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AN UPPER BOUND ON THE LAPLACIAN SPECTRAL RADIUS OF THE SIGNED GRAPHS*

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Abstract

In this paper, we established a connection between the Laplacian eigenvalues of a signed graph and those of a mixed graph, gave a new upper bound for the largest Laplacian eigenvalue of a signed graph and characterized the extremal graph whose largest Laplacian eigenvalue achieved the upper bound. In addition, an example showed that the upper bound is the best in known upper bounds for some cases.

Keywords: Laplacian matrix, signed graph, mixed graph, largest Laplacian eigenvalue, upper bound.

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1. Introduction

Let $G = (V, E)$ denote a simple graph with vertex set $V = V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \dots, e_m\}$. Then the Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(d_{v_1}, d_{v_2}, \dots, d_{v_n})$ and $A(G) = (a_{ij})$ are the diagonal matrix of degrees and the adjacency matrix of G, respectively. One may also define the Laplacian matrix by means of the oriented incidence matrix Q, that is, $L(G) = QQ^T$, where Q^T is the transpose of Q. So $L(G)$ is a symmetric, positive semidefinite, singular M-matrix. Hence the eigenvalues of $L(G)$ are nonnegative and called Laplacian eigenvalues of G. The *line graph* H of a graph G is defined by $V(H) = E(G)$, where any two vertices in H are adjacent if and only if they are adjacent as edges of G. Furthermore, let R be the $n \times m$ incidence matrix of vertices and edges of the graph G, $A(H)$ the adjacency matrix of the line graph H of G. As is well known (refer to [3], p. 61), the following relations hold:

$$RR^T = D(G) + A(G), \tag{1}$$

$$R^T R = 2I + A(H). \tag{2}$$

Note that RR^T and $R^T R$ share the same nonzero spectrum.

Some preliminary results concerning the regularity of the graph and the line graph are needed in the main section.

Lemma 1 ([12])**.** *Let G be a simple connected graph. Then the line graph H of G is regular if and only if G is regular or semiregular.*

Lemma 2 (See e.g. [3, Theorem 3.8])**.** *Let $\bar{d}$ be the mean value of the degrees, i.e., $\bar{d} = \sum_{u \in V} d_u$, and $\lambda_{\max}$ the largest eigenvalue of the adjacency matrix $A(G)$ of a graph G. Then*

$$\bar{d} \leq \lambda_{\max},$$

where equality holds if and only if G is regular.

A good deal of attention has been devoted to the largest Laplacian eigenvalue of graphs and to other spectral properties of Laplacian matrices. Recent results can be referred to [4] and the references therein, and for an extensive survey on the Laplacian matrix see [9]. Recently Laplacian eigenvalues have been generalized from simple graphs to signed graphs, mixed graphs and weighted graphs. The aim of this paper is to establish a connection between the Laplacian eigenvalues of a signed graph and those of a mixed graph, give a new upper bound for the largest Laplacian eigenvalue of a signed graph, and characterize the extremal graph whose largest Laplacian eigenvalue achieved the upper bound.

2. Signed Graphs and Mixed Graphs

Signed graphs were introduced by Harary [5] in connection with the study of theory of social balance. Let us recall some notations of signed graphs. *A signed graph* $\Gamma = (G, \sigma)$ consists of a simple graph $G = (V, E)$ and a mapping $\sigma : E \to \{+, -\}$, the edge labelling, where G is called the underlying graph of Γ. We may write $V(\Gamma)$ for the vertex set and $E(\Gamma)$ for the edge set if necessary. Denote N_u the set of vertices of Γ adjacent to the vertex u, and N_u^+ (N_u^-) the set of vertices of Γ adjacent to the vertex u by positive (negative) edges. The *degree* (*positive degree*, *negative degree*) d_u (d_u^+, d_u^-) of a vertex u is the cardinality of N_u (N_u^+, N_u^-). Obviously $d_u = d_u^+ + d_u^-$ for any u in $V(\Gamma)$. Then a signed graph $\Gamma = (G, \sigma)$ and its underlying graph G have the same degree sequence. The *average* 2*-degree* m_u of a vertex u is the average of the degrees of the vertices of Γ adjacent to u, that is, $m_u = \frac{\sum_{v \in N_u} d_v}{d_u}$.

