise $e_G(v) < r$, a contradiction.

Now consider the following graph $G - u_r$:

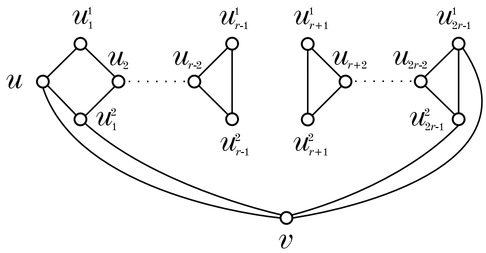

Figure 13

We have $e_{G-u_r}(v) = r - 1$ and thus $r(G - u_r) = r(G) - 1 = r - 1$. The vertex v is the unique central vertex of $G - u_r$. However, for example $e_{G-u_r-u_{r-2}u_{r-1}^1}(v) = r$. Thus $G - u_r$ is not r.e.i., a contradiction. ■

Lemma 4.12. *Let G be a cocritical radius-edge-invariant graph of radius $r \geq 8$. If $d(G - v) = 2r - 2$ and $r(G - v) = r - 1$ for any $v \in V(G)$, then $|V(G)| \geq 3r + 2$.*

Proof. Suppose u, w are two peripheral vertices of $G - v$ such that $d_{G-v}(u, w) = d(G - v)$. Again we have $|N_{i-1}(u)| > 1$, $|N_{i+1}(u)| > 1$ if

$|N_i(u)| = 1, 1 < i < d(G-v)$. Since $d(G-v) = 2r-2 = 2(r-1) = 2r(G-v)$, every central vertex of $G-v$ belongs to $N_{r-1}(u)$.

If $|N_i(u)| \geq 2$ for all $i < r-1$ or if $|N_i(u)| \geq 2$ for all $r-1 < i < 2r-2$, then $|V(G-v)| \geq (2r-1) + (r-2) + \lfloor \frac{r}{2} \rfloor = 3r + 1 + \left(\lfloor \frac{r}{2} \rfloor - 4\right) \geq 3r+1$ for $r \geq 8$. Thus $|V(G)| \geq 3r + 2$.

Now consider another case. Let $\{t_1\} \in N_k(u)$ be the vertex of the first neighbourhood of u such that t_1 is the only vertex of $N_k(u)$ adjacent to vertices of the previous neighbourhood and let $\{t_2\} \in N_l(u)$ be the last neighbourhood such that t_2 is the only vertex of $N_l(u)$ adjacent to vertices of the succeeding neighbourhood. Existence of such vertices is guaranteed by the existence of two neighbourhoods having only a single vertex. Since both $G - t_1$ and $G - t_2$ are edge 2-connected, v is adjacent to at least two vertices of $\{u\} \cup N(u) \cup \cdots \cup N_{k-1}(u)$ and to at least two vertices of $N_{l+1}(u) \cup N_{l+2}(u) \cup \cdots \cup N_{2r-2}(u)$. Moreover, since $r(G) = r$ all of these vertices adjacent to v also belong to $N_r(c)$ (see Figure 14), where c is any central vertex of both $G - v$ and G.

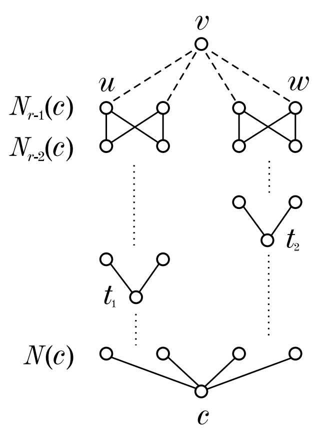

Figure 14

We have $d(c, t_1) = d(c', t_1)$ and $d(c, t_2) = d(c', t_2)$ for all $c' \in C(G-v)$. Furthermore $d(c', q) = d(c, q)$ for all $q \in N_{r-1}(c)$. Since $G - v$ is *r.e.i.* of radius $r-1$ every such q must be adjacent to at least two vertices of $N_{r-2}(c)$. It is obvious that such sets of vertices are distinct from u and w.

