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ON LOCATING AND DIFFERENTIATING-TOTAL DOMINATION IN TREES

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Abstract

A total dominating set of a graph $G = (V, E)$ with no isolated vertex is a set $S \subseteq V$ such that every vertex is adjacent to a vertex in S. A total dominating set S of a graph G is a locating-total dominating set if for every pair of distinct vertices u and v in $V - S$, $N(u) \cap S \neq N(v) \cap S$, and S is a differentiating-total dominating set if for every pair of distinct vertices u and v in V, $N[u] \cap S \neq N[v] \cap S$. Let $\gamma_t^L(G)$ and $\gamma_t^D(G)$ be the minimum cardinality of a locating-total dominating set and a differentiating-total dominating set of G, respectively. We show that for a nontrivial tree T of order n, with ℓ leaves and s support vertices, $\gamma_t^L(T) \geqslant \max\{2(n+\ell-s+1)/5, (n+2-s)/2\}$, and for a tree of order $n \geq 3$, $\gamma_t^D(T) \geq 3(n+\ell-s+1)/7$, improving the lower bounds of Haynes, Henning and Howard. Moreover we characterize the trees satisfying $\gamma_t^L(T) = 2(n+\ell-s+1)/5$ or $\gamma_t^D(T) = 3(n+\ell-s+1)/7$.

Keywords: locating-total domination, differentiating-total domination, trees.

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1. Introduction

In a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is the size of its open neighborhood. A *leaf* of a tree T

is a vertex of degree one, while a *support vertex* of T is a vertex of degree at least two adjacent to a leaf. A *strong support vertex* is adjacent to at least two leaves. We denote the order of a tree T by n, the number of leaves by ℓ, and the number of support vertices by s. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with, respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. A *subdivided star* SS_q is obtained from a star $K_{1,q}$ by subdividing each edge by exactly one vertex. A corona of a graph H is the graph G formed from H by adding a new vertex v' for each vertex $v \in V(H)$ and the edge $v'v$. For a subset $S \subseteq V$, we denote by $\langle S \rangle$ the subgraph induced by the vertices of S.

A subset S of vertices of V is a *total dominating set* of G if every vertex in V is adjacent to a vertex in S. The *total domination number*, $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G.

In this paper we are interested in two types of total-dominating sets, namely locating-total dominating sets, and differentiating-total dominating sets defined as follows: A total dominating set S of a graph G is called a *locating-total dominating set* (LTDS) if for every pair of distinct vertices u and v in $V - S$, $N(u) \cap S \neq N(v) \cap S$, and S is called a *differentiating-total dominating set* (DTDS) if for every pair of distinct vertices u and v in V, $N[u] \cap S \neq N[v] \cap S$. The *locating-total domination number*, $\gamma_t^L(G)$ is the minimum cardinality of a LTDS of G, and the *differentiating-total domination number*, $\gamma_t^D(G)$ is the minimum cardinality of a DTDS of G. A LTDS of minimum cardinality is called a $\gamma_t^L(G)$-set. Likewise we define a $\gamma_t^D(G)$-set. Note that a tree T of order n admits a LTDS (resp., DTDS) if $n \geq 2$ (resp., $n \geq 3$) since the entire vertex set is such a set. Also for every $\gamma_t^D(G)$-set D there is no component of size 2 in the subgraph induced by D, for otherwise the two vertices u, v of such a component would satisfy $N[u] \cap D = N[v] \cap D = \{u, v\}$. Locating-total domination and differentiating-total domination were introduced by Haynes, Henning and Howard [4].

