

NOTE

## A REMARK ON THE $(2, 2)$ -DOMINATION NUMBER

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### Abstract

A subset  $D$  of the vertex set of a graph  $G$  is a  $(k, p)$ -dominating set if every vertex  $v \in V(G) \setminus D$  is within distance  $k$  to at least  $p$  vertices in  $D$ . The parameter  $\gamma_{k,p}(G)$  denotes the minimum cardinality of a  $(k, p)$ -dominating set of  $G$ . In 1994, Bean, Henning and Swart posed the conjecture that  $\gamma_{k,p}(G) \leq \frac{p}{p+k}n(G)$  for any graph  $G$  with  $\delta_k(G) \geq k + p - 1$ , where the latter means that every vertex is within distance  $k$  to at least  $k + p - 1$  vertices other than itself. In 2005, Fischermann and Volkmann confirmed this conjecture for all integers  $k$  and  $p$  for the case that  $p$  is a multiple of  $k$ . In this paper we show that  $\gamma_{2,2}(G) \leq (n(G) + 1)/2$  for all connected graphs  $G$  and characterize all connected graphs with  $\gamma_{2,2} = (n + 1)/2$ . This means that for  $k = p = 2$  we characterize all connected graphs for which the conjecture is true without the precondition that  $\delta_2 \geq 3$ .

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### 1. TERMINOLOGY AND INTRODUCTION

In this paper we consider simple, finite and undirected graphs  $G = (V, E)$  with vertex set  $V$  and edge set  $E$ . The number of vertices  $|V|$  is called the *order* of  $G$  and is denoted by  $n(G)$ .

If there is an edge between two vertices  $u, v \in V$ , then we denote the edge by  $uv$ . Furthermore, we call the vertex  $v$  a *neighbor* of  $u$  and say that  $uv$  is incident with  $u$ . The *neighborhood* of a vertex  $u$  is defined as the set  $\{v \mid uv \in E\}$  and is usually denoted by  $N(u)$ . For a vertex  $v \in V$  we define the *degree* of  $v$  as  $d(v) = |N(v)|$ . If  $d(v) = 1$ , then the vertex  $v$  is called a *leaf* of  $G$ . The *minimum degree* of  $G$  is denoted by  $\delta(G) = \min\{d(v) \mid v \in V(G)\}$ .

For any positive integer  $k$  and any graph  $G$  the  $k$ -th power  $G^k$  of  $G$  is the graph with vertex set  $V(G)$  where two different vertices are adjacent if and only if the distance between them is at most  $k$  in  $G$ . Furthermore, the *minimum  $k$ -degree*  $\delta_k(G)$  of  $G$  is defined by  $\delta_k(G) = \delta(G^k)$ .

Let  $X \subseteq V$  be a subset of the vertex set of a graph  $G = (V, E)$ . Then  $G - X$  denotes the graph that is obtained by removing all vertices of  $X$  and all edges that are incident with at least one vertex of  $X$  from  $G$ . The *diameter* of a graph is defined as the maximum distance between all pairs of vertices.

For two positive integers  $k$  and  $p$  a subset  $D$  of the vertex of a graph  $G$  is a  $(k, p)$ -*dominating set* of  $G$  if every vertex  $v \in V(G) \setminus D$  is within distance  $k$  to at least  $p$  vertices in  $D$ . The parameter  $\gamma_{k,p}(G)$  denotes the minimum cardinality of a  $(k, p)$ -dominating set of  $G$  and is called the  $(k, p)$ -*domination number*.

This domination concept is a generalization of the two concepts *distance domination* and  *$p$ -domination*. For  $p = 1$  a  $(k, p)$ -dominating set of  $G$  is called a *distance- $k$  dominating set* and for  $k = 1$  a  $(k, p)$ -dominating set of  $G$  is called a  *$p$ -dominating set*.

For other graph terminologies we refer the reader to the monographs by Haynes, Hedetniemi and Slater [4, 5].

In 1994, Bean, Henning and Swart [1] posed the following conjecture for the  $(k, p)$ -domination number  $\gamma_{k,p}$ .

**Conjecture 1.** (Bean, Henning & Swart [1] 1994). Let  $k$  and  $p$  be arbitrary positive integers and let  $G$  be a graph of minimum  $k$ -degree  $\delta_k(G) \geq k+p-1$ . Then

$$\gamma_{k,p}(G) \leq \frac{p}{p+k}n(G).$$

This conjecture is valid for  $p = 1$  and all integers  $k \geq 1$  as proved by Meir and Moon [6] in 1975 (the distance- $k$  domination number is called  *$k$ -covering number* in [6]). The conjecture is also true for  $k = 1$  and all integers  $p \geq 1$  as proved by Cockayne, Gamble and Shepherd [2] in 1985.

