Note

# SOLUTION TO THE PROBLEM OF KUBESA 

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#### Abstract

An infinite family of $T$-factorizations of complete graphs $K_{2 n}$, where $2 n=56 k$ and $k$ is a positive integer, in which the set of vertices of $T$ can be split into two subsets of the same cardinality such that degree sums of vertices in both subsets are not equal, is presented. The existence of such $T$-factorizations provides a negative answer to the problem posed by Kubesa.


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## 1. Introduction

Let $K_{2 n}$ be the complete graph on $2 n$ vertices and $T$ be its spanning tree. A $T$-factorization of $K_{2 n}$ is a collection of edge disjoint factors $T_{1}, T_{2}, \ldots, T_{n}$ of $K_{2 n}$, each of which being isomorphic to $T$.

At the workshop in Krynica in 2004 D. Fronček presented the following problem originally posed by M. Kubesa [2].

Problem. Suppose that there exists a $T$-factorization of $K_{2 n}$. Is it true that the vertex set of $T$ can be split into two subsets, $V_{1}$ and $V_{2}$, such that $\left|V_{1}\right|=\left|V_{2}\right|=n$ and $\sum_{v \in V_{1}} \operatorname{deg}(v)=\sum_{v \in V_{2}} \operatorname{deg}(v) ?$

Notice that there is no requirement on connectness or disconnectness of graphs induced by $V_{1}$ or $V_{2}$.

Recently, N.D. Tan [3] solved the problem in the affirmative for two narrow classes of trees.

## 2. Constructions

A tree which becomes a star after removal of its pendant edges is called a snowflake. Its central vertex (ie. the central vertex of a star obtained in such a way) is called a root, whilst remaining vertices of degrees greater than one are called inner vertices.

We define a family of snowlakes $\tilde{T}_{2 n}$ of order $2 n=56 k$, for every positive integer $k$. There are 7 vertices of degrees: $28 k-18,28 k-20,11,10,8,7,7$, the remaining $56 k-7$ are leaves. The vertex of degree 11 is the root of $\tilde{T}_{2 n}$.

Lemma 1. For every positive integer $k$, the complete graph $K_{56 k}$ has $\tilde{T}_{56 k}$ factorization.

Proof. The snowflake $\tilde{T}_{56 k}$ is defined by listing its edges; we use the notation $u \prec u_{1}, u_{2}, \ldots, u_{m}$ if all the vertices $u_{1}, u_{2}, \ldots, u_{m}$ are adjacent to $u$. Consider two cases.

Case I. $k=1$. Let $V\left(K_{56}\right)=U \cup X \cup Y \cup Z$, where $U=\left\{u_{0}, u_{1}, \ldots, u_{13}\right\}$, $X=\left\{x_{0}, x_{1}, \ldots, x_{13}\right\}, Y=\left\{y_{0}, y_{1}, \ldots, y_{13}\right\}$ and $Z=\left\{z_{0}, z_{1}, \ldots, z_{13}\right\}$. Edges of $K_{56}$ with both endvertices either in $U$ or $X$ or $Y$ or $Z$ are called pure edges; the remaining ones are mixed edges. To indicate a required $\tilde{T}_{56}$ factorization we prescribe 28 snowflakes split into two classes: $\left\{T_{i}: i=\right.$ $0,1, \ldots, 13\}$ and $\left\{T_{i}^{\prime}: i=0,1, \ldots, 13\right\}$, each $T_{i}$ and $T_{i}^{\prime}$ being isomorphic to $\tilde{T}_{56}$.

