# A NOTE ON DOMINATION PARAMETERS IN RANDOM GRAPHS 

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#### Abstract

Domination parameters in random graphs $G(n, p)$, where $p$ is a fixed real number in $(0,1)$, are investigated. We show that with probability tending to 1 as $n \rightarrow \infty$, the total and independent domination numbers concentrate on the domination number of $G(n, p)$.


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## 1. Introduction

Domination is a central topic in graph theory, with a number of applications in computer science and engineering. A set $S$ of vertices in a graph $G$ is a dominating set of $G$ if each vertex not in $S$ is joined to some vertex of $S$.

[^0]The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. The concentration of the domination number of random graphs $G(n, p)$ was investigated in [8]. Other contributions to domination in random graph theory include $[2,6,7]$. For background on random graphs and domination, the reader is directed to $[1,5]$ and $[3,4]$, respectively. We say that an event holds asymptotically almost surely (a.a.s.) if the probability that it holds tends to 1 as $n$ tends to infinity. All logarithms are in base $e$ unless otherwise stated, and we use the notation $\mathbb{L} n=\log _{1 /(1-p)} n$.

Theorem 1 ([8]). For $p \in(0,1)$ fixed, a.a.s. $\gamma(G(n, p))$ equals

$$
\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+1 \text { or }\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2 \text {. }
$$

Despite the fact that deterministic graphs of order $n$ may have domination number equalling $\Theta(n)$ (such as a path $P_{n}$ with $\gamma\left(P_{n}\right)=\lceil n / 3\rceil$ ), Theorem 1 demonstrates that a.a.s. $G(n, p)$ has domination number equalling $(1+o(1)) \mathbb{L} n=\Theta(\log n)$.

A set $S$ is said to be an independent dominating set of $G$ if $S$ is both an independent set and a dominating set of $G$ (that is, $S$ is a maximal independent set). A total dominating set $S$ in a graph $G$ is a subset of $V(G)$ satisfying that every $v \in V(G)$ is joined to at least one vertex in $S$. The independent domination number of $G$, written $\gamma_{i}(G)$, is the minimum order of an independent dominating set of $G$; the total domination number, written $\gamma_{t}(G)$, is defined analogously. It is straightforward to see that $\gamma(G) \leq \gamma_{i}(G)$ and $\gamma(G) \leq \gamma_{t}(G)$. However, the domination number may be of much smaller order than either the independent or total domination numbers; see for example, [3, 4]. As proved in [9], there are cubic graphs where the difference between $\gamma_{i}$ and $\gamma$ is $\Theta(n)$.

Our goal in this note is to demonstrate that in $G(n, p)$ with $p$ fixed, asymptotically the independent and total domination numbers concentrate on $(1+o(1)) \mathbb{L} n$. In particular, we prove the following theorems.

Theorem 2. A.a.s. $\gamma_{t}(G(n, p))$ equals

$$
\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+1 \text { or }\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2 \text {. }
$$

Theorem 3. A.a.s. we have that

$$
\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+1 \leq \gamma_{i}(G(n, p)) \leq\lfloor\mathbb{L} n\rfloor .
$$

As the proofs of the theorems are technical-though elementary-we present them in the next section. For both proofs, we compute the asymptotic expected value of each domination parameter, and then analyze its variance. The second moment method (see Chapter 4 of [1], for example) completes the proofs.

All graphs we consider are finite, undirected, and simple. If $A$ is an event in a probability space, then we write $\mathbb{P}(A)$ for the probability of $A$ in the space. We use the notation $\mathbb{E}(X)$ and $\operatorname{Var}(X)$ for the expected value and variance of a random variable $X$ on $G(n, p)$, respectively. Throughout, $n$ is a positive integer, all asymptotics are as $n \rightarrow \infty$, and $p \in(0,1)$ is a fixed real number.