In this paper we establish sharp bounds on $\gamma_t^L(T)$, and $\gamma_t^D(T)$ for trees T. More precisely, we show that if T is a tree of order $n \geq 2$, with ℓ leaves and s support vertices, then $\gamma_t^L(T) \geqslant \max\{2(n+\ell-s+1)/5, (n+2-s)/2\}$ and if T is a tree of order $n \geq 3$, then $\gamma_t^D(T) \geq 3(n+\ell-s+1)/7$. Then we give a characterization of trees with $\gamma_t^L(T) = 2(n+\ell-s+1)/5$, or $\gamma_t^D(T) = 3(n+\ell-s+1)/7$.

We sometimes consider the removing of an edge of a tree T. If uv is an edge of T, then we denote by T_u (resp., T_v) the subtree of T that contains

u (resp., v) obtained by removing uv. The following notation and fact will be used in the proofs. Let n_1, ℓ_1, s_1 be the order, the number of leaves and support vertices of T_u, respectively, and likewise let n_2, ℓ_2, s_2 for T_v. Clearly $n_1 + n_2 = n$, and if n_1 and $n_2 \geq 3$, then $\ell_1 + \ell_2 \geq \ell + q$, and $s_1 + s_2 = s + q$, where q is the number of new support vertices in T_u and T_v with $0 \leq q \leq 2$. Also if D is a $\gamma_t^L(T)$-set or $\gamma_t^D(T)$-set, then let $D_u = D \cap V(T_u)$, and $D_v = D \cap V(T_v)$.

2. Lower Bounds on $\gamma_t^L(T)$

In [4], Haynes, Henning and Howard gave two lower bounds on the locating-total domination number for trees and characterized extremal trees for each lower bound. Let $G = P_n$ be the path on n vertices.

Theorem 1 (Haynes, Henning and Howard [4]).

(1) *If T is a tree of order $n \geq 2$, then $\gamma_t^L(T) \geq 2(n+1)/5$.*

(2) *For $n \geq 2, \gamma_t^L(P_n) = \gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

Theorem 2 (Haynes, Henning and Howard [4]). *If T is a tree of order $n \geq 3$ with ℓ leaves and s support vertices, then $\gamma_t^L(T) \geq (n+2(\ell-s)+1)/3$.*

Our next result improves the lower bound of Theorem 1 for every nontrivial tree T. It also improves Theorem 2 for trees of order $n \geq 4\ell - 4s$. Let $\mathcal{F}$ be the family of trees that can be obtained from r disjoint copies of P_4 and P_3 by first adding $r-1$ edges so that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 3. *If T is a tree of order $n \geq 2$, then*

$$\gamma_t^L(T) \geq 2(n + \ell - s + 1)/5,$$

with equality if and only if $T = P_2$ or $T \in \mathcal{F}$.

Proof. We proceed by induction on the order of T. If $n = 2$, then $T = P_2$ and $\gamma_t^L(P_2) = 2(n + \ell - s + 1)/5 = 2$. Every star $K_{1,p}$ ($p \geq 2$) satisfies $\gamma_t^L(K_{1,p}) = p \geq 2(n + \ell - s + 1)/5$ with equality if and only if $p = 2$, that is $T = P_3 \in \mathcal{F}$. This establishes the base cases. Assume that every tree T' of order $2 \leq n' < n$ satisfies $\gamma_t^L(T') \geq 2(n' + \ell' - s' + 1)/5$. Let T be a tree of order n. Among all $\gamma_t^L(T)$-sets, let D be one that contains as few leaves as possible. Note that every vertex x of D has at most one private neighbor in $V-D$ for if it had two private neighbors x', x'', then we would have $N(x') \cap D = N(x'') \cap D = \{x\}$. If $\ell = 2$, then T is a path P_n, and by Theorem 1, $\gamma_t^L(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor \geq 2(n+\ell-s+1)/5$ with equality if and only if $T = P_4 \in \mathcal{F}$. Thus we may assume that $\ell \geq 3$.