We construct the first class. The vertex $u_{12}$ of degree 11 is the root of $T_{0}$ and its inner vertices: $u_{0}, x_{1}, x_{2}, y_{0}, y_{1}, z_{7}$ have degrees $8,8,7,10,7,10$, respectively. The remaining pendant edges are: $u_{12} \prec u_{1}, u_{2}, u_{4}, u_{7}, u_{11}$; $u_{0} \prec x_{8}, x_{9}, x_{11}, x_{12}, x_{13}, y_{4}, y_{5} ; x_{1} \prec u_{5}, u_{9}, u_{10}, u_{13}, y_{3}, y_{8}, y_{10} ; x_{2} \prec x_{3}, x_{4}$, $x_{5}, x_{6}, x_{7}, x_{10} ; y_{0} \prec u_{6}, u_{8}, z_{0}, z_{1}, z_{2}, z_{3}, z_{4}, z_{6}, z_{9} ; y_{1} \prec y_{2}, y_{6}, y_{7}, y_{11}, y_{12}, y_{13}$; $z_{7} \prec u_{3}, x_{0}, y_{9}, z_{5}, z_{8}, z_{10}, z_{11}, z_{12}, z_{13}$. Snowflakes $T_{1}, T_{2}, \ldots, T_{13}$ can be obtained from $T_{0}$ by applying the cyclic permutation $\varphi=(0,1, \ldots, 13)$ in parallel on the indices of vertices in the sets $U, X, Y$ and $Z$. One can easily check that the lengths $1,2,3,4,5,6$ of all pure edges in $K_{56}$ have been already covered, as well as the following lengths of mixed edges for types: $U X: 2,3,4,5,6,8,9,10,11,12,13 ; U Y: 2,3,4,5,6,8 ; U Z: 4,9 ; X Y: 2,7,9 ;$ $X Z: 7 ; Y Z: 0,1,2,3,4,6,9,12$.

To construct the second class we need the snowflake $T_{0}^{\prime}$. Let the vertex $u_{7}$ of degree 11 be the root and $x_{0}, x_{8}, y_{2}, y_{3}, z_{0}, z_{1}$ be the inner vertices of degrees $8,8,7,7,10,10$, respectively. The remaining pendant edges are: $u_{7} \prec u_{0}, z_{3}, z_{4}, z_{5}, z_{6} ; x_{0} \prec x_{7}, y_{0}, y_{8}, y_{10}, y_{11}, y_{12}, y_{13} ; x_{8} \prec z_{2}, z_{8}, z_{9}, z_{10}$, $z_{11}, z_{12}, z_{13} ; y_{2} \prec x_{1}, x_{10}, x_{11}, x_{12}, x_{13}, y_{9} ; y_{3} \prec u_{2}, u_{3}, u_{4}, u_{5}, u_{6}, u_{10} ; z_{0} \prec$ $u_{8}, u_{9}, u_{11}, u_{12}, u_{13}, y_{1}, y_{6}, y_{7}, z_{7} ; z_{1} \prec u_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{9}, y_{4}, y_{5}$. Six snowflakes $T_{i}^{\prime}$, for $i=2,4, \ldots, 12$, can be obtained from $T_{0}^{\prime}$ by applying $i$ th power of $\varphi$ in parallel on the sets $U, X, Y$ and $Z$. Thus the length 7 of all pure edges is covered completely and still remaining lengths of mixed edges, except the lengths 0 of type $U X$ and 5 of type $Y Z$, are covered in a half. Seven remaining snowflakes $T_{j}^{\prime}$ for $j=1,3, \ldots, 13$ are obtained from $T_{0}^{\prime}$ by replacing the edges $u_{0} u_{7}, x_{0} x_{7}, y_{2} y_{9}$ and $z_{0} z_{7}$ with the edges $u_{0} x_{0}, u_{7} x_{7}, y_{2} z_{7}$ and $y_{9} z_{0}$, respectively, and then by applying the permutation $(\varphi)^{j}$ in parallel on the sets $U, X, Y$ and $Z$. Notice that such a replacement does not result in changing the structure of snowflake, i.e., all $T_{j}^{\prime}$ are isomorphic to $T_{0}^{\prime}$. In this way we cover all remaining lengths of mixed edges.

Case II. $k \geq 2$. Let $V\left(K_{56 k}\right)=\bigcup_{l=1}^{k}\left(U^{l} \cup X^{l} \cup Y^{l} \cup Z^{l}\right)$, where $U^{l}=\left\{u_{0}^{l}, u_{1}^{l}, \ldots, u_{13}^{l}\right\}, X=\left\{x_{0}^{l}, x_{1}^{l}, \ldots, x_{13}^{l}\right\}, Y=\left\{y_{0}^{l}, y_{1}^{l}, \ldots, y_{13}^{l}\right\}$ and $Z=\left\{z_{0}^{l}, z_{1}^{l}, \ldots, z_{13}^{l}\right\}, l=1,2, \ldots, k$. In what follows subscripts should be read modulo 14.