In 2005, Fischermann and Volkmann [3] confirmed that the conjecture is valid for all integers  $k$  and  $p$ , where  $p$  is a multiple of  $k$ , and presented weaker statements in the remaining cases.

Note that if  $k = p = 2$ , then Conjecture 1 requires that  $\delta_2(G) \geq 3$ . In this paper, we shall show that the conjecture is true for  $k = p = 2$  without the precondition that  $\delta_2(G) \geq 3$  for all connected graphs with the exception of the following class.

**Definition 2.** A *spider* is a graph  $G$  with vertex set  $V = \{x\} \cup \{y_i \mid i = 1, 2, \dots, k\} \cup \{z_i \mid i = 1, 2, \dots, k\}$  and edge set  $E = \{xy_i \mid i = 1, 2, \dots, k\} \cup \{y_i z_i \mid i = 1, 2, \dots, k\}$ , where  $k \geq 1$  is an integer. The vertex  $x$  is called the *centre* of  $G$ .

In particular, note that if  $G$  is a spider, then  $\delta_2(G) = 2$ . We can calculate the  $(2, 2)$ -domination number of spiders as follows.

**Theorem 3.** *If  $G$  is a spider with  $n$  vertices, then  $\gamma_{2,2}(G) = \frac{n+1}{2}$ .*

**Proof.** Let  $G$  be a spider as defined in Definition 2. Then it is easy to see that  $\{x\} \cup \{y_i \mid i = 1, 2, \dots, k\}$  is a  $(2, 2)$ -dominating set of  $G$ .

It remains to prove that there exists no  $(2, 2)$ -dominating set  $D$  of  $G$  such that  $|D| < \frac{n+1}{2}$ . Assume to the contrary that  $D$  is a  $(2, 2)$ -dominating set of  $G$  such that  $|D| < \frac{n+1}{2}$ . Note that for each pair  $y_i, z_i$  of vertices of  $G$  the vertex  $y_i$  or the vertex  $z_i$  or both belong to  $D$ . Since  $|D| < \frac{n+1}{2}$ , it follows that  $|D \cap \{y_i, z_i\}| = 1$  for each  $i = 1, 2, \dots, k$ . If  $D = \{z_1, z_2, \dots, z_k\}$ , then  $y_1$  is not  $(2, 2)$ -dominated by  $D$ , a contradiction. Otherwise let  $i$  be an integer such that  $y_i \in D$ . But then  $z_i$  is not  $(2, 2)$ -dominated by  $D$ , again a contradiction. This completes the proof of this theorem. ■

To prove our main result we need the following graph operations.

**Definition 4.** Let  $G$  be a connected graph and let  $x$  be a vertex of  $G$ .

- (i) The graph  $G_x$  is obtained from  $G$  by adding two leaves as neighbors to  $x$ , i.e.,  $V(G_x) = V(G) \cup \{y, z\}$  and  $E(G_x) = E(G) \cup \{xy, xz\}$ .
- (ii) The graph  $G^x$  is obtained from  $G$  by adding a path  $yz$  of length 1 to  $G$  such that  $y$  is a neighbor of  $x$ , i.e.,  $V(G^x) = V(G) \cup \{y, z\}$  and  $E(G^x) = E(G) \cup \{xy, yz\}$ .

## 2. RESULTS

We first prove a structural result.

**Theorem 5.** *Let  $G$  be a connected graph and let  $D$  be a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $G$ . If  $x$  is an arbitrary vertex of  $G$ , then either  $D \cup \{x\}$  or  $D \cup \{y\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $G_x$  and  $D \cup \{z\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $G^x$ .*

**Proof.** Let  $x$  be an arbitrary vertex of  $G$  and let  $D$  be a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $G$ .

We first consider  $G^x$ . If  $x \in D$ , then both neighbors of  $y$  in  $G^x$  belong to  $D \cup \{z\}$ . Otherwise  $x$  has a neighbor  $v \in D$  which naturally has distance 2 from  $y$ . Therefore  $D \cup \{z\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $G^x$ .

We now consider  $G_x$ . If  $x \in D$ , then, since  $z$  is a neighbor of  $x$  and has distance 2 from  $y$ , the set  $D \cup \{y\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $G_x$ . Otherwise  $x$  has a neighbor  $v \in D$  which naturally has distance 2 from  $y$  and  $z$ . Therefore  $D \cup \{x\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $G_x$ . ■

Our main result follows.

**Theorem 6.** *If  $T$  is a tree on  $n \geq 3$  vertices, then there exists a minimum  $(1, 1)$ - and  $(2, 2)$ -dominating set  $D$  of  $T$  such that  $|D| \leq \frac{n+1}{2}$ . In addition, equality holds if and only if  $T$  is a spider.*

**Proof.** We shall prove the proposition by induction on  $n$ .

The only tree  $T$  with  $n = 3$  vertices is the path  $xyz$  of length 2. This means that  $T$  is a spider and two arbitrary vertices of  $T$  are a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $T$ .