Assume that T contains a strong support vertex y adjacent to at least three leaves. Then D contains y and all its leaves except possibly one. Let $y' \in D$ be any leaf adjacent to y, and let $T' = T - \{y\}$. Clearly $D - \{y'\}$ is a LTDS of T', $n' = n-1, \ell' = \ell - 1$, and $s' = s$. By induction on T', we have $|D - \{y'\}| \geq \gamma_t^L(T') \geq 2(n' + \ell' - s' + 1)/5$, implying that $|D| > 2(n+\ell-s+1)/5$. Thus every support vertex is adjacent to at most two leaves.

Assume that $\langle D \rangle$ contains a connected component $\langle D_i \rangle$ of diameter at least 3. Then there exists an edge uv, such that $\langle D_i - uv \rangle$ contains no isolated vertices. Clearly each of T_u and T_v has order at least three, D_u and D_v are two LTDS of T_u and T_v, respectively. Recall that $n_1 + n_2 = n$, $\ell_1 + \ell_2 \geq \ell + q$, and $s_1 + s_2 = s + q$, where $0 \leq q \leq 2$ is defined above. Applying the inductive hypothesis to T_u and T_v, we obtain

$$|D| = |D_u| + |D_v| \geq 2(n_1 + \ell_1 - s_1 + 1)/5 + 2(n_2 + \ell_2 - s_2 + 1)/5 > 2(n + \ell - s + 1)/5.$$

Thus every component of $\langle D \rangle$ has diameter one or two.

If each $w \in V - D$ is a leaf, then D contains for each support vertex all its leaves except possibly one. Hence $|D| \geq n - s$. Since $\ell \geq 3$ and $n - s \geq \ell$, it follows $|D| \geq n - s > 2(n + \ell - s + 1)/5$. Thus there exists a vertex $w \in V - D$ such that w is not a leaf. Assume now that w has a neighbor say $v \in V - D$. Then each of T_w and T_v has order at least three, D_w and D_v are two LTDS of T_w and T_v, respectively. By induction on T_w and T_v, we obtain

$$|D| = |D_w| + |D_v| \geq 2(n_1 + \ell_1 - s_1 + 1)/5 + 2(n_2 + \ell_2 - s_2 + 1)/5 > 2(n + \ell - s + 1)/5.$$

Hence we may assume that $V - D$ is an independent set and so every private neighbor of a vertex of D is a leaf. Suppose now that w has degree at least three and let z be any vertex of $N(w) \cap D$. By removing wz, then T_w has order a least three. If $V(T_z) = \{z, z'\}$ then z' is a leaf of T and so $\{w\} \cup D - \{z'\}$ is a $\gamma_t^L(T)$-set with less leaves than D, contradicting our assumption on D. Thus T_z has order at least three. Also D_w and D_z are two LTDS of T_w and T_z, respectively. The rest of the proof is similar to as shown above, which leads to $|D| > 2(n+\ell-s+1)/5$. Thus every vertex of $V-D$ is either a leaf or has degree two. Note that all cases considered until now do not lead to extremal trees because $|D| > 2(n+\ell-s+1)/5$. Let W be the set of vertices of $V-D$ having degree two. Since T is a tree, $|W| = k-1$ where k is the number of connected components of $\langle D\rangle$. Let T' be the forest induced by the vertices of $V(T)-W$ and let $T_1, T_2, \ldots, T_k$ the components of T'. Then $n_1+\cdots+n_k = n-|W|$, $\ell_1+\cdots+\ell_k \geq \ell+q$, and $s_1+\cdots+s_k = s+q$, where q is the number of new support vertices. Also $D \cap V(T_i) = D_i$ is a LTDS of T_i, for every $i = 1, \ldots, k$. By induction on each T_i, we obtain

$$\begin{aligned}|D| = \sum_{i=1}^{k}|D_i| &\geq \sum_{i=1}^{k}2(n_i+\ell_i-s_i+1)/5 \geq 2(n-|W|+\ell-s+k)/5\\ &= 2(n+\ell-s+1)/5.\end{aligned}$$