Given a signed graph $\Gamma = (G, \sigma)$ on n vertices, the diagonal matrix of degrees and the *adjacency matrix* of Γ are denoted by $D(\Gamma)$ and $A(\Gamma) = (\sigma(uv)a_{uv})$, respectively. The *Laplacian matrix* of Γ, denoted by $L(\Gamma) = (l_{uv})$, is defined as

$$l_{uv} = \begin{cases} 1 & \text{if } uv \in E(\Gamma) \text{ and } \sigma(uv) = -, \\ -1 & \text{if } uv \in E(\Gamma) \text{ and } \sigma(uv) = +, \\ d_u & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

It is easy to see that $L(\Gamma) = D(\Gamma) - A(\Gamma)$ which is a symmetric matrix whose row sum vector is $2(d_{v_1}^-, d_{v_2}^-, \ldots, d_{v_n}^-)^T$. Similar to simple graph, we may define the Laplacian matrix of a signed graph by means of the incidence matrix of Γ. For each edge of $e = uv$ of G, we choose one of u or v to be the head of e and the other to be the tail. We call this an orientation of Γ.

The *vertex-edge incidence matrix* $C = C(\Gamma)$ afforded by a fixed orientation of Γ is the $n \times m$ matrix $C = (c_{ve})$, where

$$c_{ve} = \begin{cases} 1 & \text{if } v \text{ is the head of } e, \\ -1 & \text{if } v \text{ is the tail of } e \text{ and } \sigma(e) = +, \\ 1 & \text{if } v \text{ is the tail of } e \text{ and } \sigma(e) = -, \\ 0 & \text{otherwise.} \end{cases}$$

While C depends on the orientation of Γ, CC^T does not, and it is easy to verify that $CC^T = L(\Gamma) = D(\Gamma) - A(\Gamma)$. So $L(\Gamma)$ is positive semidefinite, all eigenvalues of $L(\Gamma)$ are nonnegative. From the above notions, it follows that $L(G) = L(G, +)$, and $D(G) + A(G) = L(G, -)$, where $(G, +)$ and $(G, -)$ are the graphs with all positive and all negative edge labellings, respectively. The eigenvalues of $L(\Gamma)$ are called Laplacian eigenvalues of Γ. Let $\lambda_{n-1}(\Gamma) \geq \lambda_{n-2}(\Gamma) \geq \cdots \geq \lambda_0(\Gamma)$ denote the Laplacian eigenvalues of the signed graph $\Gamma = (G, \sigma)$ with n vertices, and let $\lambda_{n-1}(\Gamma)$, $\lambda_{n-1}(\sigma)$ or λ_{n-1} when no confusion appears, denote the spectral radius, i.e., the largest Laplacian eigenvalue, of the signed graph $\Gamma = (G, \sigma)$. In general, for an arbitrary symmetric matrix $A \in C^{n \times n}$, we use $\lambda_{\max}(A)$ to denote the largest eigenvalue of A.

Let C be a cycle of a signed graph $\Gamma = (G, \sigma)$, the sign of C is denoted by $\mathrm{sgn}(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is $+$ (respectively, $-$) is called *positive* (respectively, *negative*). A signed graph is called *balanced* if all its cycles are positive. Suppose $\theta : V \to \{+, -\}$ is any sign function that maps each vertex in V to a sign. Switching Γ by θ means forming a new signed graph $\Gamma^\theta = (G, \sigma^\theta)$ whose underlying graph is the same as G, but whose sign function is defined on an edge $e = uv$ by $\sigma^\theta(e) = \theta(u)\sigma(e)\theta(v)$. Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graphs with the same underlying graph. Γ_1 and Γ_2 are called *switching equivalent*, written $\Gamma_1 \sim \Gamma_2$, if there exists a switching function θ such that $\Gamma_2 = \Gamma_1^\theta$. Switching leaves the many signed-graphic characteristics invariant, such as the set of positive cycles.

The following result essentially comes from [7]:

Lemma 3. *Let $\Gamma = (G, \sigma)$ be a signed graph. Then the following conditions are equivalent:*

(1) $\Gamma = (G, \sigma) \sim (G, -)$.