In every neighbourhood $N_i(c)$ marked higher than such containing t_1 (i.e., $N_{k+1}(c), \ldots, N_{r-1}(c)$ if $t_1 \in N_k(c)$) we have at least two vertices connected to c through t_1. Otherwise $N_k(u)$ does not have the described property. Similarly in every neighbourhood $N_j(c)$ marked higher than such containing t_2 we have at least two vertices connected to c through t_2. Thus $|\{u\}|+|N(u)|+\cdots+|N_{k-1}(u)| \geq 2k$, $|N_{l+1}(u)|+|N_{l+2}(u)|+\cdots+|N_{2r-2}(u)| \geq 2(2r-2-l)$ and $k > 1, l < 2r-3$.

We have

$$|V(G-v)| \geq 2k+2(2r-2-l)+\left\lfloor \frac{3}{2}(l-k+1)\right\rfloor = 3r+\left(r-\left\lfloor \frac{l}{2}-\frac{k}{2}+\frac{5}{2}\right\rfloor\right).$$

Since $k \geq 2$ and $l \leq 2r-4$

$$3r+\left(r-\left\lfloor \frac{l}{2}-\frac{k}{2}+\frac{5}{2}\right\rfloor\right) \geq 3r+\left(r-\left\lfloor \frac{(2r-4)}{2}-\frac{2}{2}+\frac{5}{2}\right\rfloor\right) =$$

$$= 3r+\left(r-\left\lfloor r-2-1+\frac{5}{2}\right\rfloor\right) \geq 3r.$$

Thus G has at least $3r+1$ vertices and if G has exactly $3r+1$ vertices, then $k=2, l=2r-4$ and only the following configuration of vertices is possible:

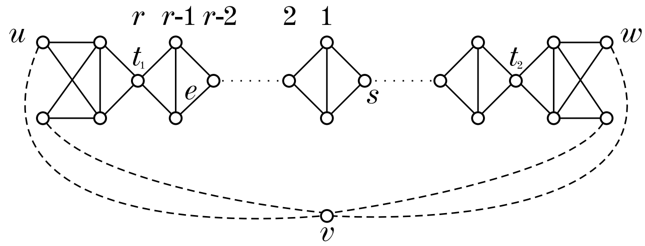

Figure 15

We have exactly $2r-7+\left\lfloor \frac{2r-7}{2}\right\rfloor$ vertices between t_1 and t_2 since there are no successive neighbourhoods of u having only one vertex. We have also five additional vertices in the set $A=\{u\}\cup N(u)\cup N_2(u)$ and five additional vertices in the set $B=N_{2r-4}(u)\cup N_{2r-3}(u)\cup N_{2r-2}(u)$. The subgraphs of G induced by A and B are not uniquely determined but v is adjacent to at least two vertices in A and to two vertices in set B.

Now consider the graph $G-t_1$. Vertices in $N_{r+2}(u)$ (as the vertex s on Figure 15, we have either one or two such vertices) have eccentricity $r-1$. All other vertices are of eccentricity greater than $r-1$ in $G-t_1$. Thus $r(G-t_1)=r-1$. By removing any edge e joining vertices from $N_3(u)$ and $N_4(u)$ we increase the radius of $G-t_1$ by one and thus G is not cocritical r.e.i. graph, a contradiction. G has at least $3r+2$ vertices. ■

Lemma 4.13. *Let G be a cocritical radius-edge-invariant graph with central vertex c and radius $r \geq 7$. If $|N_{\lceil \frac{r}{2} \rceil - 1}(c)| = |N_{\lceil \frac{r}{2} \rceil}(c)| = |N_{\lceil \frac{r}{2} \rceil + 1}(c)| = 3$, then $|V(G)| \geq 3r + 2$.*

Proof. According to Lemma 4.10 it is possible to find two vertices $v_1 \in N_{\lceil \frac{r}{2} \rceil}(c)$, $w_3 \in N_{\lceil \frac{r}{2} + 1 \rceil}(c)$ such that $r(G - v_1 - w_3) = \infty$. Moreover, if $N_{\lceil \frac{r}{2} \rceil - 1}(c) = \{u_1, u_2, u_3\}$ and $N_{\lceil \frac{r}{2} \rceil}(c) = \{v_1, v_2, v_3\}$, then $u_1v_1, u_2v_1, u_3v_2, u_3v_3, v_2w_3, v_3w_3 \in E(G)$ and there is no other edge connecting $N_{\lceil \frac{r}{2} \rceil - 1}(c)$ with $N_{\lceil \frac{r}{2} \rceil}(c)$ and w_3 with $N_{\lceil \frac{r}{2} \rceil}(c)$ (see Figure 16).