## 2. Proofs of Theorems 2 and 3

The proofs are presented in the following two subsections. We note the following facts from [8]. For $r \geq 1$, let $X_{r}$ be the number of dominating sets of size $r$. Fix an $r$-set $S_{1}$. Denote by $S(j)$ the set of $r$-sets which intersect $S_{1}$ in $j$ elements. Let $I_{1}$ and $I^{j}$ be indicator random variables, where the events $I_{1}=1$ and $I^{j}=1$ represent that $S_{1}$ and $S_{j} \in S(j)$ are dominating sets, respectively. Let

$$
A=\binom{n}{r} \sum_{j=0}^{r-1}\binom{r}{j}\binom{n-r}{r-j} \mathbb{E}\left(I_{1} I^{j}\right)
$$

Lemma 4 ([8]). The random variable $X_{r}$ satisfies the following properties.
(1) $\mathbb{E}\left(X_{r}\right)=\binom{n}{r}\left(1-(1-p)^{r}\right)^{n-r}$.
(2) For $r \geq\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2$, we have that $\mathbb{E}\left(X_{r}\right) \rightarrow \infty$ as $n \rightarrow \infty$.
(3) For $r \geq\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2$,

$$
A \leq \mathbb{E}^{2}\left(X_{r}\right)\left(1+2 r(1-p)^{r}-\frac{r^{2}}{n}\right)(1+o(1))+r g(1)\binom{n}{r},
$$

where

$$
\begin{equation*}
g(1)=\frac{2 r n^{r-1}}{(r-1)!} \exp \left(n(1-p)^{2 r-1}-2(1-p)^{r}\right) \tag{2.1}
\end{equation*}
$$

### 2.1. Proof of Theorem 2

For $r$ a positive integer, the random variable $X_{r}^{t}$ denotes the number of total dominating sets of size $r$. By Chebyshev's inequality, the proof of the theorem will follow once we show that $\mathbb{E}\left(X_{r}^{t}\right) \rightarrow \infty$ as $n \rightarrow \infty$, and $\operatorname{Var}\left(X_{r}^{t}\right)=o\left(\mathbb{E}^{2}\left(X_{r}^{t}\right)\right)$. (See, for example, Section 4.3 of [1].)

Lemma 5. If $r=\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2$, then

$$
\mathbb{E}\left(X_{r}^{t}\right)=\binom{n}{r}\left(1-(1-p)^{r}\right)^{n-r}\left(1-(1-p)^{r-1}\right)^{r}(1+o(1))
$$

Proof. For $1 \leq j \leq\binom{ n}{r}$, denote by $E_{j}$ the event that the subgraph induced by a given $r$-set $S_{j}$ has no isolated vertices. We have that

$$
\mathbb{E}\left(X_{r}^{t}\right)=\binom{n}{r}\left(1-(1-p)^{r}\right)^{n-r} \mathbb{P}\left(E_{j}\right)
$$

It is not hard to show that for all $j, \mathbb{P}\left(E_{j}\right) \geq 1-r(1-p)^{r-1}$, and so $\lim _{n \rightarrow \infty} \mathbb{P}\left(E_{j}\right)=1$. The proof follows since for $r=\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2$,

$$
\lim _{n \rightarrow \infty}\left(1-(1-p)^{r-1}\right)^{r}=1
$$

We next show that for a certain value of $r$, the expected value of $X_{r}^{t}$ concentrates on the expected value of $X_{r}$.

Lemma 6. If $r=\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2$, then $\mathbb{E}\left(X_{r}^{t}\right)=(1+o(1)) \mathbb{E}\left(X_{r}\right)$.
Proof. By Lemmas 4 and 5, we have that

$$
\frac{\mathbb{E}\left(X_{r}^{t}\right)}{\mathbb{E}\left(X_{r}\right)}=\left(1-(1-p)^{r-1}\right)^{r}(1+o(1))
$$

Hence,

$$
\lim _{n \rightarrow \infty} \frac{\mathbb{E}\left(X_{r}^{t}\right)}{\mathbb{E}\left(X_{r}\right)}=\lim _{n \rightarrow \infty}\left(1-(1-p)^{r-1}\right)^{r}(1+o(1))=1
$$

By Lemmas 4 and 6, the proof of the following lemma is immediate.
Lemma 7. If $r=\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2$, then $\mathbb{E}\left(X_{r}^{t}\right) \rightarrow \infty$ as $n \rightarrow \infty$.