Assume now that $\gamma_t^L(T) = 2(n+\ell-s+1)/5$. Then we have equality throughout the above inequality chain. In particular, $\gamma_t^L(T_i) = 2(n_i+\ell_i-s_i+1)/5$ for each i, and $\ell_1+\cdots+\ell_k = \ell+q$, and $s_1+\cdots+s_k = s+q$. This means that T' has a new leaf if and only if it has a new support vertex. So each T_i has order at least three. Recall that every component $\langle D_i\rangle$ has diameter one or two. Suppose that for some i, $\langle D_i\rangle$ has diameter two, that is $\langle D_i\rangle$ is a star of center vertex say, x and leaves $y_1, y_2, \ldots, y_t$ with $t \geq 2$. We distinguish between three cases. If x is a not a support vertex neither in T nor in T_i, then each y_i is support vertex of T_i, and so T_i is a subdivided star but $\gamma_t^L(T_i) > 2(n_i+\ell_i-s_i+1)/5$, a contradiction. If x is not a support vertex of T but x is a support vertex of T_i, then T_i is a corona of $K_{1,t-1}$, where $|D_i| = t+1 > 2(n_i+\ell_i-s_i+1)/5$. Now if x is a support vertex of T with at most two leaves, then every y_j is a support vertex in T_i for either $1 \leq j \leq t$ or $2 \leq j \leq t$, but then $\gamma_t^L(T_i) > 2(n_i+\ell_i-s_i+1)/5$, a contradiction.

Finally, assume that each connected subgraph $\langle D_i\rangle$ is of diameter one. Then $T_i = P_3$ or P_4, and the leaves of T_i are leaves in T. Thus every component of T' is either a path P_3 or P_4 where every vertex of W joins two support vertices of any different components T_i, T_j.

Conversely, let $T \in \mathcal{F}$ be a tree obtained from k_1 disjoint copies of P_4 and k_2 disjoint copies of P_3 with $k_1+k_2 \geq 1$, by adding k_1+k_2-1 new vertices, where each new vertex is adjacent to exactly two support vertices.

Clearly the set of all support vertices plus one leaf from each copy of P_3 forms a minimum LTDS of T of size $2(n+\ell-s+1)/5$. So extremal trees T achieving $\gamma_t^L(T)=2(n+\ell-s+1)/5$ are precisely those of $\mathcal{F}$. ■

Note that in [2], Chellali and Haynes showed that every nontrivial tree satisfies $\gamma_t(T)\geq(n+2-\ell)/2$. Since every LTDS is a total dominating set, $\gamma_t^L(T)\geq(n+2-\ell)/2$. Our next result improves this lower bound.

Theorem 4. *If T is a tree of order $n\geq 2$, then $\gamma_t^L(T)\geq(n+2-s)/2$.*

Proof. We proceed by induction on the order of T. It is a routine matter to check that the result holds if $\mathrm{diam}(T)\in\{1,2\}$. Assume that every tree T' of order $2\leq n'<n$ satisfies $\gamma_t^L(T')\geq(n'+2-s')/2$. Let T be a tree of order n and S a $\gamma_t^L(T)$-set that contains leaves as few as possible.

If the subgraph induced by $V-S$ contains some edge xy, then let T_x and T_y be the trees obtained by removing the edge xy where $x\in T_x$ and $y\in T_y$. Clearly each of T_x and T_y has order at least three, S_x and S_y are two LTDS of T_x and T_y, respectively. Also $n_1+n_2=n$, and $s_1+s_2=s+q$, where q is the number of new support vertices with $0\leq q\leq 2$. By induction on T_x and T_y, we have $|S|=|S_x|+|S_y|\geq(n_1+2-s_1)/2+(n_2+2-s_2)/2=(n+4-s-q)\geq(n+2-s)/2$. Thus $V-S$ is independent.