(2) *There exists a partition $V(\Gamma) = V_1 \cup V_2$ such that every edge between V_1 and V_2 is positive and every edge within V_1 or V_2 is negative.*

In 1999, R.B. Bapat *et al.* [1] provided a combinatorial description of an arbitrary minor of the Laplacian matrix of a mixed graph and obtained a generalized matrix tree theorem. Zhang and Li [13] discussed the Laplacian spectrum of a mixed graph. A *mixed graph* $\vec{G} = (V, \vec{E})$ is obtained from a simple graph $G = (V, E)$ through assigning some of the edges in E a direction and the resultant edges are called *oriented edges* (denoted by $u \to v$) while others remain unchanged and called *unoriented edges* (denoted by uv). G is called the underlying graph of $\vec{G}$. For an oriented edge $u \to v$ in $\vec{E}$, u and v are called the *positive* and *negative end*, respectively. We denote by d_u, d_u^+ and d_u^- the number of unoriented edges incident to a vertex u, oriented edges incident to the positive end u and the negative end u, respectively. Then $c_u = d_u + d_u^+ + d_u^-$ is the degree of a vertex u. Denote the *adjacency matrix* of G by $A(\vec{G}) = (a_{uv})$, where $a_{uv} = 1$ if uv is an unoriented edge; $a_{uv} = -1$ if $u \to v$ or $v \to u$; $a_{uv} = 0$ otherwise. The *incidence matrix* of $\vec{G}$ is defined to be $M(\vec{G}) = (m_{u,e})$, where $m_{u,e} = 1$, if u is incident to an unoriented edge or u is the positive end of an oriented edge e; $m_{u,e} = -1$, if u is the negative end of an oriented edge e; $m_{u,e} = 0$ otherwise. Then $L(G) = M(\vec{G})M(\vec{G})^T$ is the *Laplacian matrix* of $\vec{G}$. By a simple verification, we have $L(\vec{G}) = C(\vec{G}) + A(\vec{G})$, where $C(\vec{G})$ is the degree diagonal matrix of $\vec{G}$. That is, $L(\vec{G}) = (l_{uv})$, where

$$(4) \quad l_{uv} = \begin{cases} 1 & \text{if } u \neq v \text{ and } u \text{ and } v \text{ are joined by an unoriented edge,} \\ -1 & \text{if } u \neq v \text{ and } u \text{ and } v \text{ are joined by an oriented edge,} \\ c_u & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1. *Let $\Gamma = (G, \sigma)$ be a signed graph with the underlying graph G. Then there are 2^{m_+} mixed graphs corresponding to Γ satisfying $L(\vec{G}) = L(\Gamma)$, without the consideration of isomorphism, where m_+ is the number of positive edges in Γ.*

Proof. Let $\Gamma = (G, \sigma)$ be a signed graph. Construct a mixed graph $\vec{G}$ from G as follows: For each edge $e \in E(G)$ such that $\sigma(e) = +$, orient it in one of two possible ways; remain unchanged otherwise. Thus the oriented edges in $E(\vec{G})$ correspond to the positive edges in $E(\Gamma)$; the unoriented edges in $E(\vec{G})$ correspond to the negative edges in $E(\Gamma)$.

By equations (3) and (4) we know that when creating the respective Laplacian matrices in this way, there are 2^{m_+} mixed graphs $\vec{G}$ corresponding to Γ where $L(\vec{G}) = L(\Gamma)$. ■

A mixed graph is called *quasi-bipartite* if it does not contain a nonsingular cycle, that is, a cycle containing an odd number of unoriented edges. For a mixed graph $\vec{G} = (V, \vec{E})$, if there exists a partition $V = V_1 \cup V_2$ such that every edge between V_1 and V_2 is oriented and every edge within V_1 or V_2 is unoriented, it is called *pre-bipartite* .

Proposition 2. *Let $\vec{G} = (V, \vec{E})$ be a mixed graph with underlying graph G and $\Gamma = (G, \sigma)$ be the corresponding signed graph. Then $\vec{G}$ is quasi-bipartite (respectively, pre-bipartite) if and only if Γ is balanced (respectively, $(G, \sigma) \sim (G, -)$).*

Proof. By the proof of Proposition 1, the oriented edges in $E(\vec{G})$ correspond to the positive edges in $E(\Gamma)$; the unoriented edges in $E(\vec{G})$ correspond to the negative edges in $E(\Gamma)$, so nonsingular cycle in mixed graph corresponds to negative cycle in signed graph. So $\vec{G}$ is quasi-bipartite if and only if Γ is balanced.

