Discussiones Mathematicae
Graph Theory 28 (2008) 367–373

Note

ORDERED AND LINKED CHORDAL GRAPHS

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Abstract

A graph G is called k-ordered if for every sequence of k distinct vertices there is a cycle traversing these vertices in the given order. In the present paper we consider two novel generalizations of this concept, *k-vertex-edge-ordered* and *strongly k-vertex-edge-ordered*. We prove the following results for a chordal graph G:

(a) G is $(2k-3)$-connected if and only if it is k-vertex-edge-ordered ($k \geq 3$).

(b) G is $(2k-1)$-connected if and only if it is strongly k-vertex-edge-ordered ($k \geq 2$).

(c) G is k-linked if and only if it is $(2k-1)$-connected.

Keywords: paths and cycles, connectivity, chordal graphs.

2000 Mathematics Subject Classification: 05C38, 05C40.

1. Introduction and Results

All graphs considered in this paper are finite, undirected, and simple, i.e., without loops or multiple edges. For terminology not defined here we refer to [2]. A graph is *chordal* if it contains no induced cycles other than triangles, and it is called *k-linked* if for every set of k distinct pairs $L = \{(s_0, t_0), \dots, (s_{k-1}, t_{k-1})\}$ of vertices it contains k internally disjoint paths $P_0, \dots, P_{k-1}$ such that P_i links s_i to t_i for all $i \in \{0, \dots, k-1\}$. We shall call the subgraph of G formed by the union of $P_0, \dots, P_{k-1}$ an L-linkage. Jung [5] and, independently, Larman and Mani [6] proved that for every k there is an (minimal) $f(k)$ such that every $f(k)$-connected graph is k-linked. Bollobás and Thomason [1] showed that $f(k) \le 22k$. Recently, it was proved by Thomas and Wollan [8] that $f(k) \le 10k$. Our second result, Theorem 1.2 below, shows that for the special case of chordal graphs the precise value of $f(k)$ is $2k-1$.

A graph is called *k-ordered* if for every sequence $(v_0, \dots, v_{k-1})$ of k distinct vertices there is a cycle of G that contains $v_0, \dots, v_{k-1}$ in the given order. This concept was introduced by Ng and Schultz [7], and a survey of results on k-ordered graphs is given in [4]. It is easy to see that being k-linked implies being k-ordered. We generalize the concept of k-orderability as follows. Let $T = (a_0, \dots, a_{k-1})$ be a sequence of k distinct vertices and/or edges, and let $V(T)$ and $E(T)$ denote the sets of vertices and edges in T, respectively. Let $W(T)$ denote the set of all vertices that are either contained in T or incident to an edge in T. A T-cycle is a cycle in G that contains $a_0, \dots, a_{k-1}$ in the given order. The sequence T is said *admissible* if it satisfies the following conditions.

(1) If an edge $a_i \in E(T)$ is incident to a vertex $a_j \in V(T)$, then $|i-j| \equiv 1 \pmod{k}$.

(2) If two edges $a_i, a_j \in E(T)$ meet in a vertex $x \notin V(T)$, then $|i-j| \equiv 1 \pmod{k}$.

A graph is called *k-vertex-edge-ordered* if for every admissible sequence $T = (a_0, \dots, a_{k-1})$ of k distinct vertices and/or edges there is a T-cycle.

Theorem 1.1. *Let G be a chordal graph on at least $2k-2$ vertices with $k \ge 3$. Then the following two statements are equivalent:*

(a) *G is $(2k-3)$-connected.*

(b) *G is k-vertex-edge-ordered.*

Theorem 1.1 implies a conjecture of Faudree [4] for the special case of chordal graphs.

We further generalize this concept. An *orientation* of an edge $e = \{u, v\}$ is a pair (u, v); u is called the *tail* and v the *head*. Let $(a_0, \dots, a_{k-1})$ be an admissible sequence of k distinct vertices and/or edges. An orientation of the edges in this sequence is *admissible* if it satisfies the following conditions.

(3) If the vertex a_i is the tail of the edge a_j, then $i \equiv j - 1 \pmod{k}$.

(4) If the vertex a_i is the head of the edge a_j, then $i \equiv j + 1 \pmod{k}$.

(5) If two edges $a_i, a_j \in E(T)$ meet in a vertex $x \notin V(T)$ and $j \equiv i + 1 \pmod{k}$, then x is the head of a_i and the tail of a_j.

