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A NOTE ON DOMINATION PARAMETERS IN RANDOM GRAPHS

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Abstract

Domination parameters in random graphs $G(n,p)$, where p is a fixed real number in $(0,1)$, are investigated. We show that with probability tending to 1 as $n \to \infty$, the total and independent domination numbers concentrate on the domination number of $G(n,p)$.

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1. Introduction

Domination is a central topic in graph theory, with a number of applications in computer science and engineering. A set S of vertices in a graph G is a *dominating set* of G if each vertex not in S is joined to some vertex of S.

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The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. The concentration of the domination number of random graphs $G(n,p)$ was investigated in [8]. Other contributions to domination in random graph theory include [2, 6, 7]. For background on random graphs and domination, the reader is directed to [1, 5] and [3, 4], respectively. We say that an event holds *asymptotically almost surely* (*a.a.s.*) if the probability that it holds tends to 1 as n tends to infinity. All logarithms are in base e unless otherwise stated, and we use the notation $\mathbb{L}n = \log_{1/(1-p)} n$.

Theorem 1 ([8]). *For $p \in (0,1)$ fixed,* a.a.s. $\gamma(G(n,p))$ *equals*

$$\lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 1 \text{ or } \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2.$$

Despite the fact that deterministic graphs of order n may have domination number equalling $\Theta(n)$ (such as a path P_n with $\gamma(P_n) = \lceil n/3 \rceil$), Theorem 1 demonstrates that *a.a.s.* $G(n,p)$ has domination number equalling $(1+o(1))\mathbb{L}n = \Theta(\log n)$.

A set S is said to be an *independent dominating set* of G if S is both an independent set and a dominating set of G (that is, S is a maximal independent set). A *total dominating set* S in a graph G is a subset of $V(G)$ satisfying that every $v \in V(G)$ is joined to at least one vertex in S. The *independent domination number* of G, written $\gamma_i(G)$, is the minimum order of an independent dominating set of G; the total domination number, written $\gamma_t(G)$, is defined analogously. It is straightforward to see that $\gamma(G) \leq \gamma_i(G)$ and $\gamma(G) \leq \gamma_t(G)$. However, the domination number may be of much smaller order than either the independent or total domination numbers; see for example, [3, 4]. As proved in [9], there are cubic graphs where the difference between γ_i and γ is $\Theta(n)$.

Our goal in this note is to demonstrate that in $G(n,p)$ with p fixed, asymptotically the independent and total domination numbers concentrate on $(1+o(1))\mathbb{L}n$. In particular, we prove the following theorems.

Theorem 2. A.a.s. $\gamma_t(G(n,p))$ *equals*

$$\lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 1 \text{ or } \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2.$$

Theorem 3. A.a.s. *we have that*

$$\lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 1 \leq \gamma_i(G(n,p)) \leq \lfloor \mathbb{L}n \rfloor.$$

As the proofs of the theorems are technical—though elementary—we present them in the next section. For both proofs, we compute the asymptotic expected value of each domination parameter, and then analyze its variance. The second moment method (see Chapter 4 of [1], for example) completes the proofs.

All graphs we consider are finite, undirected, and simple. If A is an event in a probability space, then we write $\mathbb{P}(A)$ for the probability of A in the space. We use the notation $\mathbb{E}(X)$ and $Var(X)$ for the expected value and variance of a random variable X on $G(n,p)$, respectively. Throughout, n is a positive integer, all asymptotics are as $n \to \infty$, and $p \in (0,1)$ is a fixed real number.

2. Proofs of Theorems 2 and 3

The proofs are presented in the following two subsections. We note the following facts from [8]. For $r \geq 1$, let X_r be the number of dominating sets of size r. Fix an r-set S_1. Denote by $S(j)$ the set of r-sets which intersect S_1 in j elements. Let I_1 and I^j be indicator random variables, where the events $I_1 = 1$ and $I^j = 1$ represent that S_1 and $S_j \in S(j)$ are dominating sets, respectively. Let

$$A = \binom{n}{r} \sum_{j=0}^{r-1} \binom{r}{j} \binom{n-r}{r-j} \mathbb{E}(I_1 I^j).$$

Lemma 4 ([8])**.** *The random variable X_r satisfies the following properties.*

(1) $\mathbb{E}(X_r) = \binom{n}{r}(1-(1-p)^r)^{n-r}$.