In the same way, by Lemma 3, $\vec{G}$ is pre-bipartite if and only if $(G, \sigma) \sim (G, -)$. ■

The following are the known upper bounds for the largest Laplacian eigenvalue of signed graph and mixed graph:

(1) Hou *et al.'s* bound [7]:

(5) $\lambda_{n-1}(\sigma) \le \max\{d_u + d_v : uv \in E(\Gamma)\}$,

(6) $\lambda_{n-1}(\sigma) \le \max\{d_u + m_u : u \in V(\Gamma)\}$,

(7) $\lambda_{n-1}(\sigma) \le \max\{(d_u(d_u + m_u) + d_v(d_v + m_v))/(d_u + d_v) : uv \in E(\Gamma)\}$.

(2) Zhang and Li's bound on mixed graph [13]:

$$\lambda_{n-1}(L(\vec{G})) \le \max\{2 + \sqrt{(c_u + c_v - 2)(c_v + c_w - 2)}\}, \tag{8}$$

where the maximum is taken over all pairs $(u, v), (v, w) \in \vec{E}$ with $u \ne w$.

(3) Zhang's two bounds which also holds for mixed graph [12]:

(9) $\lambda_{n-1}(G) \le \max\{2 + \sqrt{d_i(d_i + m_i - 4) + d_j(d_j + m_j - 4) + 4} : v_i v_j \in E\}$,

(10) $\lambda_{n-1}(G) \le \max\{d_i + \sqrt{d_i m_i} : v_i \in V(G)\}$.

Wang [10] gave a new upper bound on the Laplacian spectral radius of simple graphs. In next section we show that the result still holds for the case of signed graphs. For this purpose, we list the following lemmas used in the sequel.

The directed graph of $A = (a_{ij}) \in C^{n\times n}$, denoted by $G(A)$, is the directed graph on n vertices $v_1, v_2, \ldots, v_n$. An oriented edge e_{ij} in $G(A)$ from v_i to v_j if and only if $i \neq j$ and $a_{ij} \neq 0$. Denote by $E(A)$ the set of edges in $G(A)$. The definitions above are from [8].

Lemma 4 ([8]). *Let $A = (a_{ij})$ be an irreducible complex matrix of order n, let $R_i' = \sum_{j\neq i} |a_{ij}|$, $\Omega_{ij} = \{z \in C : |z - a_{ii}||z - a_{jj}| \leq R_i'R_j', e_{ij} \in E(A)\}$. Then all the eigenvalues of A are contained in the region as follows:*

$$\Omega(A) = \bigcup_{e_{ij}\in E(A)} \Omega_{ij}\,. \tag{11}$$

Furthermore, a boundary point λ of (11) *can be an eigenvalue of A only if λ is located on the boundary of each oval region Ω_{ij} in* (11).

Lemma 5 ([8]). *If A is irreducible and λ is an eigenvalue of A satisfying*

$$|\lambda - a_{ii}||\lambda - a_{jj}| \geq R_i'R_j', \forall e_{ij} \in E(A),$$

then

$$|\lambda - a_{ii}||\lambda - a_{jj}| = R_i'R_j', \forall e_{ij} \in E(A).$$

And if there exists at least one odd cycle in $G(A)$ additionally, then

(1) *λ is located on each circle,*

$$|\lambda - a_{ii}| = R_i',\ i = 1, 2, \ldots, n,$$

(2) *for the eigenvector x corresponding to λ,*

$$|x_1| = |x_2| = \cdots = |x_n|.$$

Lemma 6 ([7]). *Let $\Gamma = (G, \sigma)$ be a connected signed graph of order n, then*

$$\lambda_{n-1}(\sigma) \leq \lambda_{n-1}(-)$$

equality holds if and only if $(G, \sigma) \sim (G, -)$.

Lemma 7 ([7])**.** *Let $\Gamma = (G, \sigma)$ be a signed graph, then*

$$\lambda_{n-1}(\Gamma) \geq \max\{d_v + 1, v \in V(\Gamma)\}.$$

Lemma 8 ([6])**.** *Let $A \in C^{n\times n}$ and suppose that A is irreducible and non-negative, $\rho(A) = \max\{|\lambda| : \lambda \in \varepsilon(A)\}$, where $\varepsilon(A)$ denote the spectrum of A. Then*

(1) $\rho(A) > 0$;

(2) $\rho(A)$ *is an eigenvalue of A;*

(3) *There is a positive vector x such that $Ax = \rho(A)x$; and*

(4) $\rho(A)$ *is an algebraically (and hence geometrically) simple eigenvalue of A.*