In order to construct $28 k$ factors, each isomorphic to $\tilde{T}_{56 k}$, we proceed in the following way. First, for every snowflake $T_{i}, i=0,1, \ldots, 13$, in the $\tilde{T}_{56}$-factorization of $K_{56}$ constructed in Case I we make $k$ copies $T_{i}^{l}, l=$ $1,2, \ldots, k$, by copying every edge $s t$ of $T_{i}$ into $k$ edges $s^{l} t^{l}$, each being an edge of appropriate $T_{i}^{l}$, where $s, t \in U \cup X \cup Y \cup Z$. Moreover, for every $T_{i}^{l}$ among $14 k$ trees obtained in this way, where $i=0,1, \ldots, 13$ and $l=1,2, \ldots, k$, we add $56(k-1)$ edges: $u_{i}^{l} \prec u_{j}^{p}, x_{j}^{p}, y_{j}^{r}, z_{j}^{r}, y_{i}^{l} \prec u_{j}^{r}, x_{j}^{r}, y_{j}^{p}, z_{j}^{p}$, where $l<p \leq k, 1 \leq r<l, j=0,1, \ldots, 13$. Thus every $T_{i}^{l}$ is a snowflake with the root $u_{12+i}^{l}$ of degree 11 , and six inner vertices $u_{i}^{l}, x_{1+i}^{l}, x_{2+i}^{l}, y_{i}^{l}$, $y_{1+i}^{l}, z_{7+i}^{l}$ od degrees $28 k-20,8,7,28 k-18,7,10$, respectively.

Similarly, for every snowflake $T_{i}^{\prime}$ constructed in Case $\mathrm{I}, i=0,1, \ldots, 13$, we built $k$ copies $T_{i}^{\prime l}, l=1,2, \ldots, k$, by copying every edge st of $T_{i}^{\prime}$ into $k$ edges $s^{l} t^{l}, s, t \in U \cup X \cup Y \cup Z$. Analogously to the above, for every $T_{i}^{l}$ of $14 k$ trees just obtained, $i=0,1, \ldots, 13$ and $l=1,2, \ldots, k$, new $56(k-1)$ edges are added: $x_{i}^{l} \prec u_{j}^{p}, x_{j}^{p}, y_{j}^{r}, z_{j}^{r}, z_{i}^{l} \prec u_{j}^{r}, x_{j}^{r}, y_{j}^{p}, z_{j}^{p}$, where $l<p \leq k$, $1 \leq r<l, j=0,1, \ldots, 13$. Every $T_{i}^{\prime l}$ obtained in this way is a snowflake
with the root $u_{7+i}^{l}$ of degree 11, and six inner vertices $x_{i}^{l}, x_{8+i}^{l}, y_{2+i}^{l}, y_{3+i}^{l}$, $z_{i}^{l}, z_{1+i}^{l}$ od degrees $28 k-20,8,7,7,28 k-18,10$, respectively.

Lemma 2. For every set $\bar{V} \subset V\left(\tilde{T}_{56 k}\right)=V\left(K_{56 k}\right)$ such that $|\bar{V}|=28 k$, $\sum_{v \in \bar{V}} \operatorname{deg}(v) \neq 56 k-1$.

Proof. One can check that there are only four sequences of length $28 k$ whose terms are degrees of $\tilde{T}_{56 k}$ and whose sum of terms is $56 k-1$ :
(1) $28 k-18,10,10,1,1, \ldots, 1$,
(2) $28 k-18,7,7,7,1,1, \ldots, 1$,
(3) $28 k-20,11,11,1,1, \ldots, 1$,
(4) $28 k-20,8,8,7,1,1, \ldots, 1$.

None of these sequences is a subsequence of degree sequence of $\tilde{T}_{56 k}$. Thus the assertion holds.

Notice that every of the sequences (1)-(4) indeed appears as a set of degrees for some vertex in factors of $\tilde{T}_{56 k}$-factorization of $K_{56 k}$. It is easily seen that all terms of (1) are degrees of the vertex $z_{i}^{l}$ in $\tilde{T}_{56 k}$-factorization, similarly (2) is a set of degrees for $y_{i}^{l}$, (3) for $u_{i}^{l}$ and (4) for $x_{i}^{l}, i=0,1, \ldots, 13$, $l=1,2, \ldots, k$.

It is still possible that a similar example for the order $2 n<56$ exists. Nevertheless, a computer was used to check that in that case $2 n$ cannot be smaller than 38.

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## References

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