We now analyze the variance of the random variable $X_{r}^{t}$.
Lemma 8. If $r=\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2$, then $\operatorname{Var}\left(X_{r}^{t}\right)=o\left(\mathbb{E}^{2}\left(X_{r}^{t}\right)\right)$.
Proof. For $1 \leq j \leq\binom{ n}{r}$, let $I^{j}$ be the corresponding indicator random variables. Hence,

$$
X_{r}^{t}=\sum_{j=1}^{\binom{n}{r}} I^{j}
$$

By the linearity of expectation, we have that

$$
\begin{align*}
\mathbb{E}\left(\left(X_{r}^{t}\right)^{2}\right) & =\sum_{j=1}^{\binom{n}{r}} \mathbb{E}\left(\left(I^{j}\right)^{2}\right)+2 \sum_{j \neq i}^{\binom{n}{r}} \mathbb{E}\left(I^{i} I^{j}\right) \\
& =\mathbb{E}\left(X_{r}^{t}\right)+2 \sum_{j \neq i}^{\binom{n}{r}} \mathbb{E}\left(I^{i} I^{j}\right) . \tag{2.2}
\end{align*}
$$

We fix an $r$-set $S_{1}$. For $0 \leq j \leq r-1$, denote by $S(j)$ the set of $r$-sets which intersect $S_{1}$ in $j$ elements. Let $I_{1}^{t}$ and $I^{j t}$ be the indicator random variables, where the events $I_{1}^{t}=1$ and $I^{j t}=1$ represent that $S_{1}$ and $S_{j} \in S(j)$ are total dominating sets, respectively. Then

$$
2 \sum_{j \neq i}^{\binom{n}{r}} \mathbb{E}\left(I^{i} I^{j}\right)=\binom{n}{r} \sum_{j=0}^{r-1}\binom{r}{j}\binom{n-r}{r-j} \mathbb{E}\left(I_{1}^{t} I^{j t}\right) .
$$

Together with (2.2), we obtain that

$$
\begin{align*}
\mathbb{E}\left(\left(X_{r}^{t}\right)^{2}\right) & =\mathbb{E}\left(X_{r}^{t}\right)+\binom{n}{r} \sum_{j=0}^{r-1}\binom{r}{j}\binom{n-r}{r-j} \mathbb{E}\left(I_{1}^{t} j^{j t}\right)  \tag{2.3}\\
& =\mathbb{E}\left(X_{r}^{t}\right)+A^{t},
\end{align*}
$$

where $A^{t}=\binom{n}{r} \sum_{j=0}^{r-1}\binom{r}{j}\binom{n-r}{r-j} \mathbb{E}\left(I_{1}^{t} I^{j t}\right)$. As each total dominating set is a dominating set, $A^{t} \leq A$. By Lemmas 4 and 6 , we therefore have that

$$
\begin{equation*}
A^{t} \leq r g(1)\binom{n}{r}+(1+o(1)) \mathbb{E}^{2}\left(X_{r}^{t}\right)\left(1+2 r(1-p)^{r}-\frac{r^{2}}{n}\right) \tag{2.4}
\end{equation*}
$$

where $g(1)$ is given in (2.1).

By (2.3) and (2.4) we have that

$$
\begin{equation*}
\frac{\operatorname{Var}\left(X_{r}^{t}\right)}{\mathbb{E}^{2}\left(X_{r}^{t}\right)} \leq \frac{1}{\mathbb{E}\left(X_{r}^{t}\right)}+(1+o(1))\left(2 r(1-p)^{r}-\frac{r^{2}}{n}\right)+\frac{r g(1)\binom{n}{r}}{\mathbb{E}^{2}\left(X_{r}^{t}\right)} \tag{2.5}
\end{equation*}
$$