If  $T$  is a tree with  $n = 4$  vertices, then either  $T$  is the path of length 3 or  $T$  is a star. In the first case the two leaves of  $T$  and in the latter case the centre of  $T$  and an arbitrary other vertex are a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $T$ .

Let  $T$  be a tree on  $n = 5$  vertices. If  $T$  is the path  $v_1v_2v_3v_4v_5$  of length 4, then  $T$  is a spider and  $\{v_1, v_3, v_5\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $T$ . If  $T$  has diameter 3, then the two vertices that are not leaves form a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $T$ . In the remaining case  $T$  has diameter 2 and thus,  $T$  is a star. Then the centre of  $T$  and another arbitrary vertex of  $T$  form a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $T$ .

Now let  $T$  be a tree on  $n \geq 6$  vertices. Note that each spider has an odd number of vertices. In addition, note that there exists a vertex  $x$  in  $T$  such that either

- (1) two leaves  $y, z$  of  $T$  are neighbors of  $x$  or
- (2) the vertex  $x$  is not a leaf and there exists a vertex  $y$  with  $d(y) = 2$  that has  $x$  and a leaf  $z$  as neighbors.

Let  $x, y, z$  be vertices of  $T$  that fulfill either (1) or (2). By the induction hypothesis, the tree  $T - \{y, z\}$  has a minimum  $(1, 1)$ - and  $(2, 2)$ -dominating set  $D$  such that

$$|D| \leq \frac{n(T - \{y, z\}) + 1}{2} = \frac{n - 1}{2}.$$

If  $x, y, z$  fulfill (1), then, by Theorem 5,  $D \cup \{x\}$  or  $D \cup \{y\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $T = (T - \{y, z\})_x$ . If  $x, y, z$  fulfill (2), then, by Theorem 5,  $D \cup \{z\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $T = (T - \{y, z\})^x$ .

If  $T - \{y, z\}$  is not a spider in one of the cases above, then, by the induction hypothesis,  $|D| \leq \frac{n-2}{2}$  and thus,

$$|D \cup \{x\}| \leq |D \cup \{y\}| = |D \cup \{z\}| = |D| + 1 \leq \frac{n}{2}.$$

Suppose now that  $T - \{y, z\}$  is a spider for all vertices  $x, y, z$  that fulfill (1) or (2). In this case we shall show that  $T$  itself is a spider or a path  $P_7$  of order 7 which has a  $(1, 1)$ - and  $(2, 2)$ -dominating set of size 3. Let  $T - \{y, z\}$  be a spider as defined in Definition 2.

Assume that  $x, y, z$  fulfill (1). Then there exists an integer  $i$  such that  $T - \{y_i, z_i\}$  is not a spider, a contradiction.

So assume now that  $x, y, z$  fulfill (2). Note that  $k \geq 2$ , since  $|V(T)| \geq 6$ .

If  $k \geq 3$  or  $k = 2$  and  $T \neq P_7$ , then either there exists an integer  $i$  such that  $T - \{y_i, z_i\}$  is not a spider, again a contradiction, or the centre of  $T$  is the only neighbor of  $y$  in  $T$ . But in the latter case it is immediate that  $T$  is a spider.

If  $k = 2$  and  $T = P_7$ , then let  $T = v_1 v_2 \dots v_7$ . In this case  $\{v_1, v_4, v_7\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $T$ , which completes the proof of this theorem. ■

Theorem 6 immediately implies the following corollaries.

**Corollary 7.** *If  $T$  is a tree on  $n \geq 3$  vertices, then  $\gamma_{2,2}(T) \leq \frac{n+1}{2}$  with equality if and only if  $T$  is a spider.*

**Corollary 8.** *If  $G$  is a connected graph on  $n \geq 3$  vertices, then there exists a minimum  $(1, 1)$ - and  $(2, 2)$ -dominating set  $D$  of  $G$  such that  $|D| \leq \frac{n+1}{2}$ . In addition, equality holds if and only if  $G$  is a spider.*

**Proof.** If  $G$  has a spanning tree that is not a spider, then the inequality is true by Theorem 6. Otherwise either  $G$  itself is a spider or  $G$  is a cycle  $v_1v_2v_3v_4v_5v_1$  of length 5. In the latter case  $\{v_1, v_3\}$  is a  $(1, 1)$ - and  $(2, 2)$ -dominating set of  $G$  with the required cardinality. ■

**Corollary 9.** *If  $G$  is a connected graph on  $n \geq 3$  vertices, then  $\gamma_{2,2}(G) \leq \frac{n+1}{2}$  with equality if and only if  $G$  is a spider.*

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