A graph is called *strongly k-vertex-edge-ordered* if for every admissible sequence $T = (a_0, \dots, a_{k-1})$ of k distinct vertices and/or edges and every admissible orientation of the edges of this sequence there is a cycle C of G that can be traversed such that $a_0, \dots, a_{k-1}$ are encountered in the given order and every edge is traversed according to its orientation, i.e., from tail to head. Clearly, C is a T-cycle.

Theorem 1.2. *Let G be a chordal graph on at least $2k$ vertices. Then the following three statements are equivalent:*

(a) *G is $(2k-1)$-connected.*

(b) *G is k-linked.*

(c) *G is strongly k-vertex-edge-ordered.*

2. Proofs

Let G be a graph and let x be a vertex of G. Then $N(x)$ denotes the set of all vertices adjacent to x in G. A vertex x of a graph G is *simplicial* if the subgraph $G[N(x)]$ of G induced by $N(x)$ is complete. The following Proposition 2.1 is a consequence of a well-known theorem of Dirac [3].

Proposition 2.1. *Let G be a k-connected chordal graph. Then the following hold:*

(a) *There is a simplicial vertex $x \in V(G)$, and $G - x$ is chordal.*

(b) *If G is not complete and x is a simplicial vertex of G, then $G - x$ is k-connected.*

The following Proposition 2.2 will be frequently used in the proof of Theorem 1.1. Its easy proof is left to the reader.

Proposition 2.2. *Let G be a graph, $T = (a_0, \dots, a_{k-1})$ be an admissible sequence of distinct vertices and/or edges, $X \subseteq V(G)$, and $J \subseteq \{0, \dots, k-1\}$.*

If for every vertex $x \in X$ there is a $j \in J$ such that either $x = a_j$ or x is incident to the edge a_j, then $|X| \leq 2|J|$.

***Proof of Theorem* 1.1.** To show that (a) implies (b), we apply induction on $|G|$. Let $T = (a_0, \dots, a_{k-1})$ be an admissible sequence. If G is complete the statement of the theorem is clearly true. Hence we may assume that G is not complete and, therefore, $|G| \geq 2k-1$. By Proposition 2.1, there is a simplicial vertex $u \in V(G)$ and $G-u$ is $(2k-3)$-connected and chordal. Note that $|N(u)| \geq 2k-3$. Let $H = G[N(u) \cup \{u\}]$. Clearly, H is complete. Consequently, the assertion is true if $W(T) \subseteq V(H)$. So, we henceforth assume that

$$W(T) \not\subseteq V(H). \tag{1}$$

If $u \notin W(T)$, then we apply the induction hypothesis to $G-u$, and we are done. If $u \in W(T)$, then we construct an admissible sequence $T' = (a'_0, \dots, a'_{k-1})$ of vertices and/or edges of $G-u$. Hence, by the induction hypothesis, there is a T'-cycle C' in $G-u$. It is easy to see that C' can be extended to a T-cycle C in G. For the construction of T' we distinguish the following cases.

Case 1. $u \in V(T)$, say $u = a_0$.

Case 1.1. u is incident to an edge in T, say a_1.
By Proposition 2.2 and (1), $N(u) \setminus W(T) \neq \emptyset$. Let $v \in N(u)$ be the end of a_1 and $w \in N(u) \setminus W(T)$. Put $a'_0 = w$, $a'_1 = \{v, w\}$, and $a'_i = a_i$ for $i \in \{2, \dots, k-2\}$. If a_{k-1} is an edge incident to u, then let $x \in N(u)$ be the end of a_{k-1} and put $a'_{k-1} = \{w, x\}$. Otherwise, let $a'_{k-1} = a_{k-1}$.

Case 1.2. u is not incident with any edge in T.
If $|N(u) \setminus W(T)| \geq 2$, then let $v, w \in N(u) \setminus W(T)$, and put $a'_0 = \{v, w\}$ and $a'_i = a_i$ for all $i \in \{1, \dots, k-1\}$. If $|N(u) \setminus W(T)| \leq 1$, then there is a vertex $v \in N(u)$ such that either $v = a_j$ or v is incident to the edge a_j and to no other edge in T where $j \in \{1, k-1\}$. If not, then all vertices but at most one in $N(u)$ are either in $V(T) \setminus \{a_0, a_1, a_{k-1}\}$ or incident to an edge in $E(T) \setminus \{a_1, a_{k-1}\}$. By Proposition 2.2 this implies that $|N(u)| - 1 \leq 2(k-3) < 2k-4$, contradicting $|N(u)| \geq 2k-3$. W.l.o.g., we may assume that $j = 1$. If $|N(u) \setminus W(T)| = 1$, then let $w \in N(u) \setminus W(T)$ and put $a'_0 = \{v, w\}$ and $a'_i = a_i$ for all $i \in \{1, \dots, k-1\}$. If $|N(u) \setminus W(T)| = 0$,