Let w be a vertex of $V-S$ different to a leaf. If w does not exist, then $|S|\geq n-s\geq(n+2-s)/2$, since $s\geq 2$. Thus w exists and has at least two neighbors in S. Let z be any neighbor of $N(w)\cap S$, and consider the trees T_w and T_z obtained by removing the edge wz where $w\in T_w$ and $z\in T_z$. If $V(T_z)=\{z,z'\}$, then z' is a leaf of T and $\{w\}\cup S-\{z'\}$ is a $\gamma_t^L(T)$-set with less leaves than S, a contradiction with our choice of S. Thus T_z has order at least three. Also S_w and S_z are two LTDS of T_w and T_z, respectively. Hence by induction on T_w and T_z and since $n_1+n_2=n$, and $s_1+s_2=s+q$, where $0\leq q\leq 2$ is defined as above, we obtain $|S|=|S_w|+|S_z|\geq(n_1+2-s_1)/2+(n_2+2-s_2)/2=(n+4-s-q)\geq(n+2-s)/2$. This achieves the proof. ■

The lower bound of Theorem 4 is sharp for the path P_n with $n\equiv 0(\mathrm{mod}\,4)$ and improves Theorem 3 for nontrivial trees with $n>4\ell+s-6$.

3. Lower Bound on $\gamma_t^D(T)$

In [4], Haynes, Henning and Howard gave a lower bound of the differentiating-total domination number of any tree with at least three vertices.

Theorem 5 (Haynes, Henning and Howard [4])**.**

(1) *If* T *is a tree of order* $n \geq 3$*, then* $\gamma_t^D(T) \geq 3(n+1)/7$.

(2) *For* $n \geq 3, \gamma_t^D(P_n) = \lceil 3n/5 \rceil + 1$ *if* $n \equiv 3 (\mathrm{mod}\, 5)$ *and* $\gamma_t^D(P_n) = \lceil 3n/5 \rceil$ *, otherwise.*

Note that as mentioned in [4], since $\gamma_t^D(T) \geq \gamma_t^L(T)$ for trees, the bound of Theorem 2 is also a lower bound for $\gamma_t^D(T)$.

A subset S of vertices of V is an *identifying code* (or a *differentiating domination set* as defined in [3]) if for every pair of distinct vertices u and v in V, $N[u] \cap S \neq N[v] \cap S \neq \emptyset$.

In [1], Blidia *et al.* showed for trees of order $n \geq 4$ that every identifying code contains at least $3(n+\ell-s+1)/7$ vertices. Since every differentiating-total dominating set is an identifying code, $3(n+\ell-s+1)/7$ is a lower bound for $\gamma_t^D(T)$ which improves Theorem 5. Note that the lower bound $3(n+\ell-s+1)/7$ is better than $(n+2(\ell-s)+1)/3$ for trees with $2n > 5\ell - 5s - 2$. For the purpose of characterizing extremal trees we give here a proof of $\gamma_t^D(T) \geq 3(n+\ell-s+1)/7$, by using a similar argument to that used in the proof of Theorem 3.

Let $\mathcal{G}$ be the family of trees that can be obtained from r disjoint copies of a corona of P_3, a double star $S_{2,1}$ and a star $K_{1,3}$ by first adding $r-1$ edges so that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 6. *If* T *is a tree of order* $n \geq 3$, *then*

$$\gamma_t^D(T) \geq 3(n+\ell-s+1)/7,$$

with equality if and only if $T \in \mathcal{G}$.