Corollary 1. *Suppose that A is an irreducible and non-negative matrix of order n and there exists a real vector $x \neq 0$ such that $Ax = \rho(A)x$. Then this vector x is either positive, or negative.*

Proof. It follows immediately from the theorem above. ■

3. Upper Bound on the Largest Laplacian Eigenvalue

Theorem 1. *Let $\Gamma = (G, \sigma)$ be a connected signed graph of order n. Then,*

$$\lambda_{n-1}(\sigma) \leq \leq \max\left\{\frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4\sqrt{d_u m_u d_v m_v}}}{2} : uv \in E(\Gamma)\right\}, \tag{12}$$

where equality holds if and only if $(G, \sigma) \sim (G, -)$ and G is regular or G is semiregular.

Proof. Since Γ is connected without isolated vertices, $D^{1/2}$ is invertible. Now consider the similar matrix of $L(G, -)$, $D^{-1/2}L(G, -)D^{1/2}$, denoted by $M = (m_{uv})$, where

$$m_{uv} = \begin{cases} d_v & \text{if } u = v, \\ \frac{\sqrt{d_v}}{\sqrt{d_u}} & \text{if } uv \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Applying Lemma 4 to the spectral radius $\lambda_{n-1}(M)$, we know there exist some oval region Ω_{uv} with $\lambda_{n-1}(M) \in \Omega_{uv}$, that is,

$$|\lambda_{n-1}(M) - d_u||\lambda_{n-1}(M) - d_v| \le R_u(M)' R_v(M)'.$$

By Lemma 7, we know that both $\lambda_{n-1}(M) - d_u$ and $\lambda_{n-1}(M) - d_v$ are positive, so

$$(\lambda_{n-1}(M) - d_u)(\lambda_{n-1}(M) - d_v) \le R_u(M)' R_v(M)' = \sum_{w \in N_u} \frac{\sqrt{d_w}}{\sqrt{d_u}} \sum_{w \in N_v} \frac{\sqrt{d_w}}{\sqrt{d_v}}\,.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{w \in N_u} \frac{\sqrt{d_w}}{\sqrt{d_u}} \right)^2 \le \left(\sum_{w \in N_u} 1^2 \right) \left(\sum_{w \in N_u} \left(\frac{\sqrt{d_w}}{\sqrt{d_u}} \right)^2 \right) = d_u m_u\,.$$

That is,

$$R_u(M)' = \sum_{w \in N_u} \frac{\sqrt{d_w}}{\sqrt{d_u}} \le \sqrt{d_u m_u}\,. \tag{14}$$

Similarly,

$$R_v(M)' = \sum_{w \in N_v} \frac{\sqrt{d_w}}{\sqrt{d_v}} \le \sqrt{d_v m_v}\,. \tag{15}$$

So

$$(\lambda_{n-1}(M) - d_u)(\lambda_{n-1}(M) - d_v) \le \sqrt{d_u m_u}\sqrt{d_v m_v}\,.$$

Solving the above inequality, we have

$$\lambda_{n-1}(M) \le \frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4\sqrt{d_u m_u d_v m_v}}}{2}\,.$$

By Lemma 6, we know

$$\lambda_{n-1}(\sigma) \le \lambda_{n-1}(-) = \lambda_{n-1}(M)\,.$$

Consequently, the result in (12) holds.

Now suppose the equality in (12) holds , then all the inequalities above must be equalities. First, from $\lambda_{n-1}(\sigma) = \lambda_{n-1}(-)$, we have $(G, \sigma) \sim (G, -)$ by Lemma 6. Recall that $\lambda_{n-1}(-) = \lambda_{n-1}(M)$.

From

$$\lambda_{n-1}(M) = \max\left\{\frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4\sqrt{d_u m_u d_v m_v}}}{2} : uv \in E(\Gamma)\right\}$$

it follows that for any $st \in E$, we have

$$(\lambda_{n-1}(M) - d_s)(\lambda_{n-1}(M) - d_t) \geq \sqrt{d_s m_s}\sqrt{d_t m_t}.$$

Otherwise, suppose that for some $st \in E$,

$$(\lambda_{n-1}(M) - d_s)(\lambda_{n-1}(M) - d_t) < \sqrt{d_s m_s}\sqrt{d_t m_t}.$$

Solving the equation above, we obtain

$$\lambda_{n-1}(M) < \frac{d_s + d_t + \sqrt{(d_s - d_t)^2 + 4\sqrt{d_s m_s d_t m_t}}}{2},$$

which contradicts the hypothesis that the equality in (12) holds.