then a_1 is an edge. If not, then $a_1 = v$ and therefore, by Proposition 2.2, $W(T) = N(u) \cup \{u\}$, contradicting (1). In a similar way it can be shown that there is a vertex $w \in N(u) \setminus \{v\}$ such that either $w = a_{k-1}$ or w is incident to the edge a_{k-1} and to no other edge in T. Put $a'_0 = \{v, w\}$ and $a'_i = a_i$ for all $i \in \{1, \ldots, k-1\}$.

Case 2. $u \notin V(T)$.

Case 2.1. u is incident to two edges in T, say to $a_0, ak-1$.
Let $v \in N(u)$ be the end of a_0, and $w \in N(u)$ be the end of a_{k-1}. If $|N(u) \setminus W(T)| \geq 1$, then let $x \in N(u) \setminus W(T)$, and put $a'_0 = \{v, x\}$, $a'_{k-1} = \{x, w\}$, and $a'_i = a_i$ for $i \in \{1, \ldots, k-2\}$. If $|N(u) \setminus W(T)| = 0$, then, by Proposition 2.2, $v \neq a_1$ and $w \neq a_{k-2}$. Put $a'_0 = v$, $a'_{k-1} = \{v, w\}$ and $a'_i = a_i$ for $i \in \{1, \ldots, k-2\}$.

Case 2.2. u is incident to exactly one edge in T, say to a_0.
Let $v \in N(u)$ be the end of a_0. If $|N(u) \setminus W(T)| \geq 1$, then let $w \in N(u) \setminus W(T)$ and put $a'_0 = \{v, w\}$ and $a'_i = a_i$ for $i \in \{1, \ldots, k-1\}$. If $|N(u) \setminus W(T)| = 0$, then it follows by Proposition 2.2 and (1) that $v \neq a_1$ and $v \neq a_{k-1}$. By the essentially the same arguments as in Case 1.2 it follows, that if $v \notin V(T)$ and v is not incident to any edge in $E(T) \setminus \{a_0\}$, then there is a vertex $w \in N(u) \setminus \{v\}$ such that either $w = a_j$ or w is incident to the edge a_j and to no other edge in T where $j \in \{1, k-1\}$. We may assume w.l.o.g. that $j = 1$. Put $a'_0 = \{v, w\}$ and $a'_i = a_i$ for $i \in \{1, \ldots, k-1\}$. If v is incident to an edge in $E(T) \setminus \{a_0\}$, say a_1, then there is a vertex w such that either $w = a_{k-1}$ or w is incident to the edge a_{k-1} and to no other edge in T. Put $a'_0 = \{v, w\}$ and $a'_i = a_i$ for $i \in \{1, \ldots, k-1\}$.

Next, we prove that (b) implies (a). It is clear that every k-vertex-edge-ordered graph is connected. Let G be a connected chordal graph on at least $2k-2$ vertices that is not $(2k-3)$-connected. G has a minimal separator $S \subseteq V(G)$ with $|S| \leq 2k-4$. Let G_1, G_2 be two distinct components of $G - S$. Since G is chordal, the subgraph H of G induced by S is complete. Let $Z = \{a_1, \ldots, a_{r-2}\}$ be a collection of vertices and/or edges in H such that Z is a perfect matching of H if $|H|$ is even and a maximal matching plus the (only) unsaturated vertex, otherwise. Note that $r \leq k$. Let $T = (a_0, \ldots, a_{r-1})$ where $a_0 \in V(G_1)$ and $a_{r-1} \in V(G_2)$. It is not hard to see that there is no T-cycle in G. Hence every k-vertex-edge-ordered chordal graph with at least $2k-2$ vertices is $(2k-3)$-connected. ∎