Proof. We use an induction on the order of T. If $\mathrm{diam}(T) = 2$, then $T = K_{1,p} (p \geq 2)$. Thus $\gamma_t^D(K_{1,2}) = 3 > 3(n+\ell-s+1)/7$ and for $p \geq 3$, $\gamma_t^D(K_{1,p}) = p \geq 3(n+\ell-s+1)/7$ with equality if and only if $p = 3$, that is $T = K_{1,3} \in \mathcal{G}$. If $\mathrm{diam}(T) = 3$, then $T = S_{p,q}$. Thus $\gamma_t^D(S_{1,1}) = 3 > 3(n+\ell-s+1)/7$ and for $\max\{p,q\} \geq 2, \gamma_t^D(S_{p,q}) = p+q \geq 3(n+\ell-s+1)/7$ with equality if and only if $p+q = 3$, that is $T = S_{2,1} \in \mathcal{G}$. This establishes the base cases. Assume that every tree T' of diameter at least 4 and order n', $5 \leq n' < n$ satisfies $\gamma_t^D(T') \geq 3(n'+\ell'-s'+1)/7$. Let T be a tree of order n, and D a $\gamma_t^D(T)$-set. If T is a path P_n with $n \geq 5$, then by Theorem 5(2), $\gamma_t^D(P_n) > 3(n+\ell-s+1)/7$. Thus we assume that $\ell \geq 3$.

If any strong support vertex y is adjacent to at least four leaves, then let $T' = T - \{y'\}$, where y' is any leaf adjacent to y. Without loss of generality $y' \in D$, and then $D - \{y'\}$ is a DTDS of T'. Hence by induction on T' we have $|D| - 1 \geq \gamma_t^D(T') \geq 3(n' + \ell' - s' + 1)/7$. Since $n' = n - 1, \ell' = \ell - 1$, and $s' = s$, we obtain $|D| > 3(n + \ell - s + 1)/7$. For the next we assume that each support vertex is adjacent to at most three leaves.

Assume, the subgraph $\langle D\rangle$ contains a connected component $\langle D_i\rangle$ of diameter at least 5. Thus there exists an edge uv, such that each connected component of $\langle D_i - uv\rangle$ has diameter at least 2. Then D_u and D_v are two DTDS of T_u and T_v, respectively. Since $n_1 + n_2 = n$, $\ell_1 + \ell_2 \geq \ell + q$, and $s_1 + s_2 = s + q$, then by induction on T_u and T_v, we obtain

$$|D| = |D_u| + |D_v| \geq 3(n_1+\ell_1-s_1+1)/7 + 3(n_2+\ell_2-s_2+1)/7 > 3(n+\ell-s+1)/7.$$

Thus every component of $\langle D\rangle$ has diameter two, three or four.

Suppose that $\langle V - D\rangle$ contains some edge uv. Then by removing the edge uv, each of T_u and T_v has order at least four, D_u and D_v are two DTDS of T_u and T_v, respectively. By using the induction on T_u and T_v, it follows that $\gamma_t^D(T) > 3(n + \ell - s + 1)/7$. Thus $V - D$ is independent and hence every private neighbor of a vertex of D is a leaf.

Let w be any vertex of $V - D$ different to a leaf. If w does not exist, then $|D| \geq n - s \geq 3(n + \ell - s + 1)/7$ with equality only if T is a corona of a path P_3 or a double star $S_{2,1}$. Thus $T \in \mathcal{G}$. Now if w has degree at least three, then let z be any vertex of $N(w) \cap D$. Then by removing wz, T_w has order at least seven and T_z has order at least three, D_w and D_z are two DTDS of T_w and T_z, respectively. The rest of the proof is similar to as shown above and so $|D| > 3(n + \ell - s + 1)/7$. Thus every vertex of $V - D$ is either a leaf or has degree two.