Therefore for any $st \in E$, we have

$$(\lambda_{n-1}(M) - d_s)(\lambda_{n-1}(M) - d_t) \geq \sqrt{d_s m_s}\sqrt{d_t m_t}. \tag{16}$$

Similarly as stated in equations (14) and (15), we know $R'_s R'_t \leq \sqrt{d_s m_s}\sqrt{d_t m_t}$ (note that R'_u means $R'_u(M)$ for convenience). Combining this inequality with equation (16), we get

$$(\lambda_{n-1}(M) - d_s)(\lambda_{n-1}(M) - d_t) \geq R'_s R'_t. \tag{17}$$

By Lemma 5, $\lambda_{n-1}(M)$ is the boundary point for every oval regions and so

$$(\lambda_{n-1}(M) - d_s)(\lambda_{n-1}(M) - d_t) = R'_s R'_t = \sqrt{d_s m_s}\sqrt{d_t m_t}.$$

Hence for any u in $V(\Gamma)$, $R'_u = \sqrt{d_u m_u}$. By the Cauchy-Schwarz inequality, for every $w \in N_u$, $\sqrt{d_w}/\sqrt{d_u} = c_u$, where c_u is the constant determined by the vertex u. Especially, for any $uv \in E(\Gamma)$,

$$\sqrt{d_u}/\sqrt{d_v} = c_v, \quad \sqrt{d_v}/\sqrt{d_u} = c_u, \tag{18}$$

so $c_v c_u = 1$ for any $uv \in E(\Gamma)$.

Now we discuss it in two cases:

Case 1. There is at least one odd cycle in Γ. By the (1) of Lemma 5, we know for every $v \in V$, $|\lambda_{n-1}(M) - d_v| = R'_v$, that is, $\lambda_{n-1}(M) - d_v = R'_v$ by Lemma 7. So for any choosen $v \in V$, $\lambda_{n-1}(M) = d_v + R'_v = d_v + \sum_{wv\in E} \frac{\sqrt{d_w}}{\sqrt{d_v}} = d_v(c_v+1)$. Suppose the vector x is the eigenvector associated with $\lambda_{n-1}(-)$, that is,

$$(D(G) + A(G))x = \lambda_{n-1}(-)x = \lambda_{n-1}(M)x = d_v(c_v + 1)x,$$

for any choosen $v \in V$.

For any edge $wu \in E$, we have

$$\begin{aligned}|\lambda_{n-1}(-) - d_w||\lambda_{n-1}(-) - d_u| &= |d_w(c_w + 1) - d_w||d_u(c_u + 1) - d_u| \\ &= d_w c_w d_u c_u \\ &= d_w d_u.\end{aligned}$$

It is easy to calculate that $R_w(L(G,-))' R_u(L(G,-))' = d_w d_u$. Therefore for any edge $wu \in E$,

$$|\lambda_{n-1}(-) - d_w||\lambda_{n-1}(-) - d_u| = R_w(L(G,-))' R_u(L(G,-))'.$$

By the case (2) of Lemma 5 and the hypothesis that there is one odd cycle in Γ, we know that for the eigenvector x associated with $\lambda_{n-1}(-)$,

$$|x_1| = |x_2| = \cdots = |x_n|.$$

By Corollary 1, there must be that $x = ae$ for some real constant $a \neq 0$, where $e = (1, 1, \ldots, 1)^T$. Hence $(D(G) + A(G))e = \lambda_{n-1}(-)e$. That is for any vertex $u \in V$,

$$2d_u = d_u(c_u + 1).$$

So $c_u = 1$ for any vertex $u \in V$. Therefore G is regular.

Case 2. There exists no odd cycle in Γ. In this case, Γ is bipartite. Let $V = S \cup T$ be a bipartition of $V(\Gamma)$. For arbitrary two vertices u, v in S, there is a path of even length between u and v. Without loss of generality, that is $v_1(= u)v_2 \cdots v_{2m+1}(= v)$, where $v_{2i-1} \in S, v_{2i} \in T$, $i = 1, \ldots, m$. Since $u, v_3 \in N_{v_2}$, $\frac{d(u)}{d(v_2)} = c_{v_2}^2 = \frac{d(v_3)}{d(v_2)}$ by equation (18) and so $d_u = d_{v_3}$. In the same way, $d_{v_3} = d_{v_5}, \ldots, d_{v_{2m-1}} = d_{v_{2m+1}}$. Finally we come to that $d_u = d_v$. That is, all vertices have the equal degree in S. By the same procedure as above, we can show that all vertices have the same degree in T, too. Therefore G is semiregular.