***Proof of Theorem* 1.2.** To show that (a) implies (b), we apply induction on $|G|$. Since G is $(2k-1)$-connected, $|G| \geq 2k$. If $|G| = 2k$, then G is complete, and hence it is k-linked. If $|G| > 2k$, then it follows from Proposition 2.1 that G has a simplicial vertex x and $G-x$ is $(2k-1)$-connected and chordal. Let $L = \{(s_0,t_0),\dots,(s_{k-1},t_{k-1})\}$ be a set of k distinct pairs of vertices of G. Let l denote the number of pairs in L containing x. If $l = 0$, we apply the induction hypothesis to $G - x$, and we are done. We may therefore assume that $l \geq 1$, say $x = s_0 = \ldots = s_{l-1}$. Let $A = \{t_0,\dots,t_{l-1}\}$, and suppose that $A' = \{t_0,\dots,t_{m-1}\} = A \cap N(x)$. If there is a $t_i \in A$ such that $t_i = x$, then suppose that $i = l-1$. Consequently, $A'' = A \setminus (A' \cup \{x\}) = \{t_m,\dots,t_{n-1}\}$ where $n = l-1$ if $x = t_{l-1}$ and $n = l$, otherwise. Since $|N(x)| \geq 2k-1$, $|N(x) \setminus (A' \cup \{s_l,\dots,s_{k-1},t_l,\dots,t_{k-1}\}| \geq 2k-1-m-2(k-l) = 2l-m-1 \geq l-m$. Consequently, there is a subset $B \subseteq N(x)\setminus(A' \cup \{s_l,\dots,s_{k-1},t_l,\dots,t_{k-1}\}$ such that $|B| \geq l-m$. Let $B = \{y_m,\dots,y_{n-1}\}$, and let $B' = A' \cup B$. It follows from the induction hypothesis, that $G-x$ contains pairwise disjoint paths $Q_0,\dots,Q_{n-1},P_l,\dots,P_{k-1}$ such that Q_i is the trivial path consisting of t_i for $i \in \{0,\dots,m-1\}$, Q_i links y_i to t_i for $i \in \{m,\dots,n-1\}$, and P_i links s_i to t_i for $i \in \{l,\dots,k-1\}$. For $i \in \{0,\dots,n-1\}$ let P_i be the path obtained from Q_i by adding the edge $\{y_i,x\}$. If $t_{l-1} = x$, let P_{l-1} be the trivial path consisting of x. Obviously, the paths $P_0,\dots,P_{k-1}$ form the desired L-linkage in G.

Next, we prove that (b) implies (c). Let G be k-linked and let $T = (a_0,\dots,a_{k-1})$ be an admissible sequence together with an admissible orientation of the edges. A vertex in $V(T)$ is said to be *isolated* if it is not incident with any edge in $E(T)$. Let M denote the set of all isolated vertices in $V(T)$, and let $T' = (a_{i_0},\dots,a_{i_{r-1}})$ be the subsequence of T obtained by deleting all elements $a_i \in V(T) \setminus M$. For $e \in E(T)$ let $s(e)$ and $t(e)$ denote the head and the tail of e, respectively, and set $s(x) = t(x) = x$ for all $x \in M$. Let $L = \{(s_0,t_0),\dots,(s_{r-1},t_{r-1})\}$ where $s_j = s(a_{i_j})$ for $0 \leq j \leq r-1$, $t_j = t(a_{i_{j+1}})$ for $0 \leq j \leq r-2$, and $t_{r-1} = t(a_{i_0})$. Since G is k-linked there is an L-linkage, and it is not hard to see that the union of an L-linkage and $E(T)$ forms the desired cycle.

Eventually, we prove that (c) implies (a) It is clear that every strongly k-vertex-edge-ordered graph is connected. Let G be a connected chordal graph on at least $2k$ vertices that is not $(2k-1)$-connected. G has a minimal separator $S \subseteq V(G)$ with $r = |S| \leq 2k-2$. Let G_1, G_2 be two distinct components of $G - S$. Since G is chordal, the subgraph H of G induced by S is complete. Let $Q = v_1,\dots,v_r$ be a Hamiltonian path of H, and let

u_1 and u_2 be vertices of G_1 and G_2, respectively, such that u_1 is adjacent to v_1 and u_2 is adjacent to v_r in G. For $1 \leq i \leq \lfloor \frac{r-1}{2} \rfloor$, let e_i denote the oriented edge (v_{2i}, v_{2i+1}). Furthermore, let e_0 and $e_{\lfloor \frac{r-1}{2} \rfloor+1}$ denote the oriented edges (u_1, v_1) and (v_r, u_2), respectively. It is not hard to see, that G does not contain a cycle that can be traversed such that $e_0, \ldots, e_{\lfloor \frac{r-1}{2} \rfloor+1}$ are encountered in the given order and every edge is traversed according to its orientation. Since $\lfloor \frac{r-1}{2} \rfloor + 1 \leq k$, this shows that G is not strongly k-vertex-edge-ordered. Hence every strongly k-vertex-edge-ordered chordal graph is $(2k-1)$-connected. ■

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Received 20 September 2007
Revised 1 April 2008
Accepted 2 April 2008