Let H be a subgraph of G induced by the vertex set $\{c\} \cup N(c) \cup \cdots \cup N_{\lceil \frac{r}{2} \rceil - 1}(c) \cup N_{\lceil \frac{r}{2} \rceil}(c) \cup \{w_3\}$. Observe that $d_H(v_1, v_2) = d_H(v_1, v_3) = d_H(v_1, w_3) - 1 \geq r - 1$. Otherwise there exists $x \in H$, $d_H(x, v_1) \leq \lceil \frac{r}{2} \rceil - 1$, $d_H(x, v_2) \leq \lceil \frac{r}{2} \rceil - 1$ for which $e_G(x) < r$, a contradiction. Since $d_H(v_1, w_3) \leq 2\lceil \frac{r}{2} \rceil + 1 \leq r + 2$, we have $r - 1 \leq d(H) = d_H(v_1, w_3) \leq r + 2$.

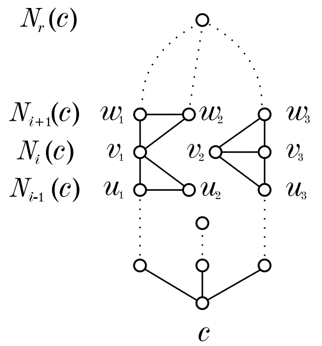

Figure 16. $(i = \lceil \frac{r}{2} \rceil)$

Now consider the graph $F = H - v_1$ and the subgraph J of G induced by the vertex set $N_{\lceil \frac{r}{2} \rceil + 1}(c) \cup N_{\lceil \frac{r}{2} \rceil + 2}(c) \cup \cdots \cup N_r(c)$. If $e_J(w_3) + e_F(w_3) < 2r - 2$, then for at least one vertex z of the set $\{w_3, v_2, v_3, u_3\}$ we have $e_F(z) \leq r - 1$ and $d_G(z, v_1) \leq r - 1$. In that case $e_G(z) \leq r - 1$, a contradiction. Otherwise $d(G - v_1) = 2r - 2, r(G - v_1) = r - 1$ or $d(G - v_1) \geq 2r - 1$. It follows from Lemma 4.11 and Lemma 4.12 that in both cases we have $|V(G)| \geq 3r + 2$. ∎

Theorem 4.14. *Every cocritical radius-edge invariant graph of radius $r \geq 15$ has at least $3r + 2$ vertices. This bound is sharp.*

Proof. Suppose c is a central vertex of G. We have $|N(c)| \geq 3$. According to previous lemmas we have

$$|V(G)| \geq |\{c\}| + |N(c)| + \cdots + N_{r-3}(c) + N_{r-2}(c) + N_{r-1}(c) + N_r(c)$$

$$\geq 1 + 3(r-3) + 8 \geq 3r.$$

Now we prove that there is no such graph having $3r$ or $3r+1$ vertices.

Suppose $|V(G)| = 3r$. The only possible configuration is the following: $|N(c)| = |N_1(c)| = |N_2(c)| = \cdots = |N_{r-3}(c)| = 3$, $|N_{r-2}(c)| + |N_{r-1}(c)| + |N_r(c)| = 8$. But from the previous lemma we know that such graph must have at least $3r+2$ vertices.

Now we prove that there is no cocritical r.e.i. graph of radius greater than fourteen having $3r+1$ vertices. If $|V(G)| = 3r+1$ then either

(1) $|N_{r-2}(c)| + |N_{r-1}(c)| + |N_r(c)| = 9$ and $|N(c)| = |N_2(c)| = \cdots = |N_{r-3}(c)| = 3$ or
(2) $|N_{r-2}(c)| + |N_{r-1}(c)| + |N_r(c)| = 9$ and $|N(c)| = |N_2(c)| = \cdots = |N_{(r-3)-2i}(c)| = 3$, $|N_{(r-3)-2i+2}(c)| = |N_{(r-3)-2i+4}(c)| = \cdots = |N_{r-3}(c)| = 2$, $|N_{(r-3)-2i+1}(c)| = |N_{(r-3)-2i+3}(c)| = \cdots = |N_{r-4}(c)| = 4$, $i \in \mathbb{N} + \{0\}$ or
(3) $|N_{r-2}(c)| + |N_{r-1}(c)| + |N_r(c)| = 8$ and $|N(c)| = |N_2(c)| = \cdots = |N_i(c)| = 3$, $|N_{i+1}(c)| = |N_{i+3}(c)| = \cdots = |N_{i+2k+1}(c)| = 4$, $|N_{i+2}(c)| = |N_{i+4}(c)| = \cdots = |N_{i+2k}(c)| = 2$, $|N_{i+2k+2}(c)| = |N_{i+2k+3}(c)| = \cdots = |N_{r-3}(c)| = 3$, $i, k \in \mathbb{N} + \{0\}$.