Let W be the set of vertices of $V - D$ having degree two. Since T is a tree, $|W| = k-1$ where k is the number of connected components of $\langle D\rangle$. Let T' be the forest induced by the vertices of $V(T) - W$ and let $T_1, T_2, \ldots, T_k$ the components of T'. Then $n_1 + \cdots + n_k = n - |W|$, $\ell_1 + \cdots + \ell_k \geq \ell + q$, and $s_1 + \cdots + s_k = s + q$, where q is the number of new support vertices. Also $D \cap V(T_i) = D_i$ is a DTDS of T_i, for every $i = 1, \ldots, k$. By induction on each T_i, we obtain

$$\begin{aligned}|D| &= \sum_{i=1}^{k}|D_k| \geq \sum_{i=1}^{k}3(n_i + \ell_i - s_i + 1)/7 \geq 3(n - |W| + \ell - s + k)/7\\ &= 3(n + \ell - s + 1)/7.\end{aligned}$$

Assume now that $\gamma_t^D(T) = 3(n + \ell - s + 1)/7$. Then we have equality throughout this inequality chain. In particular, $\gamma_t^D(T_i) = 3(n_i + \ell_i - s_i + 1)/7$ for each i, and $\ell_1 + \cdots + \ell_k = \ell + q$, and $s_1 + \cdots + s_k = s + q$. Thus T' contains a new leaf if and only if it has a new support vertex. So each T_i has order at least four. Recall that each component of $\langle D \rangle$ has diameter two, three or four. We first assume that the subgraph $\langle D_i \rangle$ has diameter three or four. We will show that no leaf of T_i is contained in $\langle D_i \rangle$. Assume to the contrary that a leaf $y \in V(T_i) \cap D_i$ and let $z \in D_i$ be its support vertex. Note that y may be a new leaf in T_i. Consider the tree $T_i' = T_i - \{y\}$. Then $\langle D_i - \{y\} \rangle$ has diameter at least two and $D_i - \{y\}$ is a DTDS of T_i', with $n_i' = n_i - 1, \ell_i' \geq \ell_i - 1$, and $s_i' \leq s_i$. It follows that $|D_i - \{y\}| \geq \gamma_t^D(T_i') \geq 3(n_i' + \ell_i' - s_i' + 1)/7$ and so $|D_i| > 3(n_i + \ell_i - s_i + 1)/7$, a contradiction since $|D_i| = 3(n_i + \ell_i - s_i + 1)/7$. Thus $\langle D_i \rangle$ contains no leaf of T_i and hence every support vertex of T_i is adjacent to exactly one leaf. Now let k_1 be the number of support vertices of T_i. Thus T_i has k_1 leaves. Let $k_2 = n_i - 2k_1$. Clearly $k_1 + k_2 \geq 4$ since $\langle D_i \rangle$ is a component of diameter three or four, but then $|D_i| = k_1 + k_2 > 3(n_i + \ell_i - s_i + 1)/7$, a contradiction.

Thus for each $i = 1, \ldots, k$, the subgraph $\langle D_i \rangle$ has diameter two, and so $\langle D_i \rangle$ is a star of center vertex x and leaves $y_1, y_2, \ldots, y_t$ with $t \geq 2$. Note that $|D_i| = t + 1$. If x is not a support vertex neither in T nor in T_i, then each y_i is support vertex of T_i. Hence T_i is a subdivided star with $|D_i| > 3(n_i + \ell_i - s_i + 1)/7$, a contradiction. If x is not a support vertex of T but it is a vertex support of T_i, then T_i is a corona of $K_{1,t-1}$, where $|D_i| > 3(n_i + \ell_i - s_i + 1)/7$. Now if x is a support vertex of T with at most three leaves, then every y_j is a support vertex in T_i for either $1 \leq j \leq t$, $2 \leq j \leq t$, or $3 \leq j \leq t$, but then $\gamma_t^L(T_i) = 3(n_i + \ell_i - s_i + 1)/7$ if and only if $T_i = K_{1,3}$, $S_{2,1}$ or T_i is a corona of a path P_3. Thus every component of T' is either a path $K_{1,3}, S_{2,1}$ or corona of P_3 where every vertex of W joins two support vertices. Therefore extremal trees T achieving $\gamma_t^L(T) = 3(n + \ell - s + 1)/7$ are precisely those of $\mathcal{G}$.

The converse is easy to show. ■

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