Conversely, if $(G, \sigma) \sim (G, -)$ and G is regular of degree r or G is semiregular of degrees r_1, r_2, by Lemma 1, we know that the line graph H of G is regular of degree $2r - 2$ or $r_1 + r_2 - 2$. Since $(G, \sigma) \sim (G, -)$, $\lambda_{n-1}(\sigma) = \lambda_{n-1}(-)$. Applying equations (1) and (2), we obtain

$$\lambda_{n-1}(D(G) + A(G)) = \lambda_{\max}(RR^T) = \lambda_{\max}(R^T R) = \lambda_{\max}(2I + A(H)).$$

Since the line graph H is regular of degree $2r-2$ or r_1+r_2-2, by Lemma 2, the largest eigenvalue of the adjacency matrix $A(H)$ is $2r - 2$ or $r_1 + r_2 - 2$ and so $\lambda_{\max}(2I + A(H)) = 2r$ or $r_1 + r_2$, respectively according to that G is regular of degree r or G is semiregular of degrees r_1, r_2, respectively. It is easy to verify that when G is regular of degree r or G is semiregular of degrees r_1, r_2,

$$\max\left\{\frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4\sqrt{d_u m_u d_v m_v}}}{2} : uv \in E(\Gamma)\right\} = 2r \text{ or } r_1 + r_2.$$

Therefore the equality in (12) holds. ■

By Proposition 2 and the Theorem above, we have the following result.

Corollary 2. *Let $\vec{G} = (V, \vec{E})$ be a connected mixed graph of order n. Then*

$$(19)\quad \lambda_{n-1}(\vec{G}) \leq \max\left\{\frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4\sqrt{d_u m_u d_v m_v}}}{2} : uv \in \vec{E}\right\},$$

equality holds if and only if $\vec{G}$ is pre-bipartite and G is regular or semiregular.

Example 1. Let Γ be the tree as the following

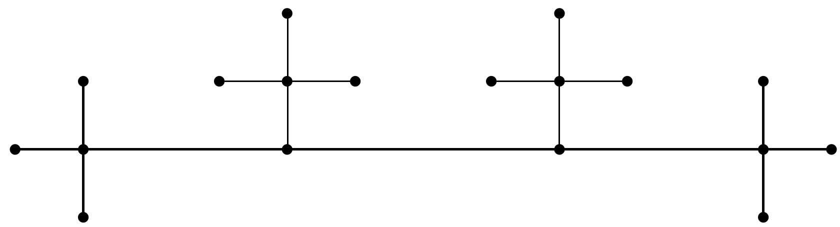

Figure 1. Signed graph Γ. Thick line: $-$, thin line: $+$.

The all known better upper bounds for the largest Laplacian eigenvalue of the signed graph are listed as follows:

$\lambda_1(\Gamma)$	(7)	(8)	(9)	(10)	(12)
5.764	6.667	7	6.472	6.449	6.394

Hence bound (12) is the best in all known upper bounds for Γ.

Remark 1. Obviously, (8) is always better than (5), and (7) is better than (6). It is easy to see that (12) is better than (10). Obviously the bound (12) is the maximum root of the following equation

$$(z - d_u)(z - d_v) = \sqrt{d_u m_u}\sqrt{d_v m_v},$$

for some edge $uv \in E$.

Let ρ denote (10). We have

$$\sqrt{d_u m_u} \leq \rho - d_u, \qquad \sqrt{d_v m_v} \leq \rho - d_v\,.$$

Consequently

$$\sqrt{d_u m_u}\sqrt{d_v m_v} \leq (\rho - d_u)(\rho - d_v).$$

From the above inequality, we have

$$\rho \geq \frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4\sqrt{d_u m_u d_v m_v}}}{2}$$

Thus (12) is better than (10).

4. Appendix

Cauchy-Schwarz inequality

$$\Big(\sum_i a_i b_i\Big)^2 \leq \Big(\sum_i a_i^2\Big)\Big(\sum_i b_i^2\Big),$$

with equality when b_i/a_i is a constant.

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