(1) This case is not possible according to Lemma 4.13.
(2) and (3) We will distinguish the following cases:

Case 1. $|N_{\lceil \frac{r}{2} \rceil}(c)| = 2$. We denote by $N_i(c) = N_{\lceil \frac{r}{2} \rceil}(c) = \{v_1, v_2\}$. Let H be a subgraph generated by the vertex set $\{c\} \cup N(c) \cup \cdots \cup N_i(c)$ and J be a subgraph generated by $V(G) - V(H) \cup v_1$. Observe that $r+1 \geq d_H(v_1, v_2) \geq 2i-1 \geq r-1$. Otherwise there exists $x \in H$, $d_H(x, v_1) \leq i-1$, $d_H(x, v_2) \leq i-1$ for which $e_G(x) \leq r-1$, a contradiction.

Now examine the graph $G - v_2$. If $e_J(v_1) + e_H(v_1) \geq 2r-1$ then by Lemma 4.11 we have $|V(G)| \geq 3r+2$. Otherwise $e_J(v_1) \leq r-2$. Now let $y \in N_{i-1}(c)$ be a vertex adjacent to v_1. If $e_J(v_1) + e_H(v_1) = 2r-2$, then $r(G - v_2) = r-1, d(G - v_2) = 2r-2$ and $y \in C(G - v_2)$. Thus $|V(G)| \geq 3r+2$ by Lemma 4.12. If $e_J(v_1) < r-2$ then $d_G(y, v_2) \leq r-1$.

We have $e_{G-v_1}(y) = r-1$ and $d_G(y, v_2) \le r-1$. This implies $e_G(y) \le r-1$, a contradiction.

Case 2. If $|N_{\lceil \frac{r}{2} \rceil -1}(c)| = |N_{\lceil \frac{r}{2} \rceil}(c)| = |N_{\lceil \frac{r}{2} \rceil +1}(c)| = 3$ then again by Lemma 4.13 $|V(G)| \ge 3r+2$.

Case 3. At last there is $i \in \{\lceil \frac{r}{2} \rceil -1, \lceil \frac{r}{2} \rceil, \lceil \frac{r}{2} \rceil +1\}$ such that $|N_i(c)| = 4$. Since $r \ge 15$ we have $r-3-i \ge 3$ and thus there is either $|N_{i-1}(c)| = |N_{i+1}(c)| = 2$, $|N_{i-1}(c)| = 2$ and $|N_{i+1}(c)| = |N_{i+2}(c)| = |N_{i+3}(c)| = 3$, $|N_{i-3}(c)| = |N_{i-2}(c)| = |N_{i-1}(c)| = 3$ and $|N_{i+1}(c)| = 2$, or $|N_{i-3}(c)| = |N_{i-2}(c)| = |N_{i-1}(c)| = 3$ and $|N_{i+1}(c)| = |N_{i+2}(c)| = |N_{i+3}(c)| = 3$.

According to Lemma 4.10 we have at least two vertices $v_1, v_2 \in N_{i-2}(c) \cup N_{i-1}(c)$ such that $d(G - v_1, v_2) = \infty$ and at least two vertices $w_1, w_2 \in N_{i+1}(c) \cup N_{i+2}(c)$ such that $d(G - w_1, w_2) = \infty$. At least two of these vertices lie in $N_{i-1}(c) \cup N_{i+1}(c)$.

Suppose $v_1 \in N_{j_1}(c)$, $w_1 \in N_{j_2}(c)$, $v_2 \in N_{j_3}(c)$, $w_2 \in N_{j_4}(c)$, $d(v_1, w_1) = j_2 - j_1$ and $d(v_2, w_2) = j_4 - j_3$. Consider the subgraph generated by the vertex set $\{c\} \cup N(c) \cup \cdots \cup N_{i+2}(c)$. There is no path joining v_1 and w_2 not including w_1 or v_2 and no path joining v_2 and w_1 not including w_2 or v_1 in this subgraph. Such path would be of length at most $j_2 - j_3 + 2$ or $j_4 - j_1 + 2$. Thus if we take for example the vertex $c' \in N_{r-2-i}(c)$ lying on the c-v_1 (c-v_2) geodesic we have $e_G(c') = r-1$, a contradiction. It follows that there are two pairs $\{v_1, w_2\}$ and $\{v_2, w_1\}$ such that c and $N_r(c)$ are not connected in $G - v_1 - w_2$ and $G - v_2 - w_1$.