To show that $\operatorname{Var}\left(X_{r}^{t}\right)=o\left(\mathbb{E}^{2}\left(X_{r}^{t}\right)\right)$, it suffices by Lemma 7 to show that

$$
\frac{r g(1)\binom{n}{r}}{\mathbb{E}^{2}\left(X_{r}^{t}\right)}=o(1) .
$$

For sufficiently large $n$ we have that

$$
\begin{aligned}
\frac{r g(1)\binom{n}{r}}{\mathbb{E}^{2}\left(X_{r}^{t}\right)} & =\frac{r \times 2 r \frac{n^{r-1}}{(r-1)!} \exp \left(n\left((1-p)^{2 r-1}-2(1-p)^{r}\right)\right)}{\binom{n}{r}\left(\left(1-(1-p)^{r}\right)^{n-r}\left(1-(1-p)^{r-1}\right)^{r}\right)^{2}}(1+o(1)) \\
& \leq \frac{3 r^{3}}{n} \frac{\left(1-2(1-p)^{r}+(1-p)^{2 r-1}\right)^{n}}{\left(1-(1-p)^{r}\right)^{2 n-2 r}\left(1-(1-p)^{r-1}\right)^{2 r}}(1+o(1)) \\
& \leq \frac{3 r^{3}}{n}\left(1+\frac{p(1-p)^{2 r-1}}{\left(1-(1-p)^{\prime}\right)^{2}}\right)^{n-r}\left(1+\frac{2 p(1-p)^{r-1}}{\left(1-(1-p)^{r-1}\right)^{2}}\right)^{r}(1+o(1))
\end{aligned}
$$

where the first equality follows by (2.1) and since $\exp (x) \sim 1+x$ if $x$ is close to 0 .

Since $1+x \leq \exp (x)$ for $x \geq 0$, we obtain that

$$
\begin{aligned}
\frac{r g(1)\binom{n}{r}}{\mathbb{E}^{2}\left(X_{r}^{t}\right)} & \leq \frac{3 r^{3}}{n} \exp \left(\frac{(n-r) p(1-p)^{2 r-1}}{\left(1-(1-p)^{r}\right)^{2}}+\frac{2 r p(1-p)^{r-1}}{\left(1-(1-p)^{r-1}\right)^{2}}\right)(1+o(1)) \\
& \leq \frac{3 r^{3}}{n} \exp \left((1+o(1)) p \frac{(\mathbb{L} n)(\log n))^{2}}{n}\right)(1+o(1))=o(1)
\end{aligned}
$$

as $r=\lfloor\mathbb{L} n-\mathbb{L}((\mathbb{L} n)(\log n))\rfloor+2$.

### 2.2. Proof of Theorem 3

We use the following lemma, whose proof is straightforward and so is omitted. For $r \geq 1$, let $X_{r}^{I}$ be the random variable which denotes the number of independent dominating sets of size $r$.

Lemma 9. (1) For all $r \geq 1$

$$
\mathbb{E}\left(X_{r}^{I}\right)=\binom{n}{r}\left(1-(1-p)^{r}\right)^{n-r}(1-p)^{\binom{r}{2}}
$$

(2) Let $\lambda \in\left(\frac{1}{2}, 1\right)$ be fixed. For $\lfloor\mathbb{L} n\rfloor+1 \leq r \leq\lfloor 2 \lambda \mathbb{L} n\rfloor$, as $n \rightarrow \infty$ we have that $\mathbb{E}\left(X_{r}^{I}\right) \rightarrow \infty$.

Our final lemma estimates the variance of $X_{r}^{I}$.
Lemma 10. Let $p \in(0,1)$ and $\lambda \in\left(\frac{1}{2}, 1\right)$ be fixed. For $\lfloor\mathbb{L} n\rfloor+1 \leq r \leq$ $\lfloor 2 \lambda \mathbb{L} n\rfloor$,

$$
\operatorname{Var}\left(X_{r}^{I}\right)=\mathbb{E}^{2}\left(X_{r}^{I}\right) O\left(\frac{(\log n)^{4}}{n^{1-\lambda}}\right)
$$

By Chebyshev's inequality and Lemmas 9 and 10 , we have that

$$
\begin{aligned}
\mathbb{P}\left(\gamma_{i}(G)>r\right) & =\mathbb{P}\left(X_{r}^{I}=0\right) \leq \mathbb{P}\left(\left|X_{r}^{I}-\mathbb{E}\left(X_{r}^{I}\right)\right| \geq \mathbb{E}\left(X_{r}^{I}\right)\right) \\
& \leq \frac{\operatorname{Var}\left(X_{r}^{I}\right)}{\mathbb{E}^{2}\left(X_{r}^{I}\right)}=o(1) .
\end{aligned}
$$

The assertion of Theorem 3 follows, therefore, once we prove Lemma 10.
Proof of Lemma 10. We denote by $\mathbb{E}\left(\left(X_{r}^{I}\right)^{2}\right)$ the expectation of the number of ordered pairs of independent domination sets of size $r$ in $G \in$ $G(n, p)$. The expectation satisfies

$$
\begin{align*}
\mathbb{E}\left(\left(X_{r}^{I}\right)^{2}\right)= & \sum_{j}\binom{n}{r}\binom{r}{j}\binom{n-r}{r-j}\left(1-(1-p)^{r}\right)^{2(n-2 r+j)}  \tag{2.6}\\
& \left.\times\left(1-(1-p)^{r-j}\right)^{2(r-j)}(1-p)^{22^{r}} \begin{array}{l}
2 \\
2
\end{array}\right)\binom{j}{2}
\end{align*}
$$

The explanations for the terms in the equation (2.6) are as follows. The vertices of the first independent dominating set $S_{1}$ may be chosen in $\binom{n}{r}$ ways. The independent dominating sets $S_{1}$ and $S_{2}$ may have $j$ elements in common. These vertices may be chosen in $\binom{r}{j}$ ways. The rest of $r-j$ vertices of $S_{2}$ may have to be chosen from $V(G) \backslash S_{1}$, which gives the $\binom{n-r}{r-j}$ term. Every vertex not in $S_{1} \cup S_{2}$ must be joined to one of $S_{1}$ and one of $S_{2}$, and so we obtain the term $\left(1-(1-p)^{r}\right)^{2(n-2 r+j)}$. Every vertex in $S_{1} \backslash S_{2}$
must be joined to one of $S_{2} \backslash S_{1}$, and every vertex in $S_{2} \backslash S_{1}$ must be joined to one of $S_{1} \backslash S_{2}$, and so we have the term $\left(1-(1-p)^{r-j}\right)^{2(r-j)}$. Both sets $S_{1}$ and $S_{2}$ are independent, which supplies the last term.

Observe that $(1-p)^{r-j} \geq(1-p)^{r}$. Hence, by (2.6) and Lemma 9 (1), we have that

$$
\begin{align*}
\mathbb{E}\left(\left(X_{r}^{I}\right)^{2}\right) \leq & \mathbb{E}^{2}\left(X_{r}^{I}\right) \frac{1}{\binom{n}{r}}\left(\binom{n-r}{r}\right.  \tag{2.7}\\
& \left.+r\binom{n-r}{r-1}+\sum_{j=2}^{r}\binom{r}{j}\binom{n-r}{r-j}(1-p)^{-\binom{j}{2}}\right) .
\end{align*}
$$

By the choice of $r$ it follows that

$$
\begin{align*}
\frac{1}{\binom{n}{r}}\left(\binom{n-r}{r}+r\binom{n-r}{r-1}\right)= & \left(1-\frac{r^{2}}{n}\right)\left(1+O\left(\frac{(\log n)^{4}}{n^{2}}\right)\right) \\
& +\frac{r^{2}}{n}+O\left(\frac{(\log n)^{3}}{n^{2}}\right)  \tag{2.8}\\
= & 1+O\left(\frac{(\log n)^{4}}{n^{2}}\right)
\end{align*}
$$

and

$$
\begin{equation*}
\frac{1}{\binom{n}{r}} \sum_{j=2}^{r}\binom{r}{j}\binom{n-r}{r-j}(1-p)^{-\binom{j}{2}}=O\left(\frac{(\log n)^{4}}{n^{1-\lambda}}\right) . \tag{2.9}
\end{equation*}
$$

By (2.7), (2.8), and (2.9) the assertion follows.

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