(2) *For $r \geq \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$, we have that $\mathbb{E}(X_r) \to \infty$ as $n \to \infty$.*

(3) *For $r \geq \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$,*

$$A \leq \mathbb{E}^2(X_r)\left(1 + 2r(1-p)^r - \frac{r^2}{n}\right)(1+o(1)) + rg(1)\binom{n}{r},$$

where

$$g(1) = \frac{2rn^{r-1}}{(r-1)!} \exp\left(n(1-p)^{2r-1} - 2(1-p)^r\right). \tag{2.1}$$

2.1. Proof of Theorem 2

For r a positive integer, the random variable X_r^t denotes the number of total dominating sets of size r. By Chebyshev's inequality, the proof of the theorem will follow once we show that $\mathbb{E}(X_r^t) \to \infty$ as $n \to \infty$, and $Var(X_r^t) = o(\mathbb{E}^2(X_r^t))$. (See, for example, Section 4.3 of [1].)

Lemma 5. *If* $r = \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$, *then*

$$\mathbb{E}(X_r^t) = \binom{n}{r}(1-(1-p)^r)^{n-r}(1-(1-p)^{r-1})^r(1+o(1)).$$

Proof. For $1 \leq j \leq \binom{n}{r}$, denote by E_j the event that the subgraph induced by a given r-set S_j has no isolated vertices. We have that

$$\mathbb{E}(X_r^t) = \binom{n}{r}(1-(1-p)^r)^{n-r}\mathbb{P}(E_j).$$

It is not hard to show that for all j, $\mathbb{P}(E_j) \geq 1 - r(1-p)^{r-1}$, and so $\lim_{n\to\infty} \mathbb{P}(E_j) = 1$. The proof follows since for $r = \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$,

$$\lim_{n\to\infty}(1-(1-p)^{r-1})^r = 1.$$

■

We next show that for a certain value of r, the expected value of X_r^t concentrates on the expected value of X_r.

Lemma 6. *If* $r = \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$, *then* $\mathbb{E}(X_r^t) = (1+o(1))\mathbb{E}(X_r)$.

Proof. By Lemmas 4 and 5, we have that

$$\frac{\mathbb{E}(X_r^t)}{\mathbb{E}(X_r)} = (1-(1-p)^{r-1})^r(1+o(1)).$$

Hence,

$$\lim_{n\to\infty}\frac{\mathbb{E}(X_r^t)}{\mathbb{E}(X_r)} = \lim_{n\to\infty}(1-(1-p)^{r-1})^r(1+o(1)) = 1.$$

■

By Lemmas 4 and 6, the proof of the following lemma is immediate.

Lemma 7. *If* $r = \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$, *then* $\mathbb{E}(X_r^t) \to \infty$ *as* $n \to \infty$.

We now analyze the variance of the random variable X_r^t.

Lemma 8. *If* $r = \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$, *then* $Var(X_r^t) = o(\mathbb{E}^2(X_r^t))$.

Proof. For $1 \le j \le \binom{n}{r}$, let I^j be the corresponding indicator random variables. Hence,

$$X_r^t = \sum_{j=1}^{\binom{n}{r}} I^j.$$

By the linearity of expectation, we have that

$$\begin{aligned} \mathbb{E}((X_r^t)^2) &= \sum_{j=1}^{\binom{n}{r}} \mathbb{E}\left((I^j)^2\right) + 2\sum_{j \neq i}^{\binom{n}{r}} \mathbb{E}\left(I^i I^j\right) \\ &= \mathbb{E}\left(X_r^t\right) + 2\sum_{j \neq i}^{\binom{n}{r}} \mathbb{E}\left(I^i I^j\right). \end{aligned} \tag{2.2}$$

We fix an r-set S_1. For $0 \le j \le r-1$, denote by $S(j)$ the set of r-sets which intersect S_1 in j elements. Let I_1^t and I^{jt} be the indicator random variables, where the events $I_1^t = 1$ and $I^{jt} = 1$ represent that S_1 and $S_j \in S(j)$ are total dominating sets, respectively. Then

$$2\sum_{j \neq i}^{\binom{n}{r}} \mathbb{E}(I^i I^j) = \binom{n}{r} \sum_{j=0}^{r-1} \binom{r}{j} \binom{n-r}{r-j} \mathbb{E}(I_1^t I^{jt}).$$

Together with (2.2), we obtain that

$$\begin{aligned} \mathbb{E}((X_r^t)^2) &= \mathbb{E}(X_r^t) + \tbinom{n}{r} \textstyle\sum_{j=0}^{r-1} \tbinom{r}{j} \tbinom{n-r}{r-j} \mathbb{E}(I_1^t I^{jt}) \\ &= \mathbb{E}\left(X_r^t\right) + A^t, \end{aligned} \tag{2.3}$$

where $A^t = \binom{n}{r} \sum_{j=0}^{r-1} \binom{r}{j} \binom{n-r}{r-j} \mathbb{E}(I_1^t I^{jt})$. As each total dominating set is a dominating set, $A^t \le A$. By Lemmas 4 and 6, we therefore have that

$$A^t \le \; rg(1)\binom{n}{r} + (1+o(1))\mathbb{E}^2(X_r^t)\left(1 + 2r(1-p)^r - \frac{r^2}{n}\right), \tag{2.4}$$

where $g(1)$ is given in (2.1).

By (2.3) and (2.4) we have that

$$\frac{Var(X_r^t)}{\mathbb{E}^2(X_r^t)} \leq \frac{1}{\mathbb{E}(X_r^t)} + (1+o(1))\left(2r(1-p)^r - \frac{r^2}{n}\right) + \frac{rg(1)\binom{n}{r}}{\mathbb{E}^2(X_r^t)}. \tag{2.5}$$

To show that $Var(X_r^t) = o(\mathbb{E}^2(X_r^t))$, it suffices by Lemma 7 to show that

$$\frac{rg(1)\binom{n}{r}}{\mathbb{E}^2(X_r^t)} = o(1).$$

For sufficiently large n we have that

$$\begin{aligned}
\frac{rg(1)\binom{n}{r}}{\mathbb{E}^2(X_r^t)} &= \frac{r \times 2r\frac{n^{r-1}}{(r-1)!}\exp\left(n\left((1-p)^{2r-1} - 2(1-p)^r\right)\right)}{\binom{n}{r}\left(\left(1-(1-p)^r\right)^{n-r}\left(1-(1-p)^{r-1}\right)^r\right)^2}(1+o(1)) \\
&\leq \frac{3r^3}{n}\frac{\left(1-2(1-p)^r+(1-p)^{2r-1}\right)^n}{\left(1-(1-p)^r\right)^{2n-2r}\left(1-(1-p)^{r-1}\right)^{2r}}(1+o(1)) \\
&\leq \frac{3r^3}{n}\left(1+\frac{p(1-p)^{2r-1}}{(1-(1-p)^r)^2}\right)^{n-r}\left(1+\frac{2p(1-p)^{r-1}}{(1-(1-p)^{r-1})^2}\right)^r(1+o(1)),
\end{aligned}$$

where the first equality follows by (2.1) and since $\exp(x) \sim 1 + x$ if x is close to 0.

Since $1 + x \leq \exp(x)$ for $x \geq 0$, we obtain that

$$\begin{aligned}
\frac{rg(1)\binom{n}{r}}{\mathbb{E}^2(X_r^t)} &\leq \frac{3r^3}{n}\exp\left(\frac{(n-r)p(1-p)^{2r-1}}{\left(1-(1-p)^r\right)^2} + \frac{2rp(1-p)^{r-1}}{(1-(1-p)^{r-1})^2}\right)(1+o(1)) \\
&\leq \frac{3r^3}{n}\exp\left((1+o(1))p\frac{((\mathbb{L}n)(\log n))^2}{n}\right)(1+o(1)) = o(1),
\end{aligned}$$

as $r = \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n))\rfloor + 2$. ■

2.2. Proof of Theorem 3

We use the following lemma, whose proof is straightforward and so is omitted. For $r \geq 1$, let X_r^I be the random variable which denotes the number of independent dominating sets of size r.

Lemma 9. (1) *For all* $r \geq 1$

$$\mathbb{E}(X_r^I) = \binom{n}{r}(1-(1-p)^r)^{n-r}(1-p)^{\binom{r}{2}}.$$

(2) *Let* $\lambda \in (\frac{1}{2}, 1)$ *be fixed. For* $\lfloor \mathbb{L}n \rfloor + 1 \leq r \leq \lfloor 2\lambda \mathbb{L}n \rfloor$, *as* $n \to \infty$ *we have that* $\mathbb{E}(X_r^I) \to \infty$.

Our final lemma estimates the variance of X_r^I.

Lemma 10. *Let* $p \in (0,1)$ *and* $\lambda \in (\frac{1}{2}, 1)$ *be fixed. For* $\lfloor \mathbb{L}n \rfloor + 1 \leq r \leq \lfloor 2\lambda \mathbb{L}n \rfloor$,

$$Var\left(X_r^I\right) = \mathbb{E}^2(X_r^I) O\left(\frac{(\log n)^4}{n^{1-\lambda}}\right).$$

By Chebyshev's inequality and Lemmas 9 and 10, we have that

$$\begin{aligned} \mathbb{P}(\gamma_i(G) > r) &= \mathbb{P}(X_r^I = 0) \leq \mathbb{P}(|X_r^I - \mathbb{E}(X_r^I)| \geq \mathbb{E}(X_r^I)) \\ &\leq \frac{Var(X_r^I)}{\mathbb{E}^2\left(X_r^I\right)} = o(1). \end{aligned}$$

The assertion of Theorem 3 follows, therefore, once we prove Lemma 10.

***Proof of Lemma* 10.** We denote by $\mathbb{E}((X_r^I)^2)$ the expectation of the number of ordered pairs of independent domination sets of size r in $G \in G(n,p)$. The expectation satisfies

$$\begin{aligned} \mathbb{E}\left(\left(X_r^I\right)^2\right) &= \sum_j \binom{n}{r}\binom{r}{j}\binom{n-r}{r-j}\left(1-(1-p)^r\right)^{2(n-2r+j)} \\ &\quad \times \left(1-(1-p)^{r-j}\right)^{2(r-j)}(1-p)^{2\binom{r}{2}-\binom{j}{2}}. \end{aligned} \tag{2.6}$$

The explanations for the terms in the equation (2.6) are as follows. The vertices of the first independent dominating set S_1 may be chosen in $\binom{n}{r}$ ways. The independent dominating sets S_1 and S_2 may have j elements in common. These vertices may be chosen in $\binom{r}{j}$ ways. The rest of $r-j$ vertices of S_2 may have to be chosen from $V(G)\backslash S_1$, which gives the $\binom{n-r}{r-j}$ term. Every vertex not in $S_1 \cup S_2$ must be joined to one of S_1 and one of S_2, and so we obtain the term $(1-(1-p)^r)^{2(n-2r+j)}$. Every vertex in $S_1 \backslash S_2$

must be joined to one of $S_2 \backslash S_1$, and every vertex in $S_2 \backslash S_1$ must be joined to one of $S_1 \backslash S_2$, and so we have the term $(1-(1-p)^{r-j})^{2(r-j)}$. Both sets S_1 and S_2 are independent, which supplies the last term.

Observe that $(1-p)^{r-j} \geq (1-p)^r$. Hence, by (2.6) and Lemma 9 (1), we have that

$$\begin{aligned} \mathbb{E}\left(\left(X_r^I\right)^2\right) \leq \mathbb{E}^2\left(X_r^I\right) \frac{1}{\binom{n}{r}} \Bigg(\binom{n-r}{r} \\ + r\binom{n-r}{r-1} + \sum_{j=2}^{r} \binom{r}{j}\binom{n-r}{r-j}(1-p)^{-\binom{j}{2}} \Bigg). \end{aligned} \tag{2.7}$$

By the choice of r it follows that

$$\begin{aligned} \frac{1}{\binom{n}{r}}\left(\binom{n-r}{r} + r\binom{n-r}{r-1}\right) &= \left(1-\frac{r^2}{n}\right)\left(1+O\left(\frac{(\log n)^4}{n^2}\right)\right) \\ &\quad + \frac{r^2}{n} + O\left(\frac{(\log n)^3}{n^2}\right) \\ &= 1 + O\left(\frac{(\log n)^4}{n^2}\right), \end{aligned} \tag{2.8}$$

and

$$\frac{1}{\binom{n}{r}} \sum_{j=2}^{r} \binom{r}{j}\binom{n-r}{r-j}(1-p)^{-\binom{j}{2}} = O\left(\frac{(\log n)^4}{n^{1-\lambda}}\right). \tag{2.9}$$

By (2.7), (2.8), and (2.9) the assertion follows. ∎

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