Let H be a subgraph generated by the vertex set $\{c\} \cup N(c) \cup \cdots \cup N_{i-1}(c)$. We have $d_H(v_1, v_2) \ge j_1 + j_3 - 1$. Otherwise there is a vertex y such that y belongs to the v_1-v_2 geodesic and $d(y, v_1) \le j_1 - 1$, $d(y, v_2) \le j_3 - 1$. Such a vertex would have $e_G(y) < r$, a contradiction.

Since $i \in \{\lceil \frac{r}{2} \rceil - 1, \lceil \frac{r}{2} \rceil, \lceil \frac{r}{2} \rceil + 1\}$, at least one vertex of v_1, v_2 belongs to $N_{i-1}(c)$ and at least one vertex of w_1, w_2 belongs to $N_{i+1}(c)$, there is a pair a, b of vertices such that $a \in \{v_1, w_1\}$, $b \in \{v_2, w_2\}$ and $r \le d(a, c) + d(c, b) \le r + 2$. Without loss of generality assume that $d(a, c) \ge d(b, c)$. Since $d_H(v_1, c) + d_H(v_2, c) - 1 \le d_H(v_1, v_2) \le d_H(v_1, c) + d_H(v_2, c)$, there is no a-b path containing vertex of H shorter than $d_G(a, c) + d_G(b, c) - 1$.

Now let J be a subgraph of G such that $V(J) = \{v \in V(G), d_G(v, c) = \min\{d(v, a) + d(a, c), d(v, b) + d(b, c)\} - \{b\}$ (i.e., the subgraph generated by the set of vertices which are "successors" of a and b including a) and let K be a subgraph of G induced by the vertex set $(V(G) - V(J) - \{b\}) \cup \{a\}$. Thus $V(G - b) = V(J) \cup V(K)$ and $V(J) \cap V(K) = \{a\}$.

If $e_J(a) + e_K(a) \geq 2r - 1$ then $d(G - b) \geq 2r - 1$ and thus $|V(G)| \geq 3r + 2$ according to Lemma 4.11. If $e_J(a) + e_K(a) = 2r - 2$ then $e_J(a)$ is between $r - 2$ and $r - 4$ and there exists a vertex $z \in K$ such that $e_J(a) + d_K(a, z) = r - 1 = d(z, b)$ having $e_{G-b}(z) = r - 1$. Thus by Lemma 4.12 $|V(G)| \geq 3r + 2$. At last if $e_J(a) + e_K(a) < 2r - 2$ it is sufficient to take a vertex c' on the a-b geodesic in H such that $d_H(c', b) = r - 1$. We have $e_G(c') = r - 1$, a contradiction. We have shown that there is no cocritical r.e.i. graph having $r \geq 15$ on less than $3r + 2$ vertices.

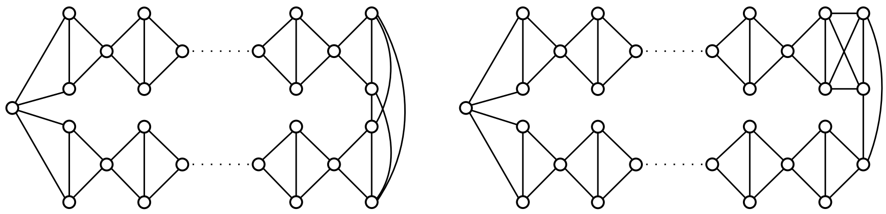

Figure 17

Possible extremal graphs for odd and even radius are depicted on Figure 17. ■

However, the previous theorem is not fully satisfactory. It is not clear if the condition for the radius being greater than 14 is necessary. We can give only the following example of cocritical r.e.i. graph having radius three on ten vertices.

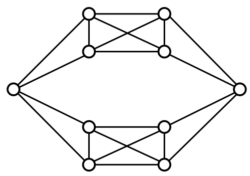

Figure 18

Conjecture. Every cocritical radius-edge invariant graph of radius $r \geq 4$ has at least $3r + 2$ vertices. This bound